The impact of software on Mathematics education in high schools

Thesis

Presented to the Department of
Mathematics and Computer Science of
University of Calabria
for the Degree of
DOCTOR OF PHILOSOPHY
IN
MATHEMATICS AND COMPUTER SCIENCE

by

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The University of Calabria

February 2017

Teachers need to integrate technology seamlessly into the curriculum instead of viewing it as an add-on, an afterthought, or an event.

Heidi-Hayes Jacobs

I am coming more and more to the conviction that the necessity of our geometry cannot be demonstrated, at least neither by, nor for, the human intellect.

Carl Friedrich Gauss

Chance only favours the mind which is prepared

Louis Pasteur

To my family

Acknowledgements

I would like to thank my thesis supervisor, Prof. Annarosa Serpe, for taking the time to understand my ideas, for providing professional advice and guidance to improve my dissertation, and for her questions and feedback, which were crucial to the completion of my work. I am so very grateful for the opportunity to work under her guidance.

I also want to extend my appreciation to the entire educational institution of "Liceo Scientifico - Fermi" of Cosenza, the head teacher and teachers who kindly welcomed me.

Special thanks to the students involved in the experimentation: this research would not have been possible without their valuable contributions. They welcomed me into their classrooms, shared their teaching and learning experiences with me, gave me the opportunity to observe them interacting with students and delivering math content in a technology driven environment, and provided me with insights and honest opinions about their everyday teaching practices. I am so very thankful.

Sommario

L'impatto dei software sulla didattica della Matematica nella scuola secondaria di II grado

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Il rapporto tra tecnologia e insegnamento-apprendimento della Matematica è un fenomeno complesso che merita di essere osservato da diversi punti di vista.

Uno di questi è il ruolo che i software svolgono, e potrebbero svolgere nella pratica didattica di Matematica.

Nell'ultimo trentennio, la disponibilità di software nella scuola secondaria di II grado italiana è notevolmente aumentata come in molti altri paesi, questo ha aperto nuove prospettive nel processo di insegnamento e apprendimento della Matematica.

A tal riguardo esiste una letteratura di ricerca vasta e crescente relativa all'integrazione di software didattici proprio perché presentano caratteristiche diverse che vanno ad incidere sulle modalità d'uso in classe: non va considerato tanto l'impatto strumentale quanto la dimensione del cambiamento generato nel processo d'insegnamento-apprendimento.

Lo scopo di questa tesi è quello di dare un posto centrale allo studio dell'impatto dei software nella didattica della Matematica nella scuola secondaria di II grado.

A tale ragione sono stati scelti ed esaminati, nelle loro intenzionalità e potenzialità didattiche, due differenti tipologie di software: il primo è il software di geometria dinamica GeoGebra, il secondo è l'ambiente di programmazione MatCos.

Lo studio è finalizzato alla disamina degli effetti prodotti dall'impatto dei suddetti software nella didattica dei seguenti argomenti: costruzione geometrica di curve piane e calcolo della probabilità.

La scelta degli argomenti è dovuta al fatto che questi nella pratica didattica in classe sono - quasi sempre - trascurati, con conseguente grave perdita nel percorso formativo degli studenti.

Lo studio di ricerca ha visto coinvolti un campione di studenti di scuola secondaria di II grado suddiviso in due gruppi: il primo ha utilizzato i due software durante tutte le attività di sperimentazione, mentre il secondo non li ha utilizzati.

L'attività di sperimentazione ha consentito di effettuare l'indagine sull'impatto dei due software nella didattica degli argomenti prescelti, nonché sulle differenze, in termini di risultati d'apprendimento, tra i due gruppi di studenti.

In definitiva, questo studio d'impatto ha preso in esame l'inquadramento dei risultati ottenuti dagli studenti e osservati sul piano epistemologico, cognitivo e didattico.

Abstract

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The relationship between technology and Mathematics teaching-learning is a complex phenomenon that deserves to be considered from different points of view.

One of these is the role that software play or could play in the practical teaching of Mathematics.

Over the last thirty years, the availability of software in Italian secondary schools, as well as in many other countries, has greatly increased. This has opened up new perspectives in the process of Mathematics teaching and learning.

These days there is vast and growing research literature available on the integration of educational software because each has different characteristics that affect how it can be used in the classroom: what needs to be considered is not its instrumental impact, but the change generated in the teaching-learning process.

The purpose of this thesis is to study the impact of software in Mathematics teaching at secondary school level.

Two different types of software were chosen and examined to this end for their intentionality and educational potential: the first is the dynamic geometry software GeoGebra, the second is the MatCos programming environment.

The study is aimed at the analysis of the effects produced by the impact of the above software in teaching the following topics: geometric construction of plane curves and calculus of probabilities. The choice of topics is due to the fact that in class teaching practice these are almost always neglected, resulting in serious gaps in student education.

The research study involved a sample of higher secondary school students, divided into two groups: the first used the two software during all testing activities, while the latter did not.

The experimentation allowed us to carry out a survey on the impact of the two software on the teaching of the chosen subjects, as well as on the differences, in terms of learning outcomes between the two groups of students.

This assessment examined the classification of the results observed and obtained by the students on the epistemological, cognitive and didactic level.

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Chapter 1: Introduction

Today more than ever Mathematics finds its *raison d'etre* in the contribution it can make to the development of thinking and personality as a whole. Thus teaching it means to take into account not only its cultural role, but also the reasons that feed this role in a constantly changing society. Hence the need for a reinterpretation of teaching, which is traditionally based on the transmission of notions, and on receptive and relatively solitary learning.

Alongside content knowledge, the acquisition of different competencies such as problem solving, reasoning and communication is aimed for. At the same time, the setting up of learning situations that favour reasoning and communication requires the integration of different areas interacting with each other to form a complex network, for example, the mental models in cognitive science, cooperative learning and technological environments. This Chapter provides the background and aim of the thesis. It also outlines the research questions.

1.1 Research context

Technological development in recent decades has changed the teaching of Mathematics in a remarkable way. In fact, the use of computers supported by appropriate software such as dynamic geometry environments, computer algebra systems, spreadsheets, programming languages, etc. has increased over time. At first - dictated by the euphoria of novelty - it was thought that this technology would free the students from the toils of data and calculus procedures, thus allowing the exploration of mathematical phenomena. However, some scholars such as Lagrange et al. (2003) pointed out some difficulties resulting from the use of technology in the Mathematics classroom. These difficulties pertain to the complexity of teaching-learning the discipline on the one hand, while on the other they are specific to the use of education tools, i.e. the risk of perceiving the instrument as the target, with learners possibly wandering through what can seem like endless possibilities. In today's society, there is broad consensus on the daily use of technology in the teaching-learning of Mathematics at both national and international level. However, issues relating to the integration of technology into institutional teaching - traditional content and above all, traditional methods- remain completely open. Also the *National Council of Teachers of Mathematics* (NCTM) stresses that:

> *Technology can facilitate mathematical problem - solving, communication, reasoning and proof; moreover, technology can provide students with opportunities to explore different representations of mathematical ideas and support them in making connections both within and outside of Mathematics*[1].

So it is not just a matter of how to use the technology, but how good teaching methods and practices should be accompanied by strong, sound and practical pedagogical thinking, able to redefine the boundaries, contents and actions of teaching according to the cognitive needs of learners.

The integration of technology in Mathematics teaching must significantly extend and enrich the traditional strategies implemented by teachers; at the same time the process must engage students in the discovery of mathematical thinking through exploration and investigation of the different representations of concepts and the simulation of real life phenomena (Doer, & Pratt, 2008; Heid, & Blume, 2008; Ritter, et al., 2008).

In Italian Secondary Schools, the combination Mathematics - sensible reality is a widely proclaimed truth, but in reality, this is seldom applied because it is considered difficult by teachers, as it requires a reinterpretation of traditional teaching practice. For

[1] National Council of Teachers of Mathematics, 2000.

example, there are some topics like plane curves and the calculus of probability which are traditionally overlooked not only because of time issues, but because they are difficult to manage in the classroom.

Outside the classic learning paths on conics, other plane curves such as spirals, epicycloids, hypocycloids, etc., are in most cases totally ignored or only touched on from an analytical point of view, thus neglecting the essential Euclidean teaching which consists in perceiving the nature of the overall structure which starts from the 'simple and obvious' and arrives at the 'complex not obvious'. Indeed, we must not forget that Euclidean geometry plays a crucial role especially for its nature as a 'model' of easily represented concrete situations. As stated by Lucio Russo:

> *It is obvious that the points, segments, triangles and other entities covered by a geometry book are not concrete objects, but it is equally clear that the ability to draw concrete figures that' approximate ' the ideal figures studied by Mathematics greatly helps intuition and is an essential key for application to theory*[2].

Plane curves offer a rich and partly unexplored field of study that can be discovered and rediscovered also in interdisciplinary contexts because they combine the physicality of objects and mechanisms with the abstraction of mathematical thinking; adequately inserted in classroom teaching practice they help students to identify the correlations between the concepts of geometry, the technical mechanisms and structures of science.

The curves refer back to the drawing that plays a fundamental role in so far as it fits between the concreteness of reality and the abstractions of the geometrical concepts it represents. The drawing serves as basis in the dialectical process between the figural and

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[2] Russo, L. (2000). *Segmenti e bastoncini. Dove sta andando la scuola?* (pp. 27). Feltrinelli Editore.

conceptual component (Mariotti, 1995); it represents the summary of an observed object into a few lines, so it is a cognitive moment, a practice that encompasses analysis and synthesis. In fact, a curve as a place of points that satisfies a given property, generally passes through the geometric construction of one of its points with the classic tools, line and compasses, which today can be integrated and/or replaced by updated educational software able to process and manipulate geometric constructions.

These translate the classical tradition into computer software. Their use in the teaching-learning of Mathematics has been well highlighted in the literature (Doerr, & Zangor, 2000; Ruthven, & Hennessy, 2002; Artigue, et al., 2003; Lagrange, et al., 2003; Monaghan, 2004; Lagrange, & Ozdemir Erdogan, 2009; Dix, et al., 2004; Guzman, & Nussbaum, 2009; Collins, et al., 1988; Costabile, & Serpe, 2012 and many others).

The calculus of probability is a topic often overlooked in teaching practice (Rossi, 1999; Vighi, 2012), although its importance in the curriculum in Italian secondary schools had already been taken into consideration in the drafting of the experimental ministerial programs PNI[3] (National Informatics Plan) in the second half of the 80s, and by Brocca[4] in the early 90s. As a result, this represent a gap in student education because the topic is well suited to a job of mathematization and formal arrangement of sensible reality, as well as the many applications which convey a more dynamic vision of Mathematics open to the real world. Other European countries have also sought to enhance the cultural debate around the meaning and interpretation of probability that has accompanied the development of mathematical theory, by including it into the school curriculum (Ahlgren, & Garfield, 1991; Shaughnessy, 1992; Batanero, et al., 2005). The efforts made in this

[3] C.M. January 28, 1989, n.35.
[4] Piani di studio della scuola secondaria superiore e programmi dei primi due anni. Le proposte della Commissione Brocca. *Studi e documenti degli annali della Pubblica Istruzione*, 56. La Monnier.

direction, however, in most cases have not produced significant results and the topic has not received proper attention for a number of reasons, among which the inadequate teacher education stands out (Stohl, 2005). Relegating the calculus of probability to a secondary, often marginal, role has given rise to a number of misconceptions that are dangerous, especially if related to the strong growth of gambling in Italy (Amodio, & Depueto, 2014; Fandiño Pinilla, 2010; Vighi, 2012). On the other hand, the specialist press does nothing but encourage false probabilistic theories while practice of the game becomes more and more widespread.

Famous mathematicians and teachers of Mathematics, such as Bruno De Finetti, have highlighted the importance of probability in thought development:

> *The calculus of probability is the logic of the probable. Just as formal logic teaches to derive the truth or falsity of certain consequences from the truth or falsity of certain premises, so the calculus of probability teaches to derive the greater or lesser likelihood or probability of certain consequences from the degree of likelihood or probability of certain premises*[5].

The calculus of probability, because of the problems that have generated and promoted its development, is a topic that lends itself well to a phenomenological-inductive approach, which is not considered in the didactic transposition. On the rare occasions when it is proposed in class, it is treated as a list of properties and results to be applied in often-limited settings, as roll of the dice, extractions of coloured balls from an urn, etc. (Rossi, 1999). Once again, the transmission of information prevails, ignoring the fact that

[5] De Finetti, B., 1981, *Scritti (1926-1930)*, con Prefazione di M. De Felice, CEDAM, Padova, pag. 261-262.

probabilistic ideas can also be explained through alternative and complementary forms: words and numbers, images and / or graphs and use of technology.

1.2 Assumption of the study

The use of technology in daily classroom practice opens up new opportunities in Mathematics teaching-learning processes, and at the same time creates active links between ideas and mathematical contents.

In this sense, technology, as part of a suitable methodological framework, constitutes a mediator for learning.

The present study fits the above framework, whose research hypotheses are:

1. Students who use this software achieve better results, in terms of learning, than those who do not;
2. The use of such software promotes the development of critical thinking and problem solving skills.

1.3 Research question

The research study is designed to show how the use of two different types of software - Dynamic geometrical software (GeoGebra) and a programming environment (MatCos) - oriented towards Mathematics will prove effective in classroom teaching.

The fundamental questions for this research study are:

1. What is the impact of GeoGebra on the construction of plane curves?
2. What is the impact of MatCos on the approach to the calculus of probability?
3. In terms of learning outcomes, are there any differences between students who use this software in classroom practice and those who do not?

1.4 SIGNIFICANCE OF THE STUDY

The research study involved a sample of students attending classes III and IV of a Cosenza High School (Italy). The study is aimed at the analysis of the effects produced by the impact of the above software in the teaching of the following subjects: geometric construction of plane curves and calculus of probabilities.

The selection of topics is motivated by the fact that in classroom practice they are almost always neglected, resulting in serious gaps in student education. The importance of geometry learning for developing thinking skills has been abundantly stressed in literature (Laborde, et al., 2006; Hilbert, et al., 2008), and still is. The studies conducted by Van Hiele go in this direction, and suggest five levels for the development of geometrical thinking. The van Hiele levels are consistent with the constructivist need to understand students' construction of geometric knowledge and provide a description of the levels of students' geometric thinking and indications of strategies used within these levels (Van Hiele, 1986; Clements & Battista, 1992; Battista, 2008). This theory is quite helpful for understanding the progress student make as their geometric thinking develops.

- ***Level 0 (Prerecognition)***.

  Students attend only to part of a shape's visual cheracteristics and are unable to identify many common shapes;

- ***Level 1 (Visual)***.

  Students recognize and mentally represent shapes such as squares and triangles as visual wholes. Their reasoning is dominated by imagery and visual perception rather than by analysis of geometric properties. When identifying shapes, students often use visual prototypes, saying that a figure is a rectangle, for instance, because 'it looks like a door';

- ***Level 2 (Descriptive/Analytic)***.

Students recognize and characterize shapes by their geometric properties, that is, by explicit describing spatial relationships between shape parts. For instance, students might think of a rectangle as a figure that has opposite sides equal and four right angles. While still important, the holistic appearance of shapes becomes secondary because students identify shapes by formal properties rather than mere reference to visual prototypes;

- ***Level 3 (Abstract/Relational)***.

  Students interrelate geometric properties, form abstract definitions, distinguish between necessary and sufficient sets of properties for a class of shapes, and understand and sometimes even provide logical arguments in the geometric domain. They meaningfully classify shapes hierarchically and give arguments to justify their classifications (e.g., a square is identified as a rhombus because it has the defining property of a rhombus, all sides congruent);

- ***Level 4 (Formal Axiomatic)***.

  Students formally prove theorems within an axiomatic system. That is, they produce a sequence of statements that logically justifies a conclusion as a consequence of the 'givens'. Thus, by necessary, thinking at level 4 is required for meaningful and full participation in a proof-oriented high school geometry course.

Scientific teaching that proposes to bring about lasting and informed learning cannot be separated from awareness of the spontaneous conceptions of probability and a reflection on its meaning and conceptual reach. Many years ago Bruno De Finetti observed:

On many occasions a lot of people reason badly because they don't know probabilistic and statistical concepts.

But it also often happens that, on such occasions, many others reason badly because they have learnt some probabilistic and statistical concepts without fully understanding them, or misunderstanding them enough to apply them wrongly.

In this tendency to err at all costs we can certainly detect [...] an effect of aversion to uncertainty: either one does not apply the concepts which express uncertainty, or applies them forcing the interpretation so as to transform uncertain predictions into certain ones, or in such a way as to obtain, thanks to the strangest misunderstandings, gratuitous or distorted conclusions[6].

In Mathematics teaching-learning integration of technology has great educational potential as far as cognitive aspects and teaching are concerned. For example, students have the opportunity to explore, investigate, engage in interactive and problem-based activities, which increases motivation and interest for the task to be accomplished (Laborde, et al., 2006; Hollebrands, et al., 2008). For the teacher, instead, technology offers the possibility to diversify methodology in relation to the learner. So all three roles need to be considered simultaneously: the teacher, the student and technology. In this research study, we chose to use two different types of software, that is GeoGebra for activities related to the geometric construction of curves, and the MatCos programming environment for the calculus of probability so as to analyse the educational and cognitive potential of both.

The choice of GeoGebra is justified by the fact that thanks to its affordances it is the ideal environment for the development of geometrical activities. Indeed, it allows students to create and manipulate constructions easily and interact with geometry, acquiring a broader vision and stimulating the imagination (Accomazzo, et al., 2013).

[6] De Finetti B. (1967). *Il "saper vedere" in matematica.* Torino: Loescher, pag.52.

The choice of the MatCos programming environment was based on its educational features, as it helps students to think actively by solving problems - a logical and sequential thinking, at the same time intuitive and creative, and encourages linguistic rigour (Costabile, & Serpe, 2012, 2013). It also involves the manipulation of mathematical objects, and subsequent drafting of a programme, and generates simulations of real objects in a virtual environment. So it provides the ideal environment for the calculus of probability.

1.5 Limitation of the study

The research study involved 85 students from the secondary school 'E. Fermi ‘, Cosenza: 45 students attending class III (Year 11) and 40 students attending class IV (Year 12). The sample of students was divided into two groups; the first group of 21 students from class III and 23 from class IV already had some prior knowledge of the two software; the second consisted of 24 students from class III and 17 students from class IV.

The first group of students used the software during the lessons, while the second group did not use the technology and took part in traditional lessons (lectures with chalk and blackboard).

The primary purpose of this research study is to compare the development of students’ mathematical thinking on the topics discussed in class in terms of learning outcomes when these are supported by the use of software, and when they are not. In this research study, the following limitation became apparent: the data obtained provide significant information on a small scale on the differences in learning outcomes from different teaching strategies in daily classroom practice when teachers used the software or when they did not.

1.6 Plan of the thesis

This thesis consists of five chapters.

Chapter 1 provides the background and the aim of the thesis. It also describes the outlines of the research questions.

Chapter 2 will try to answer the following questions:

- Why and how can technologies mediate the learning-teaching of Mathematics?
- What added value can they provide?
- How can we analyse and evaluate educational software?

In order to assess a specific teaching software product, we will introduce three levels of analysis, which will later be used as operational tools for the evaluation of GeoGebra and MatCos, two software programmes used during experimentation in the classroom.

The third chapter revisits the mode of interaction between technology, teacher and students within the discipline (Mathematics). It emphasizes the educational and pedagogical value of DGS and programming languages, and finally examines some theoretical tenets on the use of technology in modelling activities, with particular reference to issues concerning geometric construction and the introduction to the calculus of probability.

Chapter IV provides an overview of the implementation of the studies. First, it describes issues relating to the research design. Then the chapter provides a discussion of the used methods in the context of laboratory practice. To this end, it describes the sample of students involved in the experimental path and outlines the procedures. A final section of the chapter illustrates the specific experimentation activities reported in some papers, produced during PHD studies, since they are closely related to this thesis' research questions.

Finally, Chapter 5 shows the Data Analysis of the initial and final tests and discusses the results. It also provides some conclusions and suggestions for further research.

1.7 DEFINITION OF THE TERMS

For the purpose of the thesis reading it is useful to report some of the terms used.

Didactic Technology (**DT**): machines and tools (artefacts) used to implement the teaching -learning process and systematic methods, based on the application of scientific knowledge, employed in the design and management of the teaching-learning processes (Fierli, 2003).

Learning Software: software oriented towards the needs and learning style of the learners involved and valuable for the *teaching potential* it expresses (Castoldi, 2000).

Educational Software (**ES**): the software is a replacement or a subsidiary of the teacher and is characterized by its *didactic intention* (Castoldi, 2000).

Computer Algebra System (**CAS**): software system that facilitates the execution of symbolic processing (Davenport, et al., 1988).

Dynamic Geometry Software (**DGS**): software that allow you to create and later manipulate geometric constructions (Bantchev, 2010).

Students attending class III: 11^{th} grade students (16 and 17 years old).

Students attending class IV: 12^{th} grade students (17 and 18 years old).

Chapter 2: Software in Education

Why and how can technologies mediate the learning-teaching of Mathematics? What added value can they provide? How can we analyse and evaluate educational software? These are some of the questions we will try to answer in this chapter. In order to assess a specific teaching software product, we will introduce three levels of analysis, which will later be used as operational tools for the evaluation of GeoGebra and MatCos, two software programmes used during experimentation in the classroom.

2.1 Didactic Technology

In recent decades the development and evolution of technology has facilitated the creation of new and more powerful digital tools, and has promoted their multiple use in various fields of knowledge. In education, technologies have risen to the role of intellectual tools, in the broadest sense of the term, because their different usage patterns have altered the relationship with knowledge acquisition. In its broadest sense, the '*didactic technology*' expression (DT) is adopted here to refer to the relationship between technology and teaching-learning. Among the different meanings of the acronym, the most important ones are:

- machines and tools (artefacts) used to implement the teaching-learning process;
- systematic methods, based on the application of scientific knowledge, employed in the design and management of the teaching-learning processes and means used in it[7].

The circular dynamic relationship between means and methods is reflected in the interweaving that characterizes the history of DT. In order to fully understand the

[7] Fierli, M. (2003). *Tecnologie per l'educazione*, Edizioni Laterza, Bari.

educational and cognitive consequences of the potential of DT, it seems appropriate to frame reflections into the following three lines of thought: the social-technological, the psycho-pedagogical and the techno-anthropological approach.

- *Social-technological approach*

  McLuhan's studies[8] represent an essential point of reference regarding the significance of technology in the educational process. McLuhan's basic assumption is the idea that "*the medium is the message*", i.e. the ways in which information and cultural heritage are transmitted both affect them, and influence their meaning. Each medium represents an extension of either physical or mental faculties, and can change the ways in which we perceive and think about the world. Electronic media, then, qualify as "*thought amplifiers*" because they promote the overall expansion of sensory abilities. Another thread of thought highlighted by McLuhan, the development of "*imploding*" technology, is characterized by a cancellation of the space-time constraints and the multiplication of cultural perspectives. Hence, the metaphor of the "*global village*" that breaks down the boundaries between individuals and promotes exchange and dialogue between different cultures.
- *Psycho-pedagogical approach*

  Educational psychology contributions, which take into account the insights of Bruner[9] and Vygotshij[10], postulate intelligence as the internalization of the tools generated by a given culture, and therefore state that cultural media have an important role in cognitive development. The close interaction between cognitive processes and cultural context forms part of a constructivist approach and

[8] McLuhan, M. (1976). *La Galassia Gutenberg*, Armando, Roma.
[9] Bruner, J.S. (1974). *Il significato dell'educazione*, Armando, Roma.
[10] Vygotskij (1990). *Mente e linguaggio*, Laterza, Roma-Bari.

distributed knowledge: the subject constructs his or her representation of reality within a specific social and cultural environment and through the tools that are typical of that culture. Olson's thinking also fits this perspective, and sees intelligence as the progressive mastery of a cultural medium, stating that "There are as many ways to be smart as there are languages with which we can represent our intelligence"[11]. The concept has evolved with Gardner, who assumes the existence of multiple intelligences, independent of each other and arising from different sets of symbols pertaining to the different cultural contexts[12].

- *Techno-anthropological approach*

The contributions of Pierre Lévy are reflections on the relationship between mind, culture and media. Lévy speaks of "*intelligence technologies*"[13] referring to the collective perception of technology, thought and communication.

For Lévy the structures derive from "*a diverse, mixed world, whose subjectivity effects emerge from local and transient processes*". The technique is defined in relation to the social mode of use of its potential. This is a strong factor, which alters the possibility of cognitive production. Lévy's attitude towards knowledge is constructivist and situated. Indeed, it is the communicative act, which makes messages meaningful. Each new message questions the context up to that moment, and the situation is continuously redefined by the ongoing process of collective interpretation. For Lévy hypertexts explain all the aspects of reality in which meanings come into play, like communication and socio-technical processes. The hypertext is the metaphor of the hermeneutical theory of communication, whose

[11] Olson, D.R. (1979). *Linguaggi, media e processi educativi*, Loescher, Torino.
[12] Gardner, H. (1987). *Formae mentis*, Feltrinelli, Milano.
[13] Lévy, P. (1992). *Le tecnologie dell'intelligenza*, Synergon, Bologna.

focus is the significance. Each person, by making sense of a message, constructs his or her own hypertext. The collective elaboration of a hypertext is the common feeling.

The set of relationships emerging from these approaches between mind, culture and media represent the background to a debate on the nature and role of learning in a context of technological innovation. At present, what kind of software is available for educational processes?

Given the speed with which the sector has been developing, any answer to this question is likely to be inaccurate at the time of writing; that is why we shall try to provide some categories of analysis and identify some new developments in software for educational use. To this end, we shall consider the three categories of computer use for educational purposes proposed by Taylor[14] in 1980, and still valid.

The first role that software can take is replacing, entirely or in part, the role of the teacher. Taylor speaks of tutors, indicating a teaching-learning process carried out with the computer: the electronic means is superimposed on the teacher's role. Further distinctions can be made within this context adopting the classification used in the United States of America: CAI[15] and CAL[16].

The second function is that of the computer as a *tool*, in particular a *work tool*, indicating a teaching-learning process done through the computer, where the computer acts as a mediator between the learning object and the learner.

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[14] Taylor, R. (1980). *The computer in the school: tutor, tool, tutee*, Teacher College Press, Columbia University, New York.

[15] Software for replacing teaching functions such as contents of transmission control. Within the CAI, you will recognize the following software types: demo, '*drill and practice*', tutorial.

[16] Software suitable for stimulating and supporting the learning process through the creation of '*settings*' where there is enough room for initiative and student choice. Within the CAL, we can find the following software types: educational games, simulations, and consultations software and hypermedia navigation.

The last function proposed by Taylor is the computer as *tutee*, that is to say an instrument that carries out instructions, to indicate a teaching-learning process centred on the knowledge of computers and how they are used; in this case the software, with its mode of operation and use, becomes also the object of the learning. An example of this are programming languages that allow users to give instructions to a computer so that it can perform certain operations or procedures in order to reach a predefined goal.

The features of the software in the context outlined so far should be the following:

- move their focus from teaching systems towards learning environments;
- from a techno-centric approach towards an anthropocentric approach;
- from a dual relationship machine-user to several dialogic and collaborative situations both face-to-face and remote;
- from execution models to interactive and collaborative models.

Therefore, we can highlight, as Castoldi[17] says, the transition from an instructionist paradigm to a constructivist paradigm, which includes multiple modalities: expression, construction, exploration, communication, research, collaboration. As a result of such a process of evolution and research, it is appropriate to speak of '*learning software*' which is oriented towards the needs and learning style of the learners involved.

The definition of '*educational software*' tends to shift the focus on the teaching aspect, assuming that the software is a replacement or a subsidiary of the teacher, but without any profound changes to the underlying model in which knowledge is passed on.

So we can say that while *educational software* is characterized by its *didactic intention*, *learning software* is valuable for the *teaching potential* it expresses.

[17] Castoldi M. 2000, Software per l'apprendimento. Milano: Franco Angeli.

In a school of the third millennium that keeps an open and active attitude, whose aims are diversified and not exclusively restricted to the passing on of notions, the educational value of a software calls into question the whole structure of the educational process, and consequently the role of the teacher. Inevitably, the above two definitions are already obsolete, especially if we consider the sophisticated educational software currently available on the market.

2.2 Didactic Technology in Italian Schools

In the mid-eighties of the last century personal computers (PCs) were introduced in Italian schools, following similar trends in other European countries. In 1985 the National Informatics Plan (PNI) was launched by the Ministry of Public Education (MPI), involving initially teachers of Mathematics and Physics in Secondary Schools[18], with the primary purpose of achieving computer literacy. The implementation of the PNI and the large scale experiments that followed it had a big impact, and effectively sanctioned the official role of computers in schools. The PNI, however, failed to bring renewed attention to the problems of educational innovation. On the contrary, schools were faced with machines engaged in an unfriendly 'dialogue' with students, largely because of the MS/DOS operating systems, and programming languages such as BASIC, PASCAL and others.

A first attempt at improving the situation came with the introduction of the LOGO[19] programming language, which represented a significant experience in the then Elementary School, now named Primary. LOGO holds a special place in the relationship between information and education precisely because its inventor, Seymour Papert, embraces the pedagogical principle of planning as a cognitive and constructive activity.

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[18] C.M. January 28, 1989, n.35.

[19] LOGO was created in the sixties as part of research into artificial intelligence by a research team from M.I.T. (Massachusetts Institute of Technology). Seymour Papert, former student of Piaget's in Geneva, and today considered the father of this language, was a member of the team.

Papert[20] claims that a program has no material substance, but is somehow a mechanism. By analysing it and seeing how it works you can actually assimilate the model that underpins it. Despite the introduction of an educational and innovative programme such as LOGO, the use of computers in Italian schools did not produce the desired effects for several reasons (Costabile, & Serpe, 2013, Fierli, 2003):

- Computers were used with a different operating system often incompatible with the various software packages;
- Poor graphics and limited interactivity;
- Computer were regarded suspiciously as gimmicks, which would just professional complicate matters for teachers.

In addition, we must also report an initial geographical discrimination due to the high cost of the hardware. Soon the PNI was extended also to Middle School; several experiments were conducted in that context, such as the Initiatives and Research Project for Informatics in schools (IRIS), promoted by European Centre of Education Frascati (CEDE), working on the BASIC programming language (Ciarrapico, 2002; Costabile, & Serpe, 2013; Grimaldi, 2006; Menghini, 2016; Tomasi, 2012). However, despite their official inclusion in the curriculum[21], there was not a massive spread of computers in lower and upper secondary schools[22] for different reasons, not least those related to logistical problems and to the in-service training of teachers. Nevertheless, from the second half of the eighties until the early nineties the upgrading of the hardware and the development of operating systems based on graphical user interfaces, led to a rapid evolution of both professional and educational software. In addition, in the late eighties, the student learner

[20] Papert, S. (1980). *Mindstorms: Computers and Powerful Ideas*. New York: Basic Books.
[21] C.M. February 6, 1991, n. 24.
[22] Students between 11 and 18 years.

begins to be taken into consideration, and the *tutor computer* becomes a *tool-cognitive computer*, that is a tool for organizing personal knowledge and for promoting socio-cognitive interaction among peers[23]. In the early nineties with the advent of the Windows operating system, the dialogue with computers became available to everyone and technological development opened new educational scenarios.

In 1991 the second PNI was launched, which thanks to the advent of multimedia extended computer use also to the subjects humanities[24]. Attention was no longer focused only on the computer and on the understanding of its logic, but on the content of teaching actions and on the conceptual aspects that characterize it. In fact, schools recognized the importance of technology and tried to define its role (Costabile, & Serpe, 2013). However, the school system struggled to translate these principles into practice, and to significantly expand the use of technology. In that period, the greatest innovation was the emergence of the Internet. Since then the evolution of increasingly sophisticated software has been going hand in hand with the production of always-new hardware devices.

In 1996, the European Union drew up the "*White Paper on education and training. Teaching and Learning. Towards a learning society*" that outlined the scenario of the first few years of the new millennium, and indicated three major factors of change: the information society, the worldwide extension of trade, and scientific and technical culture. Even Italy aligned with European directives, and started the Educational Technology Development Program (PSTD, 1997-2000) the following year, which allocated a number of computers to each school and set up basic training for Elementary and Kindergarten school teachers. The target of the programme launched by the Ministry of Education[25] was

[23] Vygotskij, L.S. (1990). *Pensiero e linguaggio* (1934), Laterza, Roma-Bari.
[24] C.M. February 6, 1991, n. 24.
[25] C.M. April 24, 1997, n. 282.

the training of a sufficient number of teachers, able to use the Internet and multimedia resources, by the end of 2002. Traditional computer literacy training, detached from teaching, gave way to a process that seeked to combine familiarization with the new technologies with action research strategies. This was to be achieved through the design of curricular pathways which typically promoted students' divergent thinking, their language and communication skills, of graphics and sound-musical competences. The PSTD was followed the Multilab[26] project, implemented in twenty national centres.

Since 2000 there has been a spread of information technology in all areas of the Italian school system and in education in general: in October 2000 the Ministry of Education signed an agreement with RAI[27], and delivered a distance-learning training course in the use of educational multimedia technologies.

The course, called Multimedi@scuola[28], was meant to provide teachers, already equipped with basic computer literacy, with new cultural tools to be used in daily educational activities. Specific training courses focusing on the use of information technologies and their application in educational fields are now also part of initial teacher training, for example the Postgraduate Teaching in Secondary Schools (SSIS) and the Degree in Primary Education (SFP). As for the in-service training of teachers, courses at both regional and national level are nowadays available which provide instruction on the use of new technologies; among these we note the National Teacher Training Plan on Information and Communication Technologies (ForTic)[29].

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[26] C.M. January 19, 1996, n.31.
[27] Company in charge of public service broadcasting in Italy. It is one of the largest media companies in Europe, the continent's fifth television group.
[28] C.M. October 18, 2000, n. 233.
[29] C.M. May 21, 2002, n. 55.

In 2005, in order to bridge the existing gap between the widespread use of technology tools in everyday life and the insufficient space reserved to these same tools in schools, the project DiGi School[30], was implemented, sponsored by the Ministry for Reforms and Innovation in Public Administration, in collaboration with the Ministry of Education, Research and University (MIUR). The DiGi School Project saw the introduction of the digital whiteboard (IWB), which was to take a key role in the innovation of teaching instruments. In recent years many other projects we have been set up. Among them Biblioteche nelle Scuole, @urora, Scuola in Ospedale, @pprendere Digitale are worth mentioning. In 2008, the National Plan for a Digital School was launched (PNSD), which sanctioned the digitization process through the following measures:

- IWB action - capillary diffusion of IWBs in classrooms of all schools and levels;
- Action Cl@ssi 2.0 aimed at enhancing the learning environment through consistent and widespread use of technology to support teaching practice in the classroom;
- Scholastic Digital Publishing Action aimed at disseminating digital or mixed texts;
- Wifi Action, for the extension of wireless connectivity in schools;
- Formative action with the identification of educational institutions responsible for the organization and management of teacher training courses on new digital skills.

The innovation process started with IWBs led in 2008 to the InnovaScuola Project for the extension of informatics equipment in lower Secondary Schools. In subsequent years, the InnovaScuola project was extended to other types of schools. Following the analysis of the results of the monitored PNDS 2008, in 2015 PNSD[31] aimed at further expanding technological tools and laboratory instruments in schools, and improving the development of digital competences for all involved. Among the government guidelines

[30] Prot. September 4, 2006, n. 2867.
[31] Developed as part of the reform project implemented by Law 107/2015.

indicated in L.107/2015, "*Education for computational thinking and coding*" was mentioned. Moreover, the introduction of computer science in the primary and secondary schools curricula was a radical change of perspective: the importance of planning in the area of thought development - to educate informed citizens - was finally acknowledged. To this end the Ministry of Education also works with the Observatory for Technology, set up in 2000, a telematic support service which collects data relating to the digitization process of schools.

Ultimately, the inclusion of new technologies in Italian schools requires that their use becomes a permanent feature of both educational objectives and methodology. Undoubtedly, it is a complex process that requires the availability of valid instruments and, in particular, appropriate didactic software.

2.3 PROGRAMMING LANGUAGES

The translation of an algorithm into a list of binary alphabet coded instructions (compiling) is a slow process; the idea of making this step automatic stems from here. The first phase is to devise a symbolic and formal language for writing algorithms, yet similar to natural language. The symbolic language as a sequence of symbols can be introduced into the computer memory. So what is needed is a compiling programme, capable of translating the algorithm, written in symbolic language, into a programme expressed in binary and executable by the machine. In order to write a compiling programme, the symbolic language in which the algorithms are expresses should have the following characteristics:

- Expressive capacity, that is the possibility of representing, with appropriate expressions, the data and the operations to be achieved;

- Syntax clearly defined in a formal manner, avoiding ambiguity, so that that the compiler can decide whether a given expression is correct;
- Semantics, also free from ambiguity, which enables to match actions to a given expression.

In these circumstances, the symbolic language is, at all effects, a programming language and a program written like this is called a *source programme*. The first programming languages only allowed the writing of instructions with symbols, rather than directly in binary, but retained the structure of a list of instructions. The real revolution is represented by language in sentence structure, which allows you to write expressions similar to how algorithms are designed by people, and are consequently called high-level languages. The first languages with their compilers date back to the fifties of the twentieth century. Since then a few hundred have been produced, of which some are of a general type, able to deal with both numerical and character data, while others specialise in handling specific data, such as languages for mathematical processing, simulation languages, languages for graphics, for creating hypertext and many others. Most programming languages can represent different types of data and many types of processes, which can solve calculation problems. Some of the languages which have been developed and refined are based on a model, of which they are a computer version. Among the most notable examples are PASCAL, LISP, PROLOG and LOGO. PASCAL, based on the ALGOL language, was created in the seventies and was designed by Niklaus Wirth, a programming teacher who noticed the lack of a programming language suitable for teaching. PASCAL is still one of the most used programming languages and adopts a procedural paradigm[32].

[32] The programming language allows and requires the user to describe a process in order to solve a problem.

LISP was designed in 1958 by John McCarthy as a formal language for studying the computability of recursive functions on symbolic expressions. It is a programming language that adopts a functional programming[33], within a declarative paradigm[34].

PROLOG, designed in the seventies, was conceived by Robert Kowalski and Marten Van Emden and implemented by Alain Colmerauer. It represented an attempt at building a programming language that would allow the expression of problems in a logical form, rather than the translation of an algorithm into an instruction to be followed by the machine. It adopts the logic programming paradigm[35].

LOGO, as previously stated, is a language that has a special place in the relationship between computing and education. Papert revealed that the first experience with LOGO was quite unusual because it took place in a laboratory in which some tortoises, remote-controlled by a computer programmed in LOGO, were moving along the floor. Later on, the tortoise becomes virtual, and is reduced to a small arrow, drawn on the screen of a video terminal, which moves obeying LOGO commands and, while moving, leaves a trail like a drawing. LOGO is obviously not limited to the tortoise context; indeed, it is a real language, similar to LISP, which enables the user to work with any type of data and solve very complex problems.

During the nineties in Italy the MatCos programming language was devised (Costabile, & Serpe, 2012). Such language, created in the Interdepartmental Centre for Research Education (CIRD) of the University of Calabria, is at the basis of ministerial projects concerning the teaching and learning of Mathematics with the use of technology.

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[33] You need to write the result of a calculation as a function of the data.

[34] The programming language requires the programmer to specify /declare what the initial data are, and what rules enable you to obtain a result.

[35] Programming paradigm that adopts the logic of first order both to represent and to process the information.

2.4 CAS AND DGS

A *Computer Algebra System* (CAS) is a software system that facilitates the execution of symbolic processing (Davenport, et al., 1988). The basic functionality of a CAS is the manipulation of mathematical expressions in symbolic form. Early versions of CAS were available in the early seventies, as a result of research into artificial intelligence. The first systems to be widely used were REDUCE, DERIVE and MACSYMA. The current market leaders are MATHEMATICA and MAPLE, both widely used for research and development by mathematicians, scientists and engineers. Another popular commercial programme is MUPAD. A free version with a slightly limited interface is available for non-profit-making research and educational activities. In schools the most popular CAS was DERIVE. DERIVE was created in the eighties at the University of Honolulu, and was later adapted for the European market by Kutzler's team in the nineties. Different versions have been produced until the last, DERIVE 6, was replaced by the Integrated TI-Nspire™ CAS[24]. DERIVE allows you to carry out many operations such as symbolic, numerical, algebraic, trigonometric and analytical calculations. It can solve computational problems for a vector or matrix, as well as translate the data into graphs, which can then be exported in different formats. Thanks to these features DERIVE was an instant success in Higher Secondary Schools, substituting the TURBO PASCAL programming language, which not only contained actual formal and conceptual difficulties, but also needed programming. In DERIVE the immediacy of the direct command replaced the harder, but surely more formative, technique of programming.

Despite the ease of use, soon Italian teachers realized that the systematic use of a CAS like DERIVE was not suitable for classroom practice for a variety of reasons, among which were:

- The loss of many algebraic computational skills and tolerance if not encouragement of some students' laziness;
- Redefinition of objectives and policies in school curricula and of teachers' role;
- Changes in the teaching practices codified in textbooks and passed down from one generation of teachers to the next.

In other words, DERIVE on the one hand is a valuable aid which brings out the heuristic aspect of Mathematics, while on the other hand it can be 'addictive' in the construction of explanatory mathematical models because it prevents the students from perceiving the nature of the objects of study and therefore the formalization of the models in question. Meanwhile, alongside these types of software, which could provide an immediate solution to the main mathematical problems, the Dynamic Geometry Software (DGS) appeared on the market. DGS allow you to create and later manipulate geometric constructions. The first DGS was GEOMETRIC SUPPOSER, which was developed in the early eighties. This was soon replaced by CABRI in 1986 and by Geometer's SKETCHPAD. Cabri Géomètre, created in 1985 from an intuition by Jean-Marie Laborde, became instantly popular in Italian schools. Laborde proposed the creation of a *Cahier de brouillon informatique pour la géométrie* from which the software derived its name. Laborde in collaboration with three young researchers developed interactive software that allows the creation of geometrical figures on the computer, thereby transposing on computers the classic tradition of constructing figures with the use of paper, pencil, ruler and compass. The fact that the geometric figure can be manipulated without limitations represents Cabri's greatest innovation. As a result, any change in the construction is visible in real time, allowing the observation of the variants and invariants' properties. Cabri Géomètre is a micro-world where the abstract entities of elementary geometry of the plane, in the form of representations drawn on the computer screen, materialize. Its peculiarity lies in the user interface that allows the manipulation of geometric objects constructed by moving the base of the building points. Later a new version of the programme, called CABRI II, was produced, which revolutionized the earlier version based exclusively on Euclidean registry, where construction was only done with ruler and compass. The new version incorporates the use of numbers in construction, resulting in a more agile and powerful geometry in its applications. Since 2001 other variants and software applications such as CABRI II PLUS, CABRI JUNIOR and CABRI 3D have been developed. In the subsequent years many other similar software programmes were developed, such as CINDERELLA, CARTESIO, Dr. GEO and others. In January 2002 GeoGebra was developed, open source interactive software that incorporates geometry, algebra, analysis, statistics and probability. GeoGebra, developed and edited by Markus Hohenwarter at the University of Salzburg, unlike previous software packages, allows not only the construction and manipulation of geometric figures in the Euclidean and Cartesian plane, but also a good symbolic management of geometric objects and integration with the numerical environment. These features enable the user to deal with numeric variables, vectors and points. Over time, new updated versions have been developed. Here we only mention the latest: GeoGebra 5.0, available from March 20, 2016.

2.5 Assessment criteria for educational software

In section 2.1, we introduced the concepts of '*educational software*' and '*learning software*' specifying that the first qualifies for its *didactic intention*, and the second for its *teaching potential*. The continuous and rapid evolution of DT, which includes both guided didactic systems and autonomous learning systems and, regarding the role of the teacher, aids to specific functions and automatic systems as well, has made the above definitions obsolete because the boundaries between those categories have dissolved. Consequently, from now on we will use the term '*educational software*' in its broadest sense. Whatever the definition, the value of any educational software cannot be fully expressed in the absence of a training project (Castoldi, 2000; Fierli, 2003). The role of the teacher, therefore, remains crucial in setting up the most appropriate organizational, cognitive, social, emotional context in which to promote the learning process through the use of educational software (ES). A ES has two fundamental aspects: one is pedagogical, the other is structural (Castoldi, 2000). In terms of training effectiveness, the choice of a ES and its evaluation need to consider both the software *intention* and *potential*. By *intention* we mean the set of objectives and strategies adopted by the teacher; by potential we mean the educational characteristics related to the software structure. The *potential* is defined by certain features that are contextualised within the educational practice. There are thirteen identified **features** in the literature[36]:

- ***analogical***: representing the contents and operating on them in a way that maintains some links with physical reality;
- ***exploratory***: operating multiple choices in heuristic learning contexts according to theme routes;
- ***heuristic***: addressing problem solving situations with relative chances of planning solutions and real-time feedback;
- ***interactive***: enabling dialogue between machine and pupils;
- ***hypertext***: explore a body of knowledge structured into a network;
- ***ludic***: interaction through games or game-like activities that motivate and characterize the type of learning;

[36] Castoldi, M. (a cura di). (2000). *Software per l'apprendimento. Come e perché scegliere e utilizzare prodotti multimediali nella scuola*. FrancoAngeli.

- ***manipulative***: acting on what is being represented by the computer through actions done by hand;
- ***metacognitive***: control, awareness and generalization of activated cognitive processes;
- ***multimedia***: integrated and simultaneous use of multiple languages and the media;
- ***perceptive-cognitive***: developing basic perceptive functions and primary cognitive ones;
- ***personal***: calibrating the diversity of use of the software depending on the background and the cognitive mode of the user;
- ***semiotics***: interacting with semantic and structured symbolic systems for harnessing their potential and overcome their limitations in a communicative perspective of construction and negotiation of meanings;
- ***social***: interaction between students and other subjects.

The choice of a pedagogical approach and the evaluation of the potential of a ES are based on the analysis of these features, which all represent an important specific contribution to a programme. Ultimately, the approach to the assessment of ES can be configured around three actions:

- a choice dictated by a particular teaching project;
- identification of potential to plan the most appropriate use;
- evaluation of the technological and logistical context in which it is used.

So in addition to intention and potential - expressed by the different features - we should also consider the possible constraints, i.e. the limits to the scope of the software.

Specifically, the *potential* is made up of five quality elements:

- ***technical*** (degree of usability of the software);

- **structural** (degree of flexibility and customization of usage pathways);
- ***communicative*** (degree of variety and symbolic relevance of the codes);
- ***cultural*** (breadth of concerns and cultural depth of the contents);
- ***cognitive*** (contribution to student cognitive and self-reflective processes).

In the following table, we have grouped the thirteen features according to the software quality elements:

Quality elements	Features
Technical	Multimedia, Interactive, Ludic
Structural	Personal, Heuristics, Exploratory
Communicative	Semiotics, Analogical
Cultural	Hyperlinking, Social
Cognitive	Perceptual-cognitive, Meta-cognitive, Manipulative

Table 1: Elements of software quality.

Didactic intentions must comply with the general principles adopted in the educational classroom practice, and are divided into general and specific (Table 2) in the literature[37].

[37] Castoldi, M. (a cura di). (2000). *Software per l'apprendimento. Come e perché scegliere e utilizzare prodotti multimediali nella scuola*. FrancoAngeli.

DIDACTIC INTENTION	
General	**Specific**
Framing **Starting the training**	Motivating, Contextualizing
Focussing **Implementing and supporting the training**	Transmit, Deepen, Applying, Searching
Consolidating **Securing Checking the training**	Monitoring, Self-evaluating

Table 2: Software Didactic Intention.

Restrictions of use are summarized in Table 3.

RESTRICTIONS OF USE
Social structure (n° of students to the teacher)
Computer-student ratio
Network Sharing
Student degree of autonomy in the use of software (Attentive, Executive, Wizard, Free with task /exploratory)
Role taken by the teacher (Non-intervention, guidance, encouragement, evaluation)

Table 3: Restrictions of use.

2.6 Evaluation of the educational software chosen for experimentation

For the research project of this doctoral thesis, we chose two different types of ES: GeoGebra and MatCos.

The first fits the DGS category, while the second is a programming language. For the purpose of this dissertation, it is essential to identify the educational and teaching potential of the two software; that is why we have compiled an evaluation protocol based on the existing research literature, making appropriate adjustments in relation to the theoretical and practical objectives.

Specifically, the protocol has a comprehensive structure where the features are grouped according to the quality categories indicated in Table 1 (Castoldi, 2000).

For each features two analysis questions have been prepared. For the purpose of objective measuring, every question has been assigned a score on a level scale from 1 (lowest) to 5 (highest) as well as the reasons for that score. This index has been called '*User pleased with the software*' (UPS). At the end, a summarizing table shows the total and average scores referred to the five quality categories.

Finally, the evaluation protocol is summarized in 5 quality sheets (QS) listed below.

Quality Sheet 1

TECHNIQUE	MULTIMEDIA	1	Is the weight given to the different codes balanced?
		2	Are the different codes used appropriately in relation to the purposes?
	INTERACTIVE	3	Can you customize the software according to the student's needs-preferences?
		4	Can the students get feedback on their performance?
	LUDIC	5	Is the software interface appealing?
		6	Can you challenge the computer and /or other users?

Quality Sheet 2

STRUCTURAL	PERSONAL	1	Can the students follow their own paths independently?
		2	Is the student encouraged to produce messages with different codes?
	HEURISTIC	3	Does the software present problematic situations to be solved?
		4	Does the programme allow different solving options?
	EXPLORATORY	5	Is the student stimulated to adopt different mental strategies (transfer, analogical thinking, problem solving, ...)?
		6	Are students stimulated in divergent thinking and creativity?

Quality Sheet 3

COMMUNICATIVE	SEMIOTICS	1 Are access to and use of the programme easy? 2 Are the page layout and the function of its components clear?
	ANALOGICAL	3 Are the general principles illustrated by specific cases? 4 Do the images, animations, simulations help the user to grasp the ideas intuitively?

Quality Sheet 4

CULTURAL	HYPERTEXTUAL	1 Is the reticular structure of the software divided and branched? 2 Does the software consider the topic from different perspectives (points of view)?
	SOCIAL	3 Does the software facilitate the interaction between individuals and groups of students? 4 Does the software encourage the production of material in groups?

Quality Sheet 5

COGNITIVE	PERCEPTIVE-MOTOR	1 Does the software stimulate perceptive skills (discrimination, stimulus decoding, ...)? 2 Does the software stimulate basic cognitive skills (memory, classification, association between stimuli, ...)?
	METACOGNITIVE	3 Does the software allow the user to organize and plan the work to come? 4 Can students go over the pathways they have used?
	MANIPULATIVE	5 Can the perspective with which the contents are being examined be altered? 6 Can you interact with and modify what is being represented (using mouse, keyboard, and other devices)?

2.6.1 Evaluation of ES GeoGebra

GeoGebra[38] is a DGS for the construction of figures with commands that allow the user to place geometric objects (points, lines, segments, polygons, circles, etc.) on a plane.

Its main characteristic, shared by earlier DGS, is its dynamicity: the figures can be manipulated dynamically, dragged on the plane and modified without altering the construction protocol (Accomazzo, et al., 2013).

GeoGebra is different from other software because it is open-source, a feature which has made it popular with didactic researchers and schools as well.

GeoGebra allows not only the construction and manipulation of geometric figures in both the Euclidean and Cartesian planes, but also a good symbolic management of geometric objects and the integration with the numeric environment (similar to that of a spreadsheet). Finally, the files generated with GeoGebra can be uploaded to the web as interactive dynamic applets and allow the community to use and manipulate them even without opening GeoGebra. Within a few years since its production GeoGebra had spread around the world, and today it can boast translations into dozens of different languages and can be found on all the continents at any level of education. After its creation, researchers and programmers from all over the world have contributed to its evolution, providing not only new possibilities of use, but also new implementations on different media. For example, from simple dynamic geometry software, GeoGebra has been expanded to embed spreadsheets, symbolic manipulation, and simultaneous visualisations of different environments.

The software can used with IWBs, and at present is not only downloadable on computers, but also accessible on-line, or as an App on tablets and smart phones.

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[38] http://www.geogebra.org

Research concerning the teaching of Mathematics with DGS began about twenty years ago and still continues. The lines of research are identified on the basis of the DGS affordances (Accomazzo, et al., 2013) which are:

- dynamism, obtained by dragging and dropping (dragging);
- the measuring (of lengths of segments, the amplitudes of angles, shapes of areas,...);
- the tracing, the place, the animation (that let you see the evolution of models);
- the representation of functions and the investigation of their charts, at local or general level;
- the integration of different representation registers (such as the geometrical and analytical), that allows to model problematic situations.

Depending on the problem situation to be addressed in the classroom, research literature (Accomazzo, et al., 2013) has proposed the following changes:

- ***Construction Problem***: classical geometry problem dating back to the ancient Greeks, which is to construct geometric figures using a ruler and compass, therefore basing the construction on properties and principles of Euclidean geometry. Software like GeoGebra can be used in place of line and compass, provided students are aware that obtaining the required shape is not sufficient, because the figure, once dragged, must maintain the same characteristics. This problem can be supported by a theoretical justification.
- ***Exploration Problem***: a classic problem of demonstration openly formulated, that offers the chance to explore a geometric situation and make a conjecture, to validate it and then to prove it, once agreed that what has been processed with the software is not enough. Open issues yield a large mathematical production, in the sense that all the students are able to find solutions for the situation they are exploring. Also,

the proof highlights in a formal way the properties discovered and formulated using the principles of Euclidean geometry.

- ***Modelling Problem***: a revisitation of a traditional problem as it allows students to overcome computation as an end to itself through the integration of graphical, numerical and symbolic representation.

Here we report the five quality sheet (QS) developed for the assessment of the DGS GeoGebra.

QUALITY SHEET 1 (QS1)	
QUESTION No.1	**INDEX: 4**
REASON It presents 'images-icons' (symbols indicating the operations that can be performed by the programme and are drawn on buttons), which are accompanied by 'instruction texts' (texts that contain the instructions required to accomplish specific tasks). The general screen has an area devoted to drawing and to the production/reproduction of geometrical images. The drawings are made with the use of 'images-icons'. There is also an algebraic view where the symbolic/algebraic expression of the drawn objects is given. Instruments for sound and video playback are not available. In other words, more weight is given to tools which are functional to the design and processing of images.	
QUESTION No.2	**INDEX: 5**
REASON The codes (images and text) are useful for the educational purposes for which they were designed.	
QUESTION No.3	**INDEX: 5**
REASON Opportunity of creating new tools (images-icons) to be designed according to the user's needs.	
QUESTION No.4	**INDEX: 4**
REASON Option of viewing the sequence of given instructions and its result and get direct feedback.	

QUESTION No.5	INDEX: 4
REASON The software has a friendly interface.	
QUESTION No.6	**INDEX: 1**
REASON The software or other players cannot be challenged.	

Table 4: Quality Sheet 1 - GeoGebra.

QUALITY SHEET 2 (QS2)	
QUESTION No.1	**INDEX: 5**
REASON The images-icons come with the instructions carried out by the command, which allows the student to follow their own path independently.	
QUESTION No.2	**INDEX: 5**
REASON It produces messages both in symbolic (algebraic) and visual form (image and/or drawing-geometrical representation). The user (student or teacher) can simultaneously use both a computer algebra environment and an interactive geometry sheet, which favours the development of meta-cognitive skills.	
QUESTION No.3	**INDEX: 5**
REASON It provides a real chance to deal with problem-solving situations. The solution-objective can be achieved both by reproducing (the application of rules or known patterns) and by producing (that is, finding new solutions).	
QUESTION No.4	**INDEX: 5**
REASON It permits different solutions according to the strategies adopted. You can use the existing tools or create a 'new tool'.	
QUESTION No.5	**INDEX: 5**
REASON Chance of exploring and discovering the constructed objects. Possibility of construction of explanatory models of reality.	

QUESTION No.6	INDEX: 5
REASON It makes the student the protagonist of a discovery process, giving rise to new emotions and the pleasure to learn and explore.	

Table 5: Quality Sheet 2 - GeoGebra.

QUALITY SHEET 3 (QS3)	
QUESTION No.1	**INDEX: 5**
REASON GoeGebra is open source.	
QUESTION No.2	**INDICE: 5**
REASON The page structure is simple, its components clear. Its structure has a strong communicative impact.	
QUESTION No.3	**INDEX: 4**
REASON The representation of the constructed mathematical objects is true to life, with appropriate approximations in borderline cases.	
QUESTION No.4	**INDEX: 4**
REASON the analogy between mental structures and real images is quite good.	

Table 6: Quality Sheet 3 - GeoGebra.

QUALITY SHEET 4 (QS4)	
QUESTION No.1	**INDEX: 1**
REASON GeoGebra does not have a reticular structure.	
QUESTION No.2	**INDEX: 4**
REASON Possibility of discussing the items in both the algebraic and in the geometrical registers.	
QUESTION No.3	**INDEX: 5**
REASON Chance of exporting spreadsheets as web pages and exchanging between students.	
QUESTION No.4	**INDEX: 4**
REASON Opportunities for production, and for sharing material in groups.	

Table 7: Quality Sheet 4 - GeoGebra.

QUALITY SHEET 5 (QS5)	
QUESTION No.1	**INDEX: 4**
REASON Opportunity to develop basic perceptive functions.	
QUESTION No.2	**INDEX: 5**
REASON Opportunity to develop cognitive functions in a different image management.	
QUESTION No.3	**INDEX: 5**
REASON The solution of problem situations, such as geometric construction, requires organization and planning.	
QUESTION No.4	**INDEX: 4**
REASON Visualization of the construction protocol.	

QUESTION NO.5	INDEX: 5
REASON The perspective from which you analyse the content can be changed (transition from 2D to 3D).	
QUESTION NO.6	**INDEX: 5**
REASON Representations can be changed and interacted with, with mouse and keyboard.	

Table 8: Quality Sheet 5 - GeoGebra.

Below we show the summary table.

QUALITY	SUM OF INDEXES	AVERAGE
Technical	23	23:6 = 3,83
Structural	30	30:6 = 5
Communicative	18	18:4 = 4,5
Cultural	14	14:4 = 3,5
Cognitive	28	28:6 = 4,67
	Total 113	113:26 = 4,34

Table 9: Summary table regarding the GeoGebra ES evaluation protocol.

Table 9 shows that the DS GeoGebra has reached an average of 4.34, that is, it ranks quite high. Based on evaluative analysis, the software in question has high structural qualities that favour a wide range of teaching strategies, as well as the development of critical evaluation skills and creative thinking, and the investigation into the relationships between the elements involved.

2.6.2 Evaluation of ES MatCos

The MatCos programming environment was created with the dual aim (Costabile, & Serpe, 2012, 2013) of:

- introducing students to programming using mathematical concepts appropriate to their age;
- helping students to learn and internalise mathematical methods and concepts exploiting the potential of the computer.

The MatCos environment has a programming language strongly oriented towards mathematics, which uses commands specific to certain mathematical concepts.

It is in Italian, with a simple syntax and instructions, which are very close to both natural and mathematical language.

Furthermore, it is a combination of a general language and a CAS, that is it has some specific command for symbolic computation to be used mostly for verification.

MatCos consists of the following blocks of instructions:

- classic Input/Output; assignment, loop, conditional, boolean operators;
- specific geometric instructions (in the plane and in Euclidean and Cartesian space);
- arithmetic-analytical, to work with fractions, functions, derivatives, integrals, etc.;
- statistical-probabilistic, to work with diagrams, histograms, etc.

Each command contains the parameters essential to the mathematical concept to be represented. The language is modular, with each module referring to a specific school age group. It comes without declarative phase, which helps to lighten the burden of syntactic rules typical of programming languages.

The language is expressed in *step-by-step* directions that allow users to run the program using a command at a time, and visualize the partial results.

This unique feature is very important because it allows the student to check every step of the algorithm and correct any errors. With MatCos you can represent different types of data and many types of solving procedures relating to calculation, and the results are illustrated with suitable graphics and are easy to understand.

MatCos also presents the following didactic characteristics (Costabile, & Serpe, 2013):

- ***Expression***: MatCos was created in Italian; the language really helps the students, as noted by the linguist Benjamin Lee Whorf, who speaking of natural language stated that '*language shapes the way we think, and determines what we can think*'.
- ***Ease of Learning***: MatCos, without a declarative phase, helps to lighten the burden of syntactic rules typical of advanced programming languages.
- ***Step-by-step execution***: it permits constant monitoring, that is the checking of every step of the algorithm, and so the students can immediately verify the correctness of their conceptual and syntactic mental processes.
- ***Legibility***: the codes of MatCos programmes are easy to read both for programmers and for those who are new to the software.

QUALITY SHEET 1 (QS1)	
QUESTION No.1	**INDEX: 5**
REASON The codes (text and image) are balanced and perfectly integrated.	
QUESTION No.2	**INDEX: 5**
REASON The codes (images and text) are functional to the educational purposes for which they were designed.	

QUESTION No.3	INDEX: 1
REASON The software cannot be personalized.	
QUESTION No.4	**INDEX: 4**
REASON The user can get feedback (in case of wrong code, an error message comes up indicating the point at which the error occurred, and the exact code).	
QUESTION No.5	**INDEX:2**
REASON The interface is presented in a static and unattractive form.	
QUESTION No.6	**INDEX: 1**
REASON The software or other players cannot be challenged.	

Table 10: Quality Sheet 1 - MatCos.

QUALITY SHEET 2 (SQ)	
QUESTION No.1	**INDEX: 2**
REASON The students can use the software and do the work independently only if they know the commands and instructions for the language.	
QUESTION No.2	**INDEX: 5**
REASON The student is stimulated to produce messages in different registers.	
QUESTION No.3	**INDEX: 5**
REASON It provides a real chance to deal with problem-solving situations. The solution-objective can be achieved both by reproducing (the application of rules or known patterns) and by producing (that is, finding new solutions).	
QUESTION No.4	**INDEX: 5**
REASON Opportunity of achieving different solutions because the language is flexible.	

QUESTION No.5	INDEX: 5
REASON Chance to operate multiple choices in heuristic learning contexts following no predetermined paths.	
QUESTION No.6	**INDEX: 5**
REASON The need to devise an algorithm in order to solve a set problem stimulates creativity and a sense of discovery.	

Table 11: Quality Sheet 2 - MatCos.

QUALITY SHEET 3 (SQ)	
QUESTION No.1	**INDEX: 2**
REASON Online Demo Version.	
QUESTION No.2	**INDEX: 3**
REASON A clear Structure, but not very communicative.	
QUESTION No.3	**INDEX: 4**
REASON The representation of mathematical objects built is true to life with appropriate approximations in borderline cases.	
QUESTION No.4	**INDEX: 4**
REASON The analogy between mental structures and real images is quite good.	

Table 12: Quality Sheet 3 - MatCos.

QUALITY SHEET 4 (SQ)	
QUESTION No.1	**INDEX: 1**
REASON Does not have a reticular structure.	
QUESTION No.2	**INDEX: 4**
REASON Topics can be dealt with both in algebraic and geometrical register.	
QUESTION No.3	**INDEX: 3**
REASON The programming of algorithms and the transposition into a code encourages peer work.	
QUESTION No.4	**INDEX: 3**
REASON Opportunities for production, and for limited sharing of material in groups.	

Table 13: Quality Sheet 4 - MatCos.

QUALITY SHEET 5 (SQ)	
QUESTION No.1	**INDEX: 4**
REASON Opportunity to develop basic perceptive functions.	
QUESTION No.2	**INDEX: 5**
REASON Opportunities for developing cognitive functions (Combining the machine language and its representations).	
QUESTION No.3	**INDEX: 5**
REASON The implementation of algorithms requires organisation and planning.	
QUESTION No.4	**INDEX: 5**
REASON The programme instructions allow the user to go over the pathways used.	

QUESTION No.5	INDEX: 4
REASON The perspective from which you analyse the content can be changed.	
QUESTION No.6	**INDEX: 5**
REASON Representations can be changed and interacted with, with mouse and keyboard.	

Table 14: Quality Sheet 5 - MatCos.

Here is the summarising table:

QUALITY	SUM OF INDEX	AVERAGE
Technical	18	18:6 = 3
Structural	27	27:6 = 4,5
Communicative	13	13:4 = 3,25
Cultural	11	11:4 = 2,75
Cognitive	28	28:6 = 4,67
	Total 97	97:26 = 3,73

Table 15: Summary table regarding the MatCos ES evaluation protocol.

Table 15 shows that the DSe MatCos has reached an average of 3.73, a good ranking. Evaluative analysis shows that the software has a high cognitive quality, which in terms of teaching practice enriches the positive heuristic process. A trial and error process helps a full understanding of the abstract model related to the heuristic programme. To quote Papert, the student becomes an '*epistemologist*'.

Chapter 3: Review of the literature

Research literature (Artigue, et al., 2009; Biehler, 1993; Drijvers, et al., 2016) points out that in order to understand the role of technology in the teaching of Mathematics one must simultaneously consider the role of both the teacher and the students. These three agents interact within a synergic and competitive relationship that defines the classroom scenario. Teacher, student and technology together establish the learning strategy, developing the student-context interaction. Students' participation can range from simple sharing to autonomous learning. In the case of teaching directed either by the teacher and/or by the technologies, the strategy becomes a 'teaching strategy'. After considering the mode of interaction between technology, teacher, and students within the discipline of Mathematics, this chapter will highlight the educational and pedagogical value of DGS and programming languages. In the second part, we will examine some theoretical tenets on the use of technology in modelling activities, with particular reference to issues concerning the calculus of probability, and geometric constructions.

3.1 Theoretical perspective: Technology and Mathematics teaching and learning

When used as part of a suitable methodological framework, technologies help the learner to reach higher levels of competence, and at the same time, they increase motivation and interest in Mathematics, which is still often considered an irksome and unpleasant discipline.

The NCTM[39] cites:

Electronics technologies - computers and calculators - are essential tools for teaching, learning and doing Mathematics. They provide visual images of

[39] National Council of Teachers of Mathematics

Mathematics ideas, they facilitate the organisation and analysis of data, and they compute efficiently and accurately. They can support investigation by students in every area of Mathematics ... When technological tools are available, students can focus on decision-making, reflection, reasoning and problem solving.

Research literature in the teaching of Mathematics[40] has highlighted the benefits of the use of technology in the classroom; as an example, we mention:

- *Immediate feedback*: when using technological tools students get instant feedback as opposed to traditional methods, where feedback takes time[41].
- *Motivation, Interaction and Cooperation*: The use of technological instruments, especially multimedia tools, provides a good environment for group work and interaction on a given task. This ultimately boosts motivation and encourages collaboration among pupils[42].
- *Improved Skills*: The use of technology in Mathematics education helps students in the following skills[43]:
  - Use of software like drill and practice can help young children to develop skill such as counting and sorting.

[40] Momin Fasiyoddin Inayat, Shaikh Naeem Hamid (2016). Integrating new technologies and tools in teaching and learning of mathematics: An overview. *Journal of Computer and Mathematical Sciences*, *7*(3), 122-129.
[41] Yuh-Ming Cheng and Peng-Fei Chen (2007). Applied Software Agents mechanism to Mathematics Learning in Elementary School. In: *Proceeding of Innovative Computing, Information and Control (ICICIC '07) Second International Conference*, 201-201.
[42] Hudson B. (1996). Group work with Multimedia in Mathematics: Contrasting Patterns of Interaction. *British Journal of Educational Technology, 27(3)*, 172-190.
[43] Mong-Chen Chiang, Yi-Chan Deng, Han-Zen Chang, Hui-Chun Liao, Chin-Wen Ho & Tak-Wai Chan (2005). EduBingo: A Bingo-like Game for Mathematics Skill Building. In: *Proceedings of the 2005 IEEE International Workshop on Wireless and Mobile Technologies in Education (WMTE'05)*, 93-95.

- Proper use of computer games can also help young children to develop their mathematical skills.
- Through the use of technology students can work in collaboration and share with others their understanding and knowledge, which will improve their communication skills.

- *Active participation*: The interactive technology used in Mathematics helps students to become active partners in the learning process through experimentation, demonstration, exploration, and calculation. This helps them in the process of developing their understanding of a topic[44].
- *Integrate theory and practice into one*: Knowledge of discrete Mathematics is needed by computer scientists and software engineers alike, as it is centred on correctness, logic and algorithms. However, learners find it rather rigorous and challenging as it requires both reasoning and logic. Wu Xiuguo[45] in his study found that teaching discrete Mathematics with experiments using technology can integrate theory and practice into one and enhance the students' understanding of the discipline.
- *Teaching Mathematics Better and Teaching Better Mathematics*: Integrating technology into the classroom can improve Mathematics teaching and also introduce better Mathematics[46]. With the use of technology teachers can focus more

[44] Connors, M. A. (1997). Technology in Mathematics Education: A Personal Perspective. *Journal of Computing in Higher Education*, *8(2)*, 94-108.
[45] Wu Xiuguo (2009). Discrete Mathematics Teaching Reformation: Adding Experiments. *First International Workshop on Education Technology and Computer Sci.*, *IEEE*, 566-569.
[46] Roschelle J.,Pea R.,Hoadley C.,Gordin D., & Means B. (2000). Changing how and what children learn in school with computer-based technologies. *The Future of Children, 10*(2), 76-101.

on developing ideas, exploring consequences, justifying solutions, and understanding connections, which is the heart of Mathematics[47].

Alongside these benefits, however, other research studies have highlighted the complexities inherent the integration of technology in Mathematics teaching. For example, Hennessy, et al. (2005) note:

In addition to the new interpersonal and pedagogical skills which teachers require to use technology in their classrooms, other contextual factors which can act as barriers include lack of confidence, experience, motivation, and training; access to resources and timetabled use of dedicated technology classrooms; unreliability of equipment; classroom practices which clash with the culture of student exploration, collaboration, debate, and interactivity within which much technology-based activity is said to be situated.

There is also a gap between having access to technological resources and their appropriate use in the classroom; consequently, it is the teachers' responsibility to suitably integrate technology into their teaching activities in order to ensure its fruitful and meaningful use.

3.1.1 The interaction of teachers and students with technology

The use of various forms of technology in classroom teaching practice can enhance the understanding of mathematical content. The key to fostering understanding is not only in the informed development of technology, but in also how this technology is used by and with the learner; therefore, the teacher plays a central role.

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[47] Heid M. K.(1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19*(1), 3-25.

Rules for the use of technology in the Mathematics classroom are determined by the choices made by the teacher. Substantial evidence suggests there is a complex relationship between teachers' classroom practices and the interconnected knowledge and perceptions of Mathematics, technology, learning, and teaching. The integration of technology into classroom practices is influenced by teachers' background knowledge and experiences, and conceptions of technology, Mathematics, and learning (Zbiek, & Hollebrands, 2008).

Teachers' attitude towards Mathematics and its pedagogies has been indicated as a major factor in determining the way they make sense of technological tools and integrated technology in the classrooms (Ruthven, Hennessy, & Deaney, 2008; Mousley, Lambdin, & Koc, 2003).

Mouza (2009) found that teacher change was highly dependent on the continual interaction between practices and beliefs and that teachers went back and forth between implementation of practices from professional development experiences and previous practices. If teachers are to effectively use technology, changes in their knowledge, practice and beliefs are needed. Mouza also reported that teachers' beliefs about their students constituted a critical variable influencing their decision to use technology.

Technology brings changes in the roles of both teachers and students, in that teachers would act as facilitators of student discussion and collaborative exploration with peers (Heid, Sheets, & Matras, 1990).

Research shows that technology may change the nature of mathematical activity in the classroom and consequently the teaching and learning of Mathematics (Laborde, 2007). Laborde described two levels of teachers using technology: 1) teachers who master the use of technology to do an activity and 2) teachers who used the technology for organizing instruction and learning, to take advantage of the specificities of the technology in relation to the meaning of the mathematical content.

Therefore, the literature shows how the interaction between all the agents (technology, teacher, student and Mathematics) is configured as an articulated and complex system (Olive, & Lobato, 2008).

3.1.2 Changes in Mathematics curriculum in technology settings

Technology mainly plays the role of cognitive amplifier in the development of mathematical activity (Pea, 1985). Putting technological expertise at the service of cognitive activities means entering into a relationship with Mathematics, i.e. integrating the use of technology among the objectives of the discipline. This means calling into question all the traditional methods (frontal lessons), because they are being modified by the use of technologies (Heid, 1997). Ultimately, we identify technology as having a dual capacity: amplifier and re-organiser.

For example, it may happen that the relationship with technology increases the relevance of some concepts on the one hand, while on the other it makes other notions more approachable than they would be with traditional methods, and still others obsolete. In Mathematics indeed, we find all three cases: the growing importance of logic, the theory of algorithms and numerical computation, the ability to study complex geometric structures thanks to electronic graphics, and the decreasing importance of traditional computing techniques.

Today Mathematics education cannot be limited to the development of techniques and mathematical procedures, typical of the school tradition, but must also develop more sophisticated and articulate skills, like the monitoring of the features available within the systems, and of the results that their use allows to obtain, as well as support those features with a theoretical framework. All this is a real challenge to the Math curriculum and classroom practice.

In the previous chapter, we discussed the CAS, DGS and programming languages. Many research studies have focused on their impact in classroom practice. As an example we mention the research conducted on the effects produced by CAS on the teaching and learning of algebra and calculus (Heid, & Blum, 2008; Tall, et al., 2008), and on the effects produced by LOGO and DGS on the teaching and learning of geometry (Clement, et al., 2008; Hollenbrands, et al., 2008). The software different features are crucial to the way in which they can be used in the teaching of Mathematics; in particular, some features may be more attractive for pursuing specific learning objectives. Alongside their characteristics, the teaching strategies inherent the use of these software should not be neglected. For example, generally speaking the use of CAS ought to provide a dual activity for students: one based on manual development of mathematical techniques and one mediated by the mathematical techniques offered by the software. The first activity is the construction of the necessary prerequisites to access the features of the CAS. Regarding this aspect, Artigue (2002) has observed that by exploiting the functions of CAS we can witness an 'explosion' of solving techniques that can be used in a task. Intelligent use of this potential can promote better conceptualization, especially if teachers include as their method the comparison and justification of the techniques involved.

As far as programming languages are concerned, various studies have shown their potential in the classroom. For example, Dubinsky and his colleagues (Brown, Devries, Dubinsky, & Thomas, 1998) conducted experiments on the use of the programming language ISETL for the construction of mathematical concepts.

The results showed that the students involved in the study had apparently developed a greater conceptual understanding than those who had not taken part in the experimentation. A first detailed overview of computer environments for the learning of Mathematics can be found in Biehler, et al., 1993; a recent version is Bell, 2013.

Research literature has shown not only the potential, but also various difficulties inherent the educational management of software. The difficulties are due to technological, institutional and pedagogical aspects, which are interrelated.

Overall, the studies taken into consideration for this dissertation show that the use of ES can be an important element of innovation in the teaching of Mathematics provided it goes together with a significant change in teaching culture.

This change involves alterations in the educational objectives and students' competencies. The main change pertains to mathematical representations, which should be approached with multiple, connecting and dynamic skills. The transition from one representation to another is fundamental in solving problems, especially in the processes produced by the transition from a representation with its tools and methods, to another with tools and methods that offer more avenues for achieving one or more solutions (Hollenbrands, et al., 2008).

Access to multiple representations offers new possibilities for the creation of algebra curricula, compared to the traditional ones based exclusively on technical manipulation of symbols (Chazan, 1993; Head, & Blume, 2008).

As far as geometry is concerned, on the other hand, the increasing availability of DGS and their introduction in classroom practice has promoted a greater understanding of the significance and value of demonstration compared to the traditional analytical and sequential approach based on Euclidean geometry (Hollenbrands, et al., 2008).

The use of educational software supports different approaches to mathematical contents, but its use does not guarantee, by itself, enhanced learning; indeed, similar activities can produce different effects with different students.

As claimed by Dunham and Hennessy (2008):

> *Frequent use of technology in cognitively challenging ways and in supportive learning environments can reduce outcome inequities for some traditionally disadvantaged groups.*

The use of technology ought to represent the opportunity for students to make deeper reflections on the subjects on which they are working.

Consequently, the type of mathematical activity to be implemented should not be based just on students' requests, but also on the intentions and finalities of the task. For example, Doer and Pratt (2008) stress the importance of intentionality of the task in relation to the validation of the mathematical modelling process:

> *We believe that validation as an activity is intrinsically linked to the need to reveal the potential and the limitations of a model, and if validation activity is to be spontaneous then students must have a reason for wanting to pursue such a direction. Any such reason will inevitably be wrapped up in their construction of the purpose for the activity as a whole.*

The same authors claim that tasks can be designed so as to suggest the goals of the activity, a factor which attracts students to the discipline. So it becomes essential to design exploratory environments in which students have the opportunity to reflect on the objects of study. As previously pointed out DGS (such as Cabri Géomètre and GeoGebra) allow students to investigate various properties of geometric objects. Maths activities in a technological context produce greater effects when intervening elements enable a reflection on the results of the entire activity. In these cases the students assume the role of instructors, creators of tools from which they can learn and create other instruments.

3.1.3 Effects on mathematical thinking of technology activity

Research on the use of technology in secondary Mathematics education has proliferated over the last 30 years and has demonstrated that the strategic use of technological tools such as graphing calculators, DGS, programming environment and spreadsheets support students' mathematical thinking and discourse (Burrill, et al., 2002; Drijvers, et al., 2016; Heid, & Blume, 2008; Nicaise, & Barnes, 1996).

At the same time, awareness and interest in students' mathematical thinking, reasoning, and sense-making has increased (Hollebrands, Conner, & Smith, 2010; Romberg, 1994). However, research has focused on how the use of technology can support students' mathematical thinking and reasoning (Sherman, 2014).

Three themes emerge as researchers have examined mathematical thinking in the context of mathematical activity that occurs in technological environments (Heid, & Blume, 2008):

- the extent to which students develop tools for mathematical thinking and learning in these environments;
- the ways in which students engage in metacognitive activity as a result of mathematical activity in these environments;
- the level of generality of the mathematical thinking of students in these environments.

As regards the first theme, some technological instruments facilitate symbolic reasoning and/or conceptual understanding, but they can also inhibit mathematical thought. Some studies indeed highlight how symbolic manipulation skills do not automatically develop in a technology-based environment (Yerushalmy, 1991), but are the result of adequate integration of technology into classroom practice (Heid, & Blume, 2008).

As far as the second theme is concerned, technological environments have two features that may enhance metacognitive activity: when used as tools they have the capacity to offload some of the routine work associated with mathematical activity leaving more time for reflection, and with their strict communication requirements they may help bring to consciousness mathematical ideas and procedures (Clement, et al., 2008; Tall, et al., 2008).

Regarding the third theme, some studies reported a higher level of generality in the thinking of students in the technology-based activity. In the context of geometry, Hollebrand and colleagues (2008) observed that learners' generalizations were likely to be situated abstractions, not generalizing beyond the situation in which they were developed. Doerr and Pratt (2008) noted '*the lack of evidence for any claim that the students reasoned better about real-world phenomena, even when they reasoned appropriately within the microworld*' and that '*there is however little evidence that students can abstract beyond the modelling context*'.

3.1.4 Impact of DGS in Mathematics classrooms

DGS were developed in the mid 1980s, in the thread of the powerful idea of 'direct manipulation' (Laborde, & Laborde, 2008) to simulate ruler and compass constructions and assist/help in the precise design of geometric figures. The rapid evolution of *Geometer's Sketchpad* (Olive, 1998) and *Cabri Géomètre*, the first two DGS, highlighted the need for increased dynamism. Sketchpad, for example, was initially conceived as a program for drawing accurate static figures of Euclidean geometry. In the course of their development, the initial idea for the software was adapted to the emerging needs. The result was software able to build dynamic geometric shapes, in which points and segments could be dragged maintaining the properties that characterize the constructed figures. The relatively fast drag and drop operation (dragging) defined the future of DGS: the functional versatility and corresponding complexity of the operation were not anticipated, and have only gradually been treated (Hölzl, 1996; Arzarello, Olivero, Paola, & Robutti, 2002; Straesser, 2002). Interest in DGS for learning geometry was evident right from the start: DGS through dragging offered new possibilities for the visualization and verification of the properties of geometric objects. This has been recognized by the various educational segments that have integrated DGS in their curricula (Kortenkamp, et al., 2010).

Direct manipulation, a special feature of DGS, allows the student to have a simultaneous response; according to Leung (2003), this simultaneity is a key element which can shorten the distance between experimental and theoretical Mathematics, or for switching between conjecture and formalization. Laborde (2001) illustrates this potential for the teaching of geometry with an example on the 'black box'[48] situations:

> *In the black box situations, the students are given a diagram on the screen of the computer and they are asked questions about it. This kind of situation was used in our scenarios for introducing new trasformations. A point P and its image P' through the unknown transformation were given to the students. They could move P and observe the subsequent effect on P'. Students were asked to find the properties of unknown transformation by means of this black box. In such a task, students must ask themselves questions about the transformation.*

This process emphasizes the students' responsibility in the formulation of questions concerning the transformation, but at the same time provides food for thought on the fact

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[48] A strategy for investigating a complex object without knowledge or assumptions about its internal make-up, structure or parts. The method aims at either a formal description of the transformation rules linking inputs and outputs or the construction of a model exhibiting a behavior that approximates what is observable from the outside of the "black box".

that this may well distract learners from the main goal, which is to identify geometric properties.

On the other hand, before they can use the software properly, the students require a preliminary phase of instruction on the functions offered by DGS (including the dragging); only at a later time should they experience the constraints and relations that restrict the movement of the points and of the figures in general. This procedure adds useful meaning to the validation of the constructed figures (Costabile, & Serpe, 2013). In addition to the function of dragging, the DGS also have the following features/affordances:

- measuring (of lengths of segments, the amplitudes of angles, shapes of areas, ...);
- tracing, place, animation (that let you see the evolution of models);
- the representation of functions and the investigation of their graph, at local or general level;
- the integration of different representation registers (such as the geometrical and analytical), that allows to model problematic situations.

The potential of DGS is not just restricted to the field of geometry. The software can be also used to introduce the notion of function (Falcade, et al., 2007), since the 'dragging' feature can support the concept of functional dependence. Furthermore, the geometrical situations and functions can also be associated with specific problems, in which the DGS help in the visualization of the dynamic aspects of functional dependence simultaneously in different representations (Hoffkamp, 2010).

3.1.5 Impact of programming environments in Mathematics classrooms

In the last thirty years, several studies have been directed to the investigation and analysis of the role and effects of the use of technology in teaching and learning of

Mathematics at cognitive level (Bishop-Clark, 1995; Goos, et al., 2003; Lesmeister, 1996; Quesada, & Maxwell, 1994; Lye, & Koh, 2014; Weber, 1998).

In particular, the value of the programming in educational and pedagogical terms has been recognized internationally (Aydin, 2005; Dubinski, & Tall, 1991; Kuzler, 2000; Liao, & Bright, 1991; Oprea, 1988; Pea, & Kusland, 1984).

Programming is more than just coding, for, it exposes students to computational thinking, which involves problem-solving, using computer science concepts like abstraction and decomposition.

The importance of programming since the early years of schooling has also been stressed (Clements, & Gullo, 1984; Clements, 1990; Clements, & Maredith, 1993; Fessakim, et al., 2013).

Furthermore, the curriculum of mathematics has given an increasing emphasis on problem solving processes, indicating a didactic problems of intentional and functional nature than the educational goals to be achieved, in terms of knowledge, skills and abilities[49].

However, in the recent years, there has been renewed interest in introducing programming in teaching because it is a cognitive and constructive activity (Costabile, & Serpe, 2012, 2013).

Hatfield (1984) had already highlighted the pedagogical value of programming by detecting the existence of strong links between the thought processes that are triggered in students when they design and/or write computer programmes and aspects related to mathematical thinking. Even Papert (1980) emphasized the fact that the construction of a programme is configured as the construction of knowledge and, above all, warned about

[49] National Council of Teachers of Mathematics, 2000.

the use of 'pre-packaged' programmes that inhibit creativity and a sense of discovery in the learner. Similarly, Bork (1981) associates '*computer programming as a vehicle for ... trainning ... analytic thinking applicable to broad classes of problems*' and Nickerson (1982) states that we can see '*computer programming as a vehicle for teaching thinking skills*'. Students only learn when they get the chance to construct their own knowledge and re-elaborate previous knowledge. In Mathematics education, computer programming allows the learner to develop new ideas and become familiar with them, and handle mathematical objects in an informed way as a result. The use of a programming environment in the classroom promotes a shift from concrete to abstract learning, and from the iconic to the symbolic (Kelly, 1984) strengthening rational thinking.

Computer programming has been described by Nickerson (1982) as a creative endeavour that requires planning, accuracy in language use, generation and testing of hypotheses, and ability to identify action sequences. It ought to represent a fundamental part of the literacy for twenty-first century citizens (Rushkoff, 2010) as it is now a skill required for most jobs and spare time activities. While this is certainly true, one should not forget the psychological perspectives of computing in Mathematics (Hatfiel, 1984).

The basic principle underlying the practice of programming in the classroom is that students can use the computer as a Mathematics laboratory in which to experiment (Howe, et al., 1989). Devising a programme creates the opportunity for experimenting by operating a cognitive shift in focus from procedures to their effects.

This kind of learning experience develops problem-solving skills. Feurzeig, et al. (1969) - the first to propose the practice of programming in the teaching of Mathematics in the classroom - believed that a programming language would provide an excellent environment for students to explore Maths and provide an ideal context for the understanding of problem solving steps.

Programming gives the opportunity to organize, produce, understand and establish processes and procedures in order to plan, design and build computer prototypes of objects; translated in terms of student skills this means the learners can acquire intellectual education in all its components (logical, intuitive, creative and imaginative) and capacity for abstraction and modelling to solve real-world problems (Collins, & Brown, 1988).

3.2 Theoretical perspective: Mathematical modelling and technology

Modelling constitutes one of the pillars of basic scientific knowledge and therefore deserves greater consideration in the design and management of learning environments. Mathematical modelling is one of the topics in Mathematics education, which has been the object of much debate especially in recent years. In classroom practice all over the world, however, modelling still has a far less prominent role than is desirable (Blum, et al., 2007; Stillman, et al., 2015). The main reason for this gap between the goals of the educational debate and everyday school practices is that modelling is difficult for both students' and teachers. Today more than ever, the teaching of Mathematics, finds its *raison d'etre* in the contribution it can give to the formation of thought and personality as a whole, therefore, we emphasise mathematical modelling as an important element in an up-to-date Mathematics secondary curricula preparing for further education. The ICMI[50] has long promoted cultural debate at international level (Blum, et al., 2007; Stillman, 2013 and 2015; and many others) also pointing out that teaching means taking into account not only the cultural role of Mathematics (which pertains to it) and the reasons that support it in a changing society, but also School structure, practices, trends and constraints. Modelling is a lot more than a mere serial reproduction, as it educates to thoroughly reflect on a question; it helps to familiarize with many important aspects (for example, the fact that the same

[50] International Commission on Mathematical Instruction

mathematical model can describe different phenomena according to the meaning of variables) and to consider familiar concepts in a new light. Teaching while bearing in mind the binomial Mathematics - real-world means promoting active learning in the classroom, turning studying into a process of discovery and promoting the understanding of Maths concepts (Niss, 2003); also, it means providing the students with opportunities for cognitive reconstruction of mathematical structures that evoke and enhance learners' natural constructions, especially perceptual motor ones (Welsh, & Lakoff, 2005). The inclusion of modelling in school Mathematics curricula is crucial for the development of problem solving skills, and promotes a reflection on the relationship between Mathematics and sensible reality (Rivera, et al., 2015). A real problem offers a learning opportunity in three dimensions (Wedelin, & Adawi, 2015):

- ***Familiarity with real-world problems***

  A realistic problem and its solution (including any necessary derivation), acts as a representative case and contributes to a familiarity with real-world problems in the domain of interest.

- ***Supporting knowledge.***

  The concepts and methods required to solve the problem (known in advance or created as a part of the solution process.

- ***Processes and skills.***

  The particular way in which the solution (and its derivation) was found, and the modelling and problem solving techniques involved.

In addition, teaching Mathematics through problems is important because dealing with real life problems helps students to (Siller, & Greefrath, 2009):

- understand and cope with situations in their daily lives;

- acquire the necessary tools for moving from the plane of reality to that of Mathematics;
- have a clear picture of Mathematics, and in the process identify the tools they need for life;
- comprehend Mathematics and Computer Science in depth, so as to be able to recall important concepts with confidence at a later time;
- see the history of Mathematics as a kind of 'laboratory in which to explore the development of mathematical knowledge'.

Almost all of the research on learning with Mathematics modelling is based, with few exceptions, on an epistemological perspective that begins with an examination of the relationship between the real world and the world of the model.

Doerr and Pratt (2008) suggest two epistemological foundations for mathematical modelling: the first entails a separate model from the world that must be modelled, while the second sees modelling as a cyclical and interactive process. The epistemological position that separates the real world of the phenomenon from the world built with the model is in some way artificial; on the other hand, however, it is crucial for modelling. This separation is not a naive and modern version of a platonic point of view of reality, but is distinctly represented in a mathematical model. The essence of the epistemological position is that the two worlds - that of the phenomenon and the model - co-construct one another: the co-construction of the real world on the one hand, and the model world on the other, is highlighted by the way in which the models are planned and designed in the real world. The real model may be affected by results, which are extraneous to it; therefore, the separation between the real world of the phenomenon and the mathematical model of the world is in a way artificial. At the same time, the separation of the phenomenon from the model is a crucial point in the meaning of a model.

Lehrer, Horvarth, & Schauble (1994) have argued in favour of such separation and have reached the conclusion that the separation between the phenomenon and the model originates because the models are basically similar.

A model is a system of objects, relationships, and rules whose behaviour is similar in some systems. Modelling is the mapping of one system to another and is motivated by the need to describe, predict or explain a particular phenomenon. Elements from the real world are selected, organized and structured so they can be mapped in the model world.

Moreover, the world of the model simplifies and distorts certain aspects of the real world through manipulation according to certain rules.

These manipulations lead to the description and explanation that represent the results of modelling.

The second epistemological position - based on research and practice in relation to mathematical models - takes on modelling as a cyclical and interactive process (Doerr, & Pratt, 2008).

The reason for this interaction is the attempt to validate in the real world the results, manipulations and objects in the model world. The results of the validation process are derived from a model, or generate a further cyclic modelling activity. The cyclic nature of this modelling paradigm (National Council of Teachers of Mathematics, 1989; Giordano, Weir, & Fox, 1997; Galbraith, & Clatworthy, 1990; Edwards, & Hansom, 1989; Niss, Blum, & Huntley, 1991) is represented in Figure 1; the paradigm can also be subject to changes.

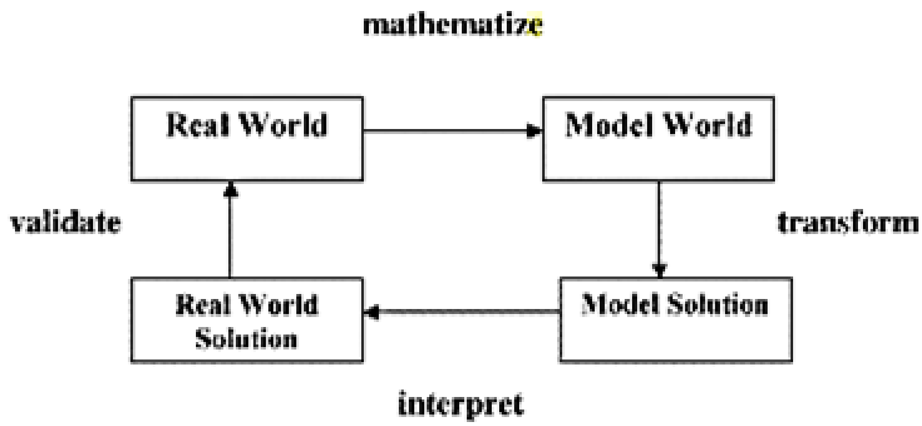

Figure 1: The cyclic nature of the modelling process.

The most common is Blum and Leiss' cyclical model (2007).

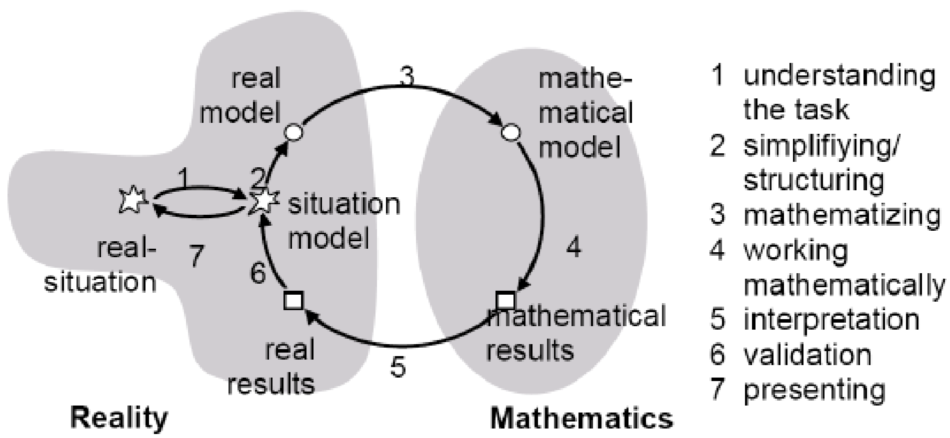

Figure 2: Modelling cycle of Blum & Leiss.

In Blum and Leiss' modelling cycle we can identify three fundamental points:

- ***Design and development.***

  Comparable to "Finding the real model" and to the step of "Translation" – Real situation to real model by including the situation model.

- ***Description.***

  Comparable to "Finding the mathematical model".

- ***Evaluation.***

  Comparable to "Finding (Calculating) mathematical results" and to the step of validating.

The modelling process is generally described as a series of steps (Edwards, & Hanson, 1989) similar to those shown in Figures 1 and Figure 2 that include:

- the understanding of a particular real phenomenon that can be modelled;
- the definition of a context and its constraints;
- the identification of key variables; the explicit definition of relations between variables;
- the transposition of these relationships;
- the analysis and interpretation of results; the refining of the model and its understanding through an iterative process, by repeating the same steps.

A psychological perspective on research in relation to the models and modelling refers more directly to issues relating to the nature of student activities.

Many research studies have challenged the linearity of the steps in a modelling process, preferring a plots model because it is more functional to learning (Bell, 1993; Leish, Surber, & Zawoyewski, 1983; Doerr, 1996a, 1996b; Lesh, & Doerr, 2000; Lester, & Kehle, 2003). This type of modelling process - characterized by nodes and interconnections - is represented in Figure 3.

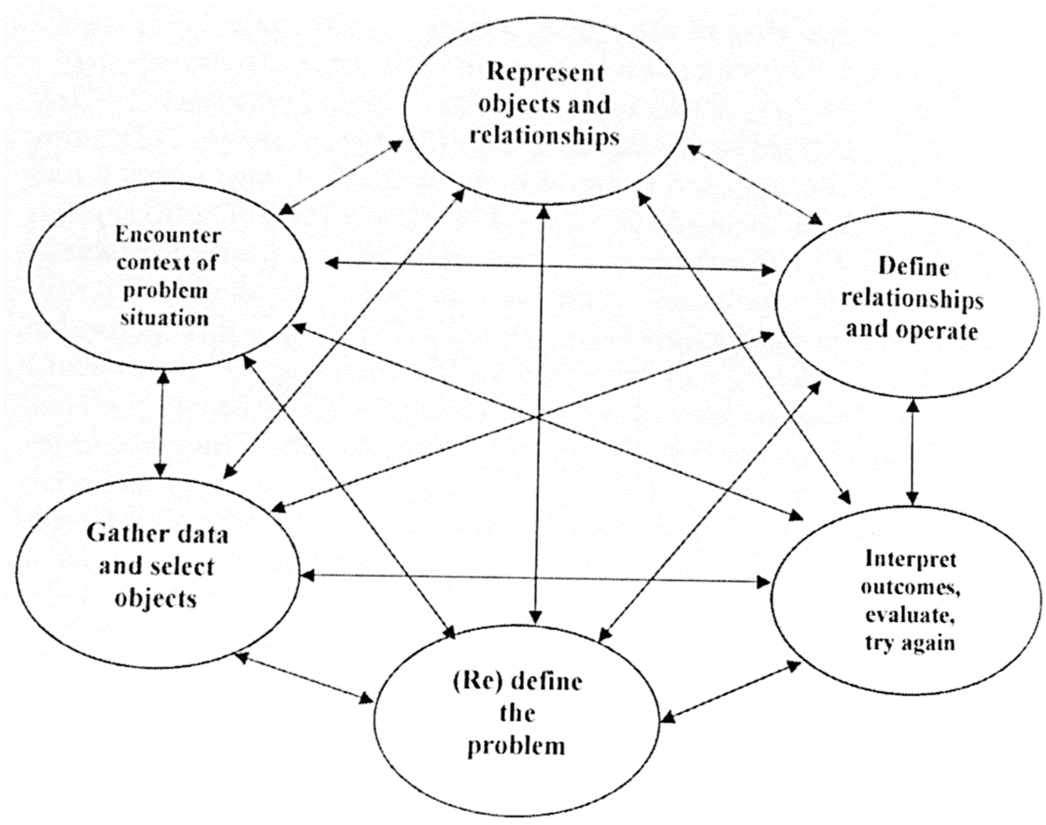

Figure 3: A nonlinear view of the modelling process.

Several research studies recognize that the development of technology creates more opportunities for practicing mathematical modelling in the classroom (Galbraith, et al., 2007; Gallegos, & Rivera, 2015; Siller, & Greefrath, 2010); this has led to the promotion of modelling activities in schools. Modelling and technology development have crossed paths in recent years thanks especially to computers. This has led to the solution of many real problems, which were difficult to solve with traditional instruments. Until only a few decades ago, technology was seen as a means to perform repetitive operations, calculations and graphics in a short time; soon, however, its usefulness in the educational process became apparent. Nowadays it is clear that computers can indeed operate 'in transformation of human activity' and lead to new forms of action (Tikhomirov, 1981;

Goos, 1998). In recent years, the practice of modelling with the use of technology has been fully included in the Mathematics school curricula (CAS, DGS, spreadsheets, and programming environments). Siller and Greefrath (2009) have implemented *Blum and Leiss' modelling cycle*, introducing the world of technology (Figure 4).

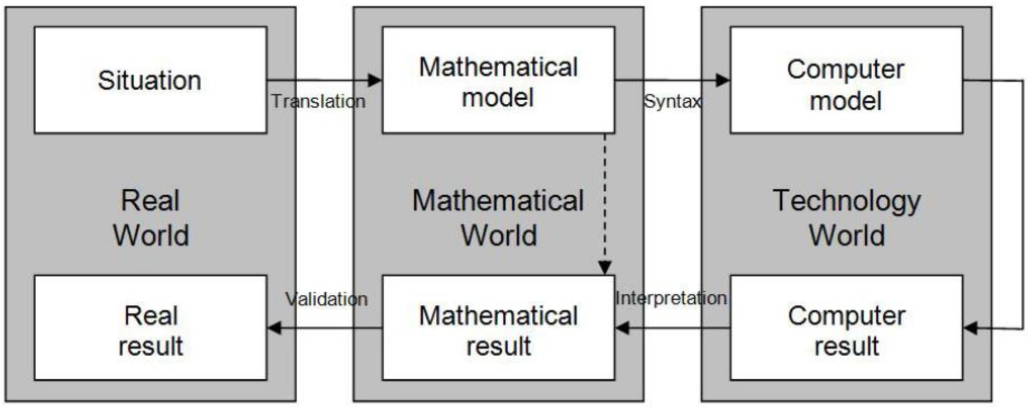

Figure 4: Extended modelling cycle-regarding technology when modelling.

The three worlds shown in Figure 4 are idealized, and influence each other. For example, the development of a mathematical model depends on mathematical knowledge on the one hand, and on the other hand is influenced by the technology.

The use of technology increases the chance to solve some mathematical models; in particular, the intelligent use of computers in teaching allows to increase motivation on the one hand and to recognize the importance of mathematics in life on the other (Gallegos, & Rivera, 2015). Unfortunately, many teachers prefer not to introduce modelling in classroom practice for different reasons, which include the lengthier learning process; at the same time, there are many reasons in favour of integration between modelling and technology. Fuchs & Blum (2008) quote Möhringer's aims (2006) which can be reached through (complex) modelling with technology:

- **Pedagogical aims:** With the help of modelling cycles, it is possible to connect skills in problem solving and argumentation. Students are able to learn application competencies in elementary or complex situations.
- **Psychological aims:** With the help of modelling the comprehension and the 71emorization of mathematical contents is supported.
- **Cultural aims:** Modelling supports a balanced picture of Mathematics as a science and its impact on culture and society (Maaβ, 2005).
- **Pragmatically aims:** Modelling problem helps to understand, cope and evaluate known situations.

The use of technology can help to simplify the difficulties typical of some modelling procedures. Siller and Greenfrath (2009) identify some key elements when you engage in modelling activities with technology: computationally-intensive or deterministic activities; working, or evaluating structuring of large data sets; visualizing processes and results; experimental working.

In classroom teaching practice, the use of technology allows not only to cope with traditional content using different methods, but also to explore and encounter new content.

3.3 TECHNOLOGY AND GEOMETRY TEACHING AND LEARNING

Geometry has increased importance in today's Mathematics curriculum, but at the same time, traditional approaches to geometry instruction do not seem to take into account the potential of today's learners (NCTM, 1985).

It has already been mentioned that technology can facilitate the learning of geometry and enhance the development of the spatial dimension of geometric objects. In particular, the characteristics of DGS were analysed.

The use of DGS continually opens up new teaching perspectives in the teaching-learning of geometry because it enhances the constructive aspect without detracting from deductive accuracy, from clarity of hypotheses and related consequences pertaining to the discipline (Hannafin, et al., 2001; Laborde, 2001b; Arzarello, et al., 2002; Hohenwarter, et al., 2008; Leikin, & Grossman, 2013). Thanks to DGS the graphic-constructive phase, both prior to the acquisition of some concepts and geometrical properties, and subsequently as verification and/or further study, is not only enjoyable, but also greatly helps teaching, as it offers both visualization and exemplification and/or exploration. On the other hand, according to Federigo Enriques (1921):

> *It does not help to develop with impeccable deduction series of theorems of Euclidean geometry, if you do not return to contemplate the constructed edifice, inviting the disciples to distinguish the truly significant geometric properties (for example: the sum of the angles of a triangle and Pythagoras' theorem) from those that have value only as links in the chain*[51].

The real world stimulates both the connection with 'significant' geometric properties, and their simulation with modern technological tools. It is not superfluous to recall here that though geometry was created as modelling of the physical world around us, the Italian teaching tradition has followed the hypothetical-deductive Euclidean interpretation and has progressively emphasized its formal character, 'decontaminated' from the figural and constructive aspect, even in the most harmless of terms. Conversely, it has appropriately and explicitly pointed out that '*geometrical concepts cannot ignore the dual conceptual and figural aspects*' (Fischbein, 1993), therefore the two should interact, and otherwise geometry activities will not increase conscious learning.

[51] Enriques, F. (1921). Insegnamento dinamico. Periodico di Matematica, Serie IV, vol.I, 6-16.

In short, the surveyor's traditional tools (ruler, square ruler, compasses), retrieved and simulated by DGS, on the one hand facilitate geometrical intuition, while on the other raise and stimulate interest and learners' imagination, enabling speculation, which is sometimes immediately verifiable, thanks to the simultaneous computer *feedback* (Jones et al, 2000; Hollebrands, 2007; Hohenwarter, et al., 2008; Ruthven, et al., 2008; Baccaglini-Frank, & Mariotti, 2010). The connection between a drawing and a geometric object in everyday teaching practice is nearly always established through a process of approximation. This is based on the idea that with subsequent, better attempts the drawing can eventually achieve something close to the ideal figure. Geometric constructions made with traditional tools (R and C) also fit this framework and are opposed to free-hand constructions in purely empirical terms of precision. In Italian higher secondary schools students come across constructions made with tools – when this happens – as part of the *Drawing and Art History* subject, something which in most cases reinforces the practical aspect of constructions and their separation from a geometric context. The use of tools is then seen in practical rather than theoretical terms (Mariotti, 1995). However, in this way a fundamental aspect is ignored and remains unknown to students: each tool contains some knowledge, which is useful for the solution of a particular class of problems. In this sense, a geometric construction appears like a geometrical problem (Mariotti, 1996) whose solution can be worked out within a given theoretical framework. Indeed, the essential didactic value of the Euclidean frame has always been the perception of its nature of a comprehensive frame, which begins with the 'simple and evident' and progresses to the complex and 'non-evident'.

Geometric construction, suitably contextualized in the teaching practice, helps the students to begin just this complex path which starts with the simple and evident (R and C) and moves on to the complex and 'non evident' in a tangible, critical and rigorous way.

The integrated tools offered by a DGS represent a valid aid along the way as they progress in the same way from what is predefined to what is made by the user.

3.4 Technology and probability teaching and learning

The breadth of applications of statistical and probabilistic tools in the contemporary world is really impressive. Indeed, it could be argued that the construction of a probability theory in some way represents an attempt at controlling the future, providing the rationale behind choices that are made in uncertain conditions. Attempts to harness uncertainty bring with them a background of insights and convictions regarding widespread random phenomena, which are rooted in mankind; but - like with every scientific theory - the consequences of probability theory laws clash with those spontaneous conceptions, and subvert those insights. Many years ago (1967) Bruno De Finetti observed:

> *On many occasions, a lot of people reason badly because they don't know probabilistic and statistical concepts.*
> *But it also often happens that, on such occasions, many others reason badly because they have learnt some probabilistic and statistical concepts without fully understanding them, or misunderstanding them enough to apply them wrongly.*
> *In this tendency to err at all costs we can certainly detect [...] an effect of aversion to uncertainty: either one does not apply the concepts which express uncertainty, or applies them forcing the interpretation so as to transform uncertain predictions into certain ones, or in such a way as to obtain, thanks to the strangest misunderstandings, gratuitous or distorted conclusions*[52].

[52] De Finetti B. (1967). *Il "saper vedere" in matematica.* Torino: Loescher, pag.52.

In the schools of the III millennium, the Mathematics of probability is at the basis of education because it helps to shape and channel the decisions and choices of future citizens (Bennet, 1998; Beltrami, 1999; Everitt, 1999; Gal, 2005).

Scientific teaching that proposes to facilitate learning and learner awareness ought to acknowledge 'spontaneous' conceptions of probability, and reflect on the meaning and conceptual reach of probability itself. Educating to probabilistic thinking is important because students get used to critical thinking and the ordering of ideas so they can rationally assess events and correctly interpret real world phenomena. The formal development of probability learning is closely related to a series of paradoxes that show the disparity between intuition and conceptual development in this field. For example, many students think that the events "Get 5 and 6" and "Get two times 6" in the launch of two dice are equally likely to occur (Borovcnik, et al., 1991 Batanero, et al., 2005). Many people's spontaneous judgments on probabilistic results are partial and sometimes erroneous; incorrect assessments of some random situations are often expressed as preconceived opinions and superstitions (Fischbein, & Gazit, 1984). Learning about probability is essential to help prepare students for life; everyday life is permeated by random events and possible phenomena (Bennett, 1998; Beltrami, 1999; Everitt, 1999).

According to Hawkins (1990) the calculus of probability cannot be reduced to the teaching of conceptual frameworks but must foster in students the development of thought processes, which lead to correct intuitions.

The need to develop appropriate probabilistic intuitions is a basic component of the teaching of probability (Feller, 1968). Intuitions in most cases cannot be modified by verbal explanations alone; new intuitive abilities can only be developed through the learner's direct involvement in practical activities. Accordingly, a teaching program that intends to develop intuitions and promote informed knowledge of probabilistic facts should provide students with active experiences of stochastic situations. Learners should experience the consistent flow of general ideas from detailed problems, and from particular observations to general theories, the continuous transition from the concrete to the abstract and from the abstract to the concrete, ideas that are based on the intuition of a single magical reality, in which all conceptual resources serve the practical issues and all the practical problems contribute to conceptual development, this and that in turn, means and end, overcoming any form of antagonism (De Finetti, 1957). In Italian secondary schools teaching practice, unfortunately, the calculus of probability has often been overlooked (Vighi, 2012). This generalized ignorance of even the most elementary notions determines the spread of wrong convictions and/or misconceptions, which border on superstition (Fischbein, et al., 1991). The calculus of probability is actually one of the most stimulating branches of mathematics, also because of its many applications (Gigerenzer, et al., 1989; Greer, & Mukhopadhyay, 2005), and includes three different approaches: classic, frequentist and subjective (Chernoff, 2008). The classic approach to probability has long been dominant in Italian secondary schools (Rossi, 1999). The majority of students find this approach based on combinatory calculus quite hard especially because of the calculations involved in solving the formulae. The difficulties in the classic approach have so far represented a real obstacle to classroom teaching and learning of the topic, not only in Italy but also in other countries (Batanero, et al., 2005). However, in recent years the growing interest for statistical methods and the use of information technology have contributed to the study of probability as a limit of stabilized frequency (Biehler, 1991). The modelling point of view was adopted in the last years, linking probability teaching with the statistical thinking. The introduction of efficient computers in secondary education allows us to simulate models resulting from statistical observations and to introduce students to the large field of statistical inference. The advantage to using simulations is that we can overcome much of the difficulty encountered when using the formal rules. However, despite its importance, in many countries the concept of probability has only recently been included in the Mathematics curriculum and has become a field of research only in the twentieth century (Gürbüs, 2010). Fischbein (1975), Steinbring (1991) and Pratt (2005) have stressed the importance of the calculus of probability for student education. In recent years, the reasons for including probability in the school curricula have been highlighted several times (Gal, 2005; Franklin, et al., 2005; Jones, 2005; Batanero, & Diaz, 2009), emphasizing also the need for a connection between theoretical and experimental activities. Furthermore, it is that reciprocal dynamics of theoretically computed probabilities and observed relative frequencies that may best contribute to the development of efficient probabilistic intuition (Fischbein, & Gazit, 1984).

The combination of games and simulation could be a successful strategy for the development of probabilistic thinking. Game-situations offer good opportunities for learning (Speiser, & Walter, 1998, Vidakovic, et al., 1998). At the same time, simulations allow to develop students' mental models in relation to complex situations or problem solving strategies. Games and simulations represent real 'experiential exercises' because they constitute a real opportunity for students to interact with knowledge. Gaming experiences and simulations offer excellent learning opportunities for all students, even from a motivational point of view. A simulation can be used as a pseudo-concrete model for different real situations, in this case offering the chance of working without mathematical formalisms. The simulation acts as a mediator between reality and the mathematical model; it is an educational tool, which serves two purposes: it helps students improve probabilistic intuitions and the teacher in the various steps of modelling (Beanery, et al., 2005). The teaching of probability is a difficult task, so it may seem counterproductive to introduce the computer in teaching because it represents an additional obstacle, which requires a specific educational effort. On the contrary, technology helps to overcome many obstacles; Biehler (1991) had already stressed the importance of using computers appropriately in probability teaching and learning to fill the following gaps:

- ***Lack of experience.***

  Time constraints and limited resources do not allow for sufficient opportunities for gaining experience to support the learning of probability. Learning environments, which allow computer simulation as well as exploration of real data, may be of considerable help here;

- ***Concept-tool gap.***

  There is a gap between the intended generality of the probability concept and the system of operations and tools students actually use. Solving problems by simulation or substituting analytical methods with computer-based graphical and numerical methods makes a new range of problems and realistic situations accessible to students' activities.

The use of the computer for the simulation of even simple situations and random processes is of great help because it allows the repetition of the phenomenon an increasing number of times, which is not always feasible. It also enables students to build models, to gain experience with random phenomena and predict the behaviour of the same, something hardly possible without fast random generators.

Ultimately, in probability teaching and learning the computer is a support tool for creating simulations and for generating data. Along these lines, studies carried out by Pratt (2000) show that probabilistic reasoning can be explained as part of a broader perspective that encompasses different types of resources for learning situations.

For example, the use of special programming environments aimed to the teaching and learning of mathematics gives learners the opportunity to build and/or process a different type of interaction that entails the writing of a resolving algorithm and the resulting implementation in language.

Compared to real or experiential simulation, virtual or symbolic simulation implies two or more variables, which means that the student is cognitively engaged in more interconnected tasks at the same time (Cosmides, & Tooby, 1996).

Virtual or symbolic simulation can also be performed with 'closed software': in this case, students would remain external to the development of the phenomenon and would thus be unable to follow the process of construction of meaning.

Probability teaching-learning can be fully achieved through strategic choices such as cooperation and constructivist activities (Barab, et al., 1999). Sensible use of technology fits a similar didactic perspective.

Chapter 4: Methods

This chapter provides an overview of the implementation of the studies. First, it describes issues relating to the research design. Then the chapter provides a discussion of the used methods in the context of laboratory practice. To this end, it is explained the sample of the students involved in the experimental path and outlined the procedures. A final section of the chapter concerns the specific activities of experimentation reported in some papers, produced during the doctoral studies, since it is closely linked to research questions of the thesis.

4.1 Research questions

As mentioned in Chapter 2 and Chapter 3 DG Software and the programming environment are effective in the teaching of Mathematics. The fundamental questions for the research study of this thesis are:

1. What is the impact of GeoGebra on the construction of plane curves?
2. What is the impact of MatCos on the approach to the calculus of probability?
3. In terms of learning outcomes, are there any differences between students who use this software in classroom practice and those who do not?

4.2 Hypothesis

Chapter 3 showed how technology as part of a suitable methodological framework constitutes a mediator for learning. The assumptions on which the following research study is based are:

1. Students who use this software achieve better results, in terms of learning, than those who do not;

2. The use of such software promotes the development of critical thinking and problem solving skills.

4.3 PARTECIPANTS

The research study involved 85 students from the "Liceo Scientifico - E. Fermi", Cosenza: 45 students attending class III (Year 11) and 40 students attending class IV (Year 12). The sample of students was divided into two groups; the first group of 21 students from class III and 23 from class IV already had some prior knowledge of the two software; the second consisted of 24 students from class III and 17 students from class IV.

The first group of students used the software during the lessons, while the second group did not use the technology and took part in traditional lessons (lectures with chalk and blackboard).

4.4 PROCEDURES

The primary purpose of the trial was to investigate the impact of software in teaching regarding the chosen mathematical topics, including the existence of differences, in terms of learning outcomes between the two groups of students. The procedures of experimentation were framed around three phases:

1. Administration to the entire sample of students of the initial test[53], consisting of three parts, each of which had 5 questions. In particular, the first part was aimed at investigating students' attitudes and thoughts on the discipline of Mathematics and the use of technology in the classroom. The second part was aimed at ascertaining their knowledge of elementary geometric objects and related skills, and of geometric constructions. The third and final part was aimed at exploring the

[53] Appendix A.

students' ideas and thoughts in relation to some probabilistic phenomena. On the basis of the results of the initial test[54], we planned some laboratory activities.

2. Creation of four educational activities[55], further divided into various dates, supported by the use of GeoGebra and MatCos software. Two meetings were devoted to the introduction of plane curves (the deltoid and Bréguet's spiral) using different techniques of geometric construction, and two to an approach to the calculus of probability.

3. Administration of the entire sample of students of the final test[56], consisting of three parts. The first part was administered only to the sample of students who followed the trial supported by software, consisting of five questions. This part was aimed at collecting the students' opinions on the educational activities.

   The second part, administered to the entire sample of students, consisted of three questions concerning geometric constructions. The third and last part of the final test, also administered to the entire student sample, consisted of three questions concerning the evaluation and comparison of some random phenomena.

4.5 INITIAL EXPERIMENTATION TEST

The initial survey aims at verifying the students' knowledge on the construction of meanings of geometric objects, ideas and interpretations regarding some elementary probabilistic phenomena.

This phase represents a cornerstone for the Design Activity, providing a logical flow throughout the entire experimental path.

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[54] Analyses and interpretation of the results obtained are reported in Chapter 5.

[55] The activities with the software have been carried out by 44 students, 41 students, instead, followed the curricular lessons without the help of software.

[56] Appendix B.

Three aspects that act as 'guiding principles' were taken into consideration on the basis of the research study: exploration, re-construction and explanation.

These included activities such as exploration, conjecturing, construction of formal proof, and exploration for further generalizations.

Therefore, the initial test was structured so as to identify gaps in the way we think about and analyze mathematical situations inherent the chosen themes. In fact, from an analysis of national and international results (Invalsi, Timms, OCSE-Pisa) some very critical aspects emerged: in short, the students are not able to apply the skills learned at school in a less structured context (Siniscalco, & Pedrizzi, 2005).

In particular, part B of the initial test has 10 questions; the first five are about the fundamental, easily represented geometric entities, while the remaining five are about the mathematical description and interpretation of some elementary random phenomena.

4.6 DESIGN ACTIVITY

All proposed activities follow the workshop methodology in order to motivate and stimulate students' interest, posing to their attention open-ended problems, such as those inherent the modeling of real situations.

4.6.1 Activity 1: The deltoid as envelope of line

Design activities in the classroom have a dual objective: raising students' curiosity and interest for geometry on the one hand, and on the other retrieving and consolidating geometric concepts and methods, already known to students in broad terms but which they have not fully mastered yet. That is why we chose a practical approach based on 'manipulative experience' implemented through problem posing and problem solving in order to gain insight into Euclidean geometry. In particular, we set up two workshop activities which entailed the construction of a deltoid curve first using traditional tools (R and C), and then virtually by using the GeoGebra spreadsheet.

Step One

The class is divided into groups, and each group gets the required material and tools (graph paper, ruler, compasses, protractor, pencils and rubbers); then the teacher sets the following task for the geometric construction:

- *Draw a circumference C with centre O and radius* $r = 2\ cm$*;*
- *Trace a diameter naming the two extremes A and B;*
- *Starting from the B point subdivide the circumference into n (n divider of 360) arches of equal width α, let* B_i *(i=1, …, n) be the second extremes of the arches (Example* $\alpha = 10°$*);*
- *Starting from the A point subdivide the circumference into n arches of equal width 2α, let* A_i *(i=1, …, n) be the second extremes of the arches;*
- *Trace the straight line passing through points* A_i *and* B_i *(i= 1, …, n).*

Materials and task interact motivating the students to come to terms with the new concept; the sheet of graph paper acts as a background which enhances the drawing and at the same time supports the students while they work with the required accuracy. In other words, making a construction with R and C to draw a deltoid curve means starting from other geometrical objects using only real tools; this requires not only good manual skills at drawing but also a first level of abstraction. The direct manual work is obviously centred on the role of the traditional tools in the construction of Euclidean geometry as a hypothetical-deductive science highlighting the process of idealization through which these real tools are turned into abstract ones, characterized by the properties expressed by the postulates.

Once all the groups have finished (Figure 5) the task the teacher (T) asks the students (S) some stimulus-questions. The examples below are to be intended as basic guidelines. Here is an excerpt from the protocol:

T: *the geometric construction obtained shows a new figure, let's try to understand better ……..*

S: *it's an unusual figure …. Because from the drawing of straight lines we get some border curved lines;*

S: *the construction obtained "generates a closed curve with three points";*

T: *have you come across any "closed curves with three points" before?*

S: *I saw a similar construction in a museum where two "pictures" had been made with taught wires, which replaced the drawn lines;*

S: *it makes me think of an equilateral triangle with the sides curved inward;*

T: *Which properties do you think verify the drawn straight lines?*

S: *all the straight lines are tangent to the curve;*

T: *Are these properties casual or do they occur regularly and if so in which conditions?*

S: *so it is a curve obtained from all these tangent lines….*

T: *what happens if the measure of the radius varies from the circumference, does the figure remain the same or does it change?*

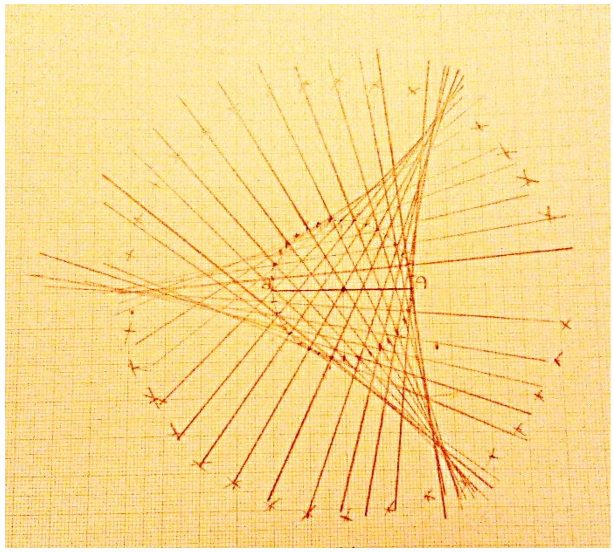


Figure 5: Construction made by a group of students.

The methodology of guided discovery encourages the students to formulate the first conjectures and to explore further; after that, the teacher introduces the concept of an envelope of curves, a term which contains simple but fascinating concepts. The set of traced straight lines forms a curve called *envelope, which* can be assimilated to an arch of circumference; the ordered composition of these arches generates a curve with different shapes, open or closed, which is quite impressive. The conversation triggered and guided by the teacher is very important because it avoids the construction of formal games and it educates to reasoning before formulating conjectures and hypotheses, stimulating creativity, intuition and the imagination. As a result, the majority of students continue to investigate, to tally up points, monitor, reflect and experiment trying to go deeply into the mathematics knowledge system; a small minority, instead, has reservations about the need to repeat the process of construction with different radius values, as it would be repetitive and therefore boring. To re-establish class motivation, the teacher can then suggest the repetition of the process of construction in the computer lab using the GeoGebra spreadsheet. To recap, the starting point of the activity is a declarative and static task of a problem posing nature which asks the students to carry out an imperative and functional piece of traditional work (tracing of the curve with R and C) finalized to a representation which has a lot of learning potential. Such a problem-posing activity in fact goes beyond the traditional logic of the repetitive execution of a drawing (same mechanical operations of subdivision) because at cognitive level it triggers the ability to understand and interpret knowledge (hermeneutics), to investigate (ability to discover and produce knowledge) and heuristic (ability to invent and create new knowledge).

Step Two

In this second activity the students, again divided into groups, are asked to repeat the previous process of construction, but this time they should do so using the GeoGebra; therefore the "predefined objects" available on the tool bar will be used: point, medium point, circumference, rotation.

The steps of the solving algorithm are as follows:

1.		Draw two points, A and B in the Euclidean plane;
2.		Determine the medium point (C) of A and B points;
3.		Trace the circumference with centre C passing by point B;
4.		Construct point B', rotated of B by 10° angle with respect to C;
5.		Construct point B_n', rotated di B by an angle $n \cdot 10°$ with respect to C ($n = 1, \ldots, 35$) (Figure 6);
6.	Trace the envelope lines (Figure 7).	

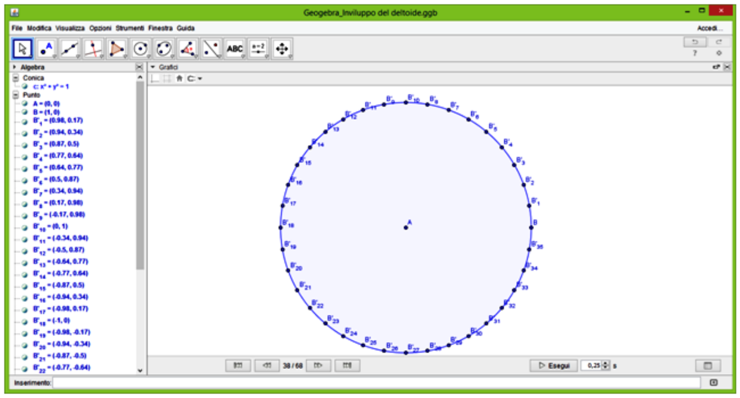


Figure 6: Subdivision of the circumference into arches with same width.

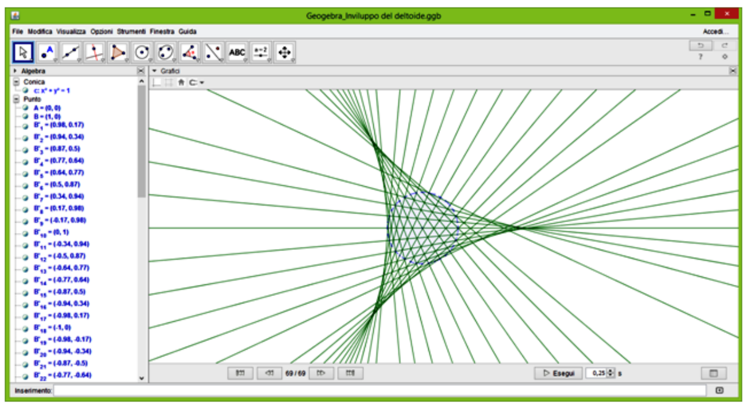


Figure 7: Output of the envelope of the deltoid curve.

The repetition of the construction algorithm helps the learner to understand and use the geometrical shapes in question; it provides the answer inherent to the question of the invariable: the drawing of the curve does not change for different values of the circumference radius. At this point the teacher tells the students that the triangular curve obtained is called deltoid due to its shape being similar to the Greek letter ‘delta’, and continues by saying that the discovery of the curve cannot be ascribed to a particular person due to its relation with another curve named cycloid (rolling curves) studied by Galileo and Mersenne as far back as 1599, and later conceived by the Danish Roemer in 1674 while studying the best shape for a gearwheel (Cresci, 2006). The first to actually seriously consider the deltoid curve was Euler in 1745 in relation to an optical problem, while later in 1856 Stainer studied the curve in such depth that it was nicknamed 'Steiner's ipocycloid' (Cresci, 2006).

In any case a more extended study of the curve within the context of its historical background could be the object of future lessons. With regard to the present, the teacher begins a problem solving activity by asking the students to compare the geometric constructions made, noting analogies and possible differences. A lively discussion takes place among the groups.

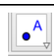
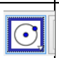
The teacher keeps to herself. At this point all the students agree that both constructions do not show any differences regarding the geometrical objects used; but they notice that "on the computer we can move points without having to start all over again". To sum up, from the discussion it becomes apparent that the repetition process should be streamlined; consequently the students are required to study a strategy for the resolution of the problem. Improving the previous process of construction becomes very important from a teaching point of view: the tools used previously are no longer sufficient. This is when the GeoGebra slider tool can represent a valid aid. A numerical slider visible on the spreadsheet is thus created, finalized to the tracing of the straight lines so as to reduce the time needed for the execution of the previous algorithm. This is a very tricky phase for the students because they have to take quite a big jump into abstraction: the difficulty lays not so much in the creation of the spider, but in identifying the object or objects to apply it to. The slider requires the allocation of a parameter specifying the numeric interval (min and max) and the increase; the obstacle is represented by the identification of the geometrical object (there can be more than one!) which will be modified by the slider.

The basic geometrical objects of the construction are points A and B on which the slider will work; therefore the following points need to be defined:

- A' (rotated by A by an angle $n \cdot 10°$ clockwise);;
- B' (rotated by B by an angle $n \cdot 10°$ anticlockwise);
- tracing of the straight line passing by points A' and B'.

The creation of the slider is fundamental because it represents the tool designated to further improve the construction process of the deltoid curve as an envelope of lines.

Below you can find the steps of the algorithm:

1.		Draw two points, A and B in the Euclidean plane;
2.		Determine the medium point (C)of A and B points;
3.		Trace the circumference with centre C passing by point B;
4.		Define a *slider* n (n whole number from 1 to 36); - Construct a point A', rotated of A of an angle $-20 \cdot n°$; - Construct a point B', rotated of B of an angle $10 \cdot n°$; - Trace the straight line passing by points A' and B' (Figure 8).

The envelope of the deltoid curve can be obtained moving the *slider* along (Figure 9).

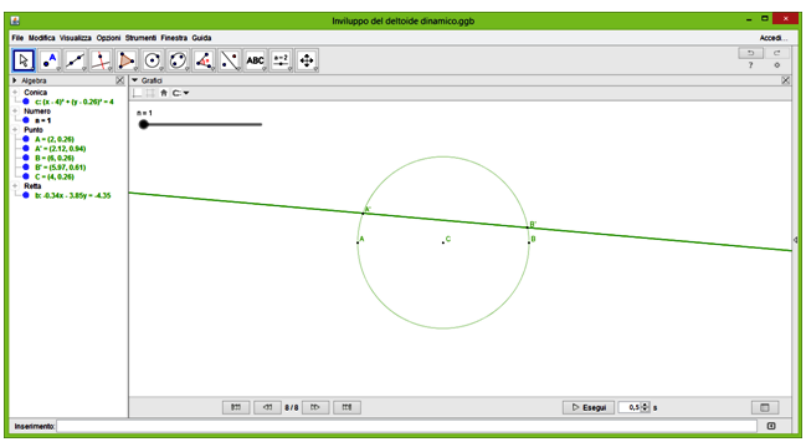


Figure 8: Partial output of the envelope of the deltoid curve.

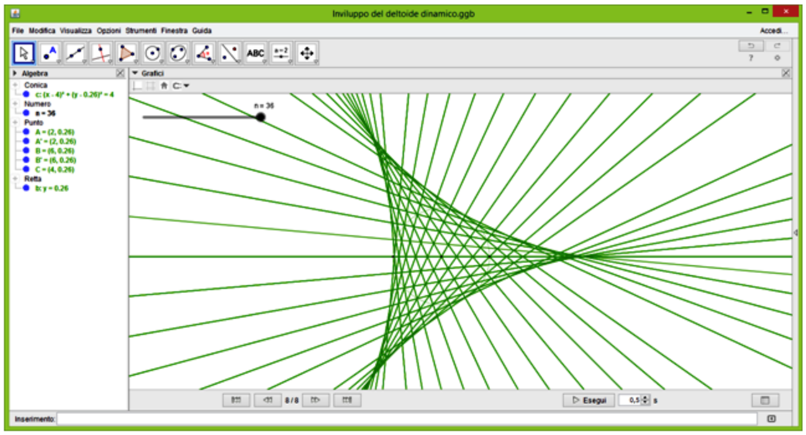


Figure 9: Output of the complete envelope of the deltoid curve.

This second activity carried out in the computer lab is very educational because it:

- implies the real comprehension of the algorithm of construction, through the simplification of the geometrical objects and their complexity;
- enables the learner to answer the question about the invariant of construction;
- adds value to the students' learning, because they get the opportunity to improve their skills and to become familiar with the formal and rigorous language of mathematics without being compelled to do so by a teacher.

To sum up, the second activity acts as support because it values intuition and at the same time enables the learners to make generalizations - impossible to attain with a static image - which lead to manage intuitive discovery within a rational framework, and ultimately to accept not only traditional deductive but also inductive processes of learning.

4.6.2 Activity 2: Bréguet spiral

The activity, designed according to the previous scheme, is aimed at the geometric construction of Bréguet's spiral (De Carle, 2014) – a little-known curve - designed to give watches greater regularity and better performance. We have focused on the digital construction of the geometrical model so as to make the representation of the physical element easier (Bréguet's spiral), which enables the proper mechanical functioning of the watch. The dichotomy between the conformative and the representative component is of fundamental importance here. The authors believe the representative component has greater impact on teaching, as it moves from the iconic to the semantic. The construction of the model indeed shows and communicates the synthesis between the structural composition (mechanical device) and the shape (physical element separated from the mechanism).

Centred on these considerations, the design activity is shown in Figure 10.

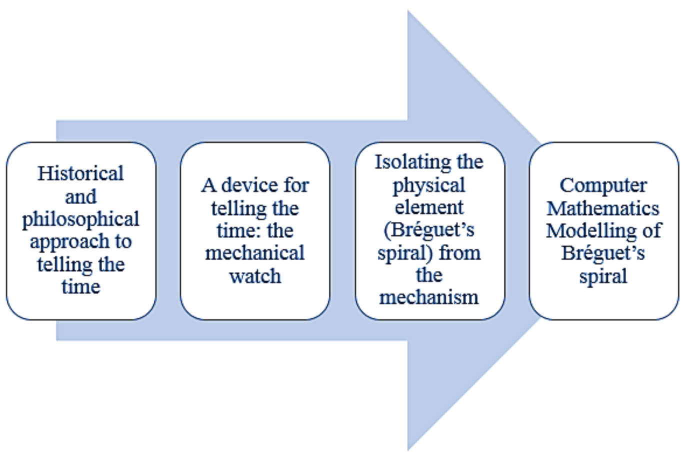


Figure 10: Design activity.

A historical and philosophical approach to telling the time implies a natural and coherent overlap with Physics, so we can start by introducing the idea of measuring the time, and after lead the students to the discovery of the mechanical watch. Examination of the device through the geometrical modelling process opens the doors to Mathematics.

Thus methodology follows the classic modelling and application process, a choice which is also supported by recent international research in the field of didactics (Arzarello, et al., 2011; Heide, & Blume, 2008).

The aim is for the students to experience the object being studied, so that knowledge is acquired from both a synthesis of the processes leading to it, and the contexts in which they are used, as opposed to the acquisition of knowledge by 'closed compartments'.

Specifically, the teaching initiative begins with the introduction and analysis of the device, which makes the basis for the mechanical watch. Attention focuses on the different components of the 'barrel-spiral' system, which makes it work. Later, the 'barrel-spiral' system is subdivided into elements, and so the barrel spiral (Bréguet's spiral) is isolated. Detaching the physical element from the mechanism is the start of a process of abstraction which is at the basis of any possible interpretation; the elementary process of Mathematical modelling follows on from this, and is carried out through the use of the GeoGebra software. So the identification of Bréguet's spiral represents a real chance for the students to work at a deeper level, while the geometric language begins to give the object a mathematical structure.

Therefore the essence of geometry, at the same time rational and objective in its structure, is the ideal tool to use for the synthesis between the technical and scientific aspects (Bréguet's spiral consists of 13 turns of which the first 12 are the same as Archimedes' turns, while the last one is curved inwards).

In the body of Euclidean geometry, the introduction of an operative phase, representative of the movement (using *GeoGebra*) during the various phases, leads to a structuring of the actions and their effects into a descriptive framework whose convincing arguments and demonstrative clarify gradually increases .

Furthermore, the graphic representation of Bréguet's spiral offers a double advantage: it adds value to intuition on the one hand, while on the other it allows a form of generalisation – impossible with a static picture – which enables the rational handling of intuitive elements and consequently encompasses not only traditional deductive processes, but also inductive ones.

Reading the time

The teaching initiative starts from a broad historical and philosophical perspective so that ideas can be correctly set against an original background of discovery and novelty and in so doing, open up interdisciplinary perspectives around concepts, meanings and applications. In particular, we begin from a philosophical viewpoint with the work '*Le Confessioni di Agostino*' (*Augustin's Confessions*), where in book eleven the question of time is taken into consideration:

> *Quid est ergo tempus?*
>
> *Si nemo ex me quaerat, scio; si quaerenti explicare velim, nescio*[57] *[12, Liber XI, 14]*.

Augustin was the first to talk about subjective time, beginning his reflection on time from the consideration that as soon as one tries to stop it in any way, to describe it or measure it, it dissolves into nothing. Time cannot be defined using the categories of space, because time is not *per se*, it is not something, but a relation, a reference to a system, something which is perceived by a sentient subject. In the fourth century BC Augustin

[57] Translation: 'What is time then? If no-one asks me, I know; if I try to explain it to those that ask me, I don't know.'

already wonders, although not in formal terms, about the chance to measure time through prefixed units of measure[58].

At this stage it seems appropriate to observe that time cannot be categorized if through categories its meanings are reorganised outside the perceiver's space. Other philosophers and scientists have dealt with this topic, so discussion on this could be taken further.

The above considerations naturally lead us to consider now the subject of Physics, a discipline that has greatly contributed to progress in this field by creating devices for the precise measuring of time since the beginning of history, as documented so far. The concept of time is obviously different from that of length; to measure the latter we can use a rigid tape measure, whereas a sample time interval can be used only once.

It must be noted then that a fundamental requirement for measuring time is a 'regular repetitive process', as well as numerable; in other words, we must turn to a periodical process. Watches are in fact devices (mechanical, electrical, digital, etc.) which accomplish an appropriate periodical phenomenon.

At this point our attention turns to the mechanical watch, that is a device made up not of a material point but a set of objects of finite dimensions, which interact according to the laws of mechanics and are driven by forces (the spring power). We have focused on the digital construction of the geometrical model so as to make the representation.

The mechanical watch

The most common mechanical watches are those to be worn on a wrist, with a barrel, anchor escapement and spring engine, the wheel work operating three hands which

[58] how do you measure "the length of an ode from the verses, and the verses from the feet, and the feet from the length of the syllables. […] But it may happen that a short verse, pronounced more slowly, is heard for a longer duration of time than a longer verse recited more quickly " [Augustine, XI, 26, p.396-398].

indicate the hours, minutes and seconds respectively. In detail, a steel spring inside a barrel, suitably charged by the powering mechanism, represents the engine which provides the energy needed to operate the wheel work and the oscillator (Figure 11).

The wheel work consists of two sets of cogwheels: the main one has four coaxial wheels and as many safety pinions, each of which engages the previous wheel ("centre wheel", whose pinion engages the barrel toothed crown, "first intermediate", "second intermediate", "escape wheel"). From the axis of the centre wheel, which extends through the central hole of the face and supports the minutes hand, through a system of friction the second series of wheels departs. This comprises the demultiplier, that makes the hour hand go round and the mechanism of 'keeping the right time' controlled by the power button.

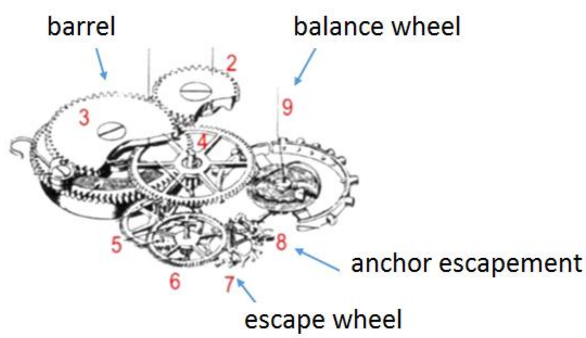


Figure 11: Mechanical watch device.

The observation of the mechanical device allows an initial generalisation; that is why at this stage we recommend that the students use the Internet to download a simple virtual simulation programme about the functioning of the 'barrel-spiral' system which will help them to better understand the mechanism.

Bréguet's spiral

In wrist watches with a mechanical movement, time beats thanks to a spring which controls the motion of a mechanism called *balance wheel*, able to oscillate on any plane. This very thin spring, coiled into a spiral shape, dictates the rhythm of the movement and regulates the marching of time thanks to its regular oscillations.

The hairspring, even though it is one the smallest components in a mechanical movement, has an essential function for the precision of the movement. It represents the heart of a watch: the central turn is fixed to the collet mounted on the axis of the balance wheel, while the external turn is fixed to the taper pin, attached to the barrel bridge (Figure 12).

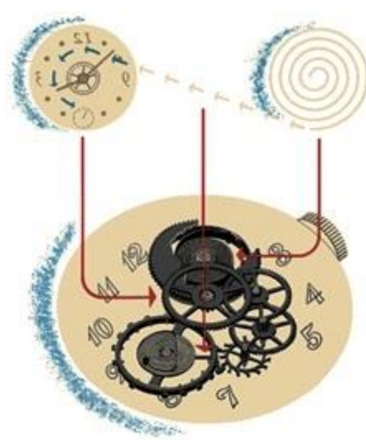


Figure 12: Barrel-spiral system.

The function of the hairspring is to make the barrel period uniform (making its oscillations *isochronous*) and with it, it forms a whole commonly called barrel-spiral system just because together they form a harmonic oscillator.

The flat spiral was applied for the first time in 1675 by Huygens. Considered as key for the accuracy of a movement, the spiral has been the object of much research, especially as far as the materials used for its construction are concerned. For obvious reasons isochronism of the barrel has always represented a challenge for watchmakers and designers of movements. The key factors which interrupt such isochronism are:

- The asymmetry of the contraction and expansion of the spiral (breath);
- the varying elasticity of the spiral to variations in temperature;
- the effect of magnetic fields;
- temperature and mechanical variations at the spiral two connecting points;
- the effect of the centrifugal force and gravity on the spiral;
- inadequate dynamic balance of the barrel.

Ever since the spiral was invented, many attempts were made to eliminate these problems and Bréguet [59], a famous and clever Swiss watchmaker, made a breakthrough in the history of watches both for the material used and as far as geometry was concerned.

In fact in 1795 Bréguet modified the shape of the hairspring moving its external end towards the balance staff, following a curve - called Lossier-Phillips - calculated with mathematical precision (De Carle, 2014). Thanks to this curve (Figure 13), the hairspring moves concentrically and the watch gains precision. Compared to a classic spiral, in this new design the external turn is slightly elevated and forms an inward angle, above the plane on which the rest of the spiral lies. This guarantees a better distribution of weight compared to traditional solutions, minimising the effects of terrestrial gravity on the working of a watch.

[59] Abraham-Louis Breguet (1747-1823) significantly and decisively contributed to the evolution of watchmaking techniques with his inventions and technical solutions. www.breguet.com/it/Il-Museo-Breguet.

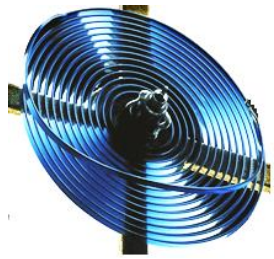


Figure 13: Bréguet's spiral.

In practice, Bréguet's spiral (Figure 13) is a spiral made up of 13 turns of which the first 12 are those of Archimedes' spiral, whereas the last turn (Lossier-Phillips' curve), is raised and bent inwards to improve isochronism.

Mathematics Modelling

The representative model of the *barrel-spiral system* as a whole provides a series of useful information on the physical significant especially as far as the characteristics of Bréguet's spiral are concerned. At this stage of the activity, mathematical modelling enables a considerable abstraction leap: a synthetic and objective description of Bréguet's spiral requires the introduction of concepts and tools which are acquired and tested in the model study phase. Afterwards the evaluation of the model allows to work to improve the tools, reflect on the theory and finally identify any further needs. So mathematical modelling in this process takes on the greater role of experimenting tool with which it becomes possible to verify the mathematical properties of the curve (Figure 14).

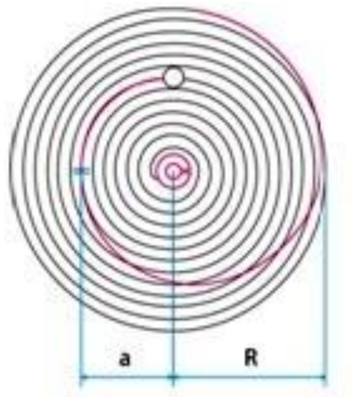


Figure 14: Bréguet's spiral.

For the reasons outlined so far the process of modelling consists of three subsequent steps, with each step representing an improvement on the previous step. In particular, in the first step the GeoGebra spreadsheet is used like a real drawing pad by proceeding to the construction of Archimedes' spiral and the Lossier-Phillips[60].

The second step has a strong teaching impact as it requires a sharper process of construction of the previous Archimedean spiral: the "predefined objects" contained in the GeoGebra toolbar are no longer sufficient. That is why a new tool has been designed which overcomes the problem of reiterating the process followed in the first step. The new tool is called "ArcSpiral" and aids the construction of the connected arcs of the spiral.

Finally, the last step further improves the process of construction started in the previous phase and the link between the Archimedean spiral and the Lossier-Phillips' curve. This highly formative process allows students to appreciate the potential of mathematical language while at the same time it offers a key for informed understanding of the theory. The three steps are presented in detail below.

---

[60] The construction procedure of Lossier-Phillips's curve was made on the basis of the theoretical study, that is mathematical, taken from Chapter XIII 'La spira superiore della Bréguet's spiral' [13, p.184-185].

Construction of the Bréguet's spiral with GeoGebra software

First step

Classic construction of Archimedes' spiral through the "predefined objects" available in the *GeoGebra* toolbar: *point*, *regular polygon*, *median point*, *half line*, *circumference*, *arc*. This procedure will allow an analysis of the construction from a mathematical point of view.

Algorithm 1. Archimedes' spiral

1. Draw two points A and B;
2. Construct the square ABCD with extreme side A and B;
3. Determine the median point (E, F, G, H) of the sides of the square;
4. Costruct the rays (e, h, g, f) with origin the median point of the square and passing by one of the vertexes of the square and pertaining to the side on which the median point lies (anticlockwise direction);
5. Costruct the circumference with centre A and passing by point B;
6. Determine the point of intersection I between the circumference and the ray with origin the median point of side AB and passing by A;
7. Draw the arc p with centre A and points B and I as extreme;
8. Repeat from 5 a 7:
    a) Costruct the circumference *CC* with centre *A*, *B*, *C*, *D* (in this order) and radius the distance between the vertex of the square and the point of intersection between the old circumference and the rays *e*, *h*, *g*, *f* (in this order);
    b) Costruct the arc with centre in the vertex of the square.

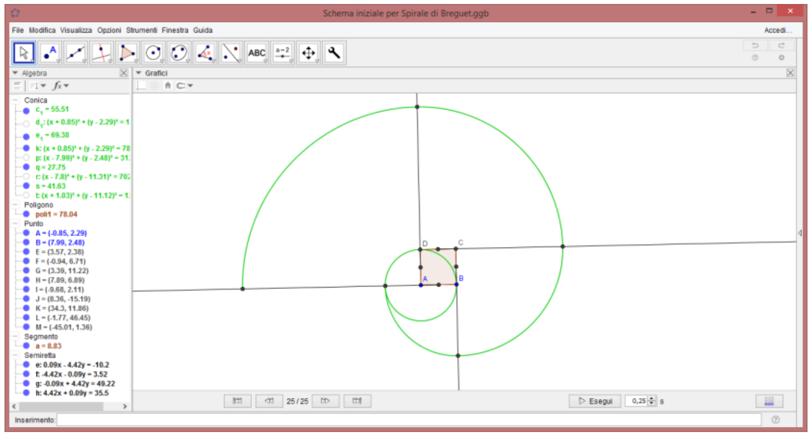


Figure 15: Output of Archimedes’ spiral.

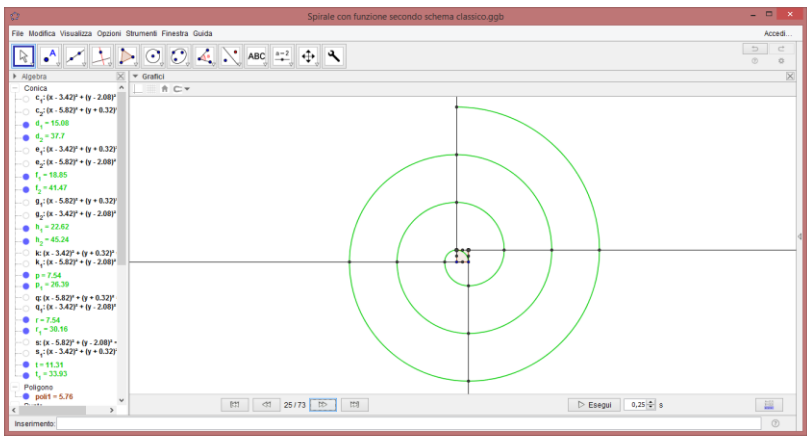


Figure 16: Partial output of Bréguet’s spiral.

To complete Bréguet’s spiral we must build the Lossier-Phillips’ curve (De Carle, 2014) using the tools made available by the GeoGebra spreadsheets.

Algorithm 2. Lossier – Phillips’ curve

1. Costruct the circumference w with centre O and radius r (OK);
2. Allocation: $d = 0{,}67\, r$;
3. Draw the diameter GK;

4. Draw the point G’, obtained from the rotation of point G respect to centre O by 83° angle clockwise;
5. Draw the segment OG’;
6. Costruct the circumference c with centre O and radius d;
7. Determine the following points of intersection: B is the point of intersection between the circumference c and the segment OG’ and D is the point of intersection between the circumference c and the segment KG;
8. Draw the arc DB;
9. Costruct the ray b passing by points B e O;
10. Determine the point of intersection C between the circumference w and the ray b;
11. Determine the median point A, between B and C;
12. Draw a semicircle with extremes B and C.

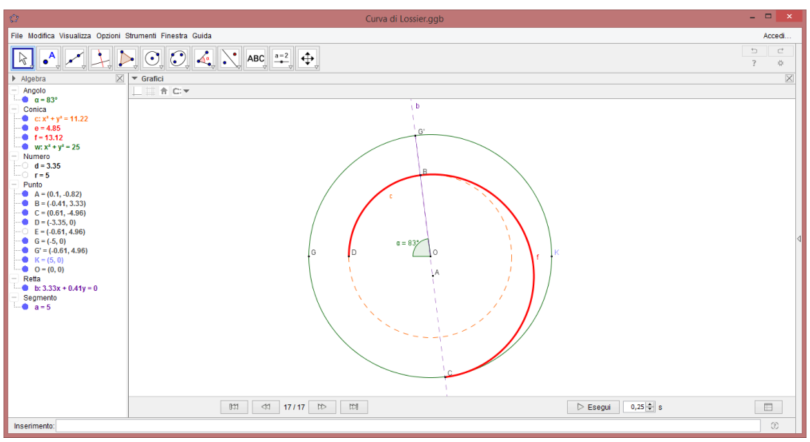


Figure 17: Output of the Lossier-Phillips’ curve.

At the end of this phase we note how the repetition of the same “solving algorithm” takes the students to a more thorough and informed understanding of the Bréguet’s spiral.

However, repeating for four/five consecutive times the same algorithm may be boring, once the learners have understood that the Bréguet's spiral consists of 12 complete turns (Archimedean spiral) plus a Lossier-Phillips's curve.

In the Archimedean spiral the execution of a turn requires the construction of four arcs traditionally constructed with ruler and compasses; so the need to simplify the repetition process about the construction of the connected arcs becomes apparent.

As a result, the students are asked to study a "geometrical strategy" for the solution of the problem. In this case, the GeoGebra software represents a valid help as it allows the creation of *new* computational *tools* into the spreadsheet interface, which can then be used as predefined objects.

Second step

Design a new tool called "ArcSpiral" used for the construction of all the connected arcs so as to reduce the execution time of the previous algorithm. This step is very sensitive because the students have to face yet another abstraction leap. They have to plan the design of a new tool called "ArcSpiral" finalised to the construction of "ArcSpiral" tool starting from the construction of the first arc, which requires the identification of the initial objects (points A, B, median point E between A and B and the half line EA) and final (point I e arc p).

Algorithm 3: The improved Bréguet's spiral

After the first seven steps of algorithm 1 we have:

1. Create a new tool "ArcSpiral", having the points A, B and median point E between A and B, and the half line passing by points E and A as initial objects, and points I and arc p as final objects. The tool thus created can be used by clicking on its icon;

2. Construct the connected spiral arcs, through the tool “ArcSpiral” created, with appropriate choice of points.
3. Steps of algorithm 2 follow.

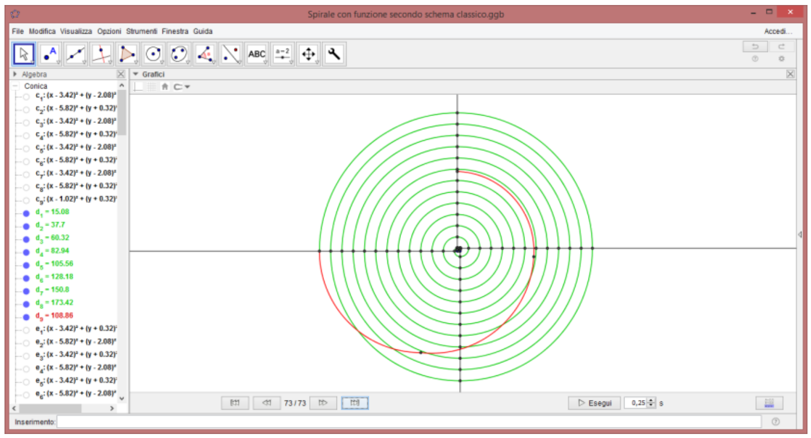


Figure 18: Final output of the complete Bréguet’s spiral.

Third step

Design a new tool “ArcSpiral” which contains a lower number of initial objects. Planning the “*ArcSpiral*” tool with only two initial objects (points A and B) and one final object (arc with centre B and points A and A’ as extremes)..

Algorithm 4. Optimised Bréguet’s spiral

1. Draw two points *A* and *B*;
2. Draw point A’, obteined from the rotation of A with respect to centre B by angle anticlockwise;
3. Costruction of the arc with centre B and points A and A’ as extremes;
4. Create a new tool called “Arcofspiral”, with points A and B as initial objects and as final objects the arc with centre B and A and A’ as extremes; the tool created can be used by clicking on its icon.

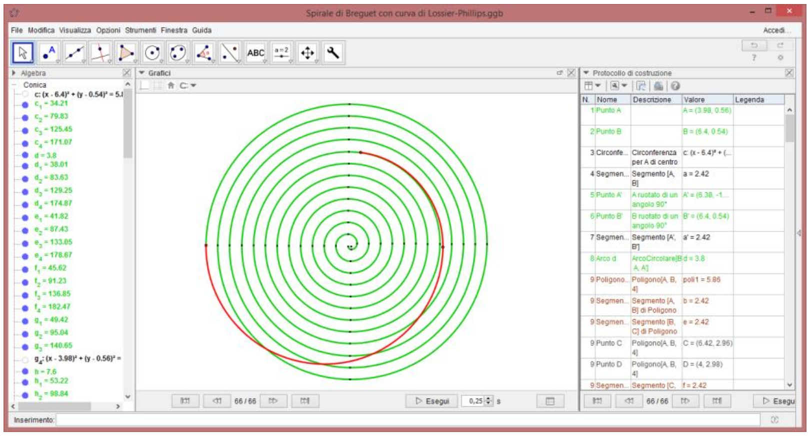


Figure 19: Output of the optimised Bréguet's spiral.

4.6.3 Activity 3: Throwing of the dice

The teaching practice takes advantage of the game intended as a "space of knowledge" that allows students the formulation of precise questions, recording his thoughts, and therefore the approach to the formulation of a common synthesis. Aspects of sensible reality, then, are reconfigured to be analyzed and studied by means of computer, how to program the machine.

The design activities is circular and follows the pattern shown in Figure 20.

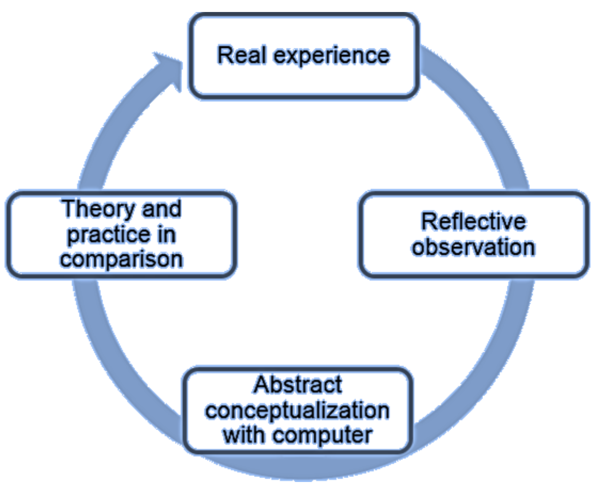


Figure 20: Design activities cycle.

The design in Figure 20 facilitates the study of random phenomena because it allows to compare the results with those dictated by the theory; the student judges that the empirical results are close to the theoretical ones the more the greater the number of trials made. The transition from the experience in the construction of the meaning takes place after the actual simulation through the implementation of simple algorithms, implemented in the programming environment MatCos.

The virtual simulation through the practice of programming is a constructive activity and cognitive as it allows the student to acquire skills, strategies and techniques to solve problems using the concepts of variable, procedure, repetition and recursion, which are also, transversal concepts to other school subjects. The design teaching setting based on game understood as "knowledge space" allows students to exercise their critical thinking skills, problem-solving, creativity and collaboration. Virtual simulation game allows students to reconfigure aspects of sensible reality to be analyzed and studied by means of computer programming environment MatCos.

The teaching action requires the simulation real and virtual about two games on the classic dice throwing regular six-sided.

Real simulation of random phenomena

The teacher, in the first part of the didactic activity, divides the class into three groups by distributing their dice and give delivery on the first game.

<u>Game 1 - Launch of 4 dice</u>

The gentleman at the counter of a table game launches 4 nuts. There is a victory of the game if you get out at least a 6 launching the four nuts.

Students motivated and intrigued begin (real simulation); the teacher observes the actions of the three groups, and suggests, if they have not already done so, to record the results in a table.

After making 20 consecutive tests, the students reveal the data obtained noting that conflict with each other or each group got different results; consequently, the teacher starts a guided discussion in order to stimulate arguments and conjectures about.

After the debate, all agree that in order to obtain more meaningful results you need to make a fairly large number of repeated trials, but to do so would require a long time; from here, the need for a tool to support the action teaching, able to simulate a large number of repeated tests in a reasonable time.

Students, digital natives, they immediately think the use of the computer and the teacher proposes them to perform the simulation using software MatCos, already used previously, in teaching situations related to other mathematical content.

Computer simulation of random phenomena

The actual simulation of the game proposed constitutes an important occasion to highlight the ability to switch from the plane of reality to that of mathematics; but in order for the teaching contributes to a real understanding of the concepts and a solid acquisition of the same, the use of computers as a tool to program, acquires great importance because it is a method, "a mental place" where students have the real chance to explore mathematical concepts, formulate conjectures to validate or refute, and then to continue the experience of problem solving. The use, then, of a programming environment, as MatCos, aimed at teaching and learning of mathematics, is an added value: the students enhance skills on the mastery of mathematical language, get used to a formal language and rigorous without any imposition from the teacher, understanding, as well as the penalty is not a useless thing.

Algorithm of Game 1

The construction of the algorithm is an important and delicate phase because the students have to design the 'finite sequence of steps' that allows the computer to get to the solution or to highlight the need for further study. The steps of the algorithm related to the simulation of Game 1 are:

1. Assignment: n (Simulation number).
2. Initialization: $cf = 0$ (counter for the number of favorable cases).
3. Cycle (simulation of launch of 4 dice)
   - creating 4 variables a, b, c, d (to which the result of the launch for each dice is assigned);
   - Control action: if($a = 6 \; o \; b = 6 \; o \; c = 6 \; o \; d = 6$) then do
     - Increase counter cf.
4. Calculation: $p = \frac{cf}{n}$.
5. Print action: cf.
6. Graphic rappresentation: histogram of the absolute and relative frequencies (Figure 21, Figure 22, Figure 23).

The algorithm is easily implemented in the programming environment MatCos.

Code MCS1 - Game 1

```
n=readnumber("number of simulation");
cf=0;
for(i from 1 to n)do;
 a=int(random(1,6.99));
 b=int(random(1,6.99));
 c=int(random(1,6.99));
 d=int(random(1,6.99));
 if ((a=6)o(b=6)o(c=6)o(d=6)) then do;
```

```
  cf=cf+1;
 end;
end;
p=cf/n;
print(cf);
v=array(2); f=array(2);
v(1)=cf; v(2)=n-cf;
f(1)=p; f(2)=1-p;
brushcolor(255,128,0);
histogram(v);
histogram(f);
```

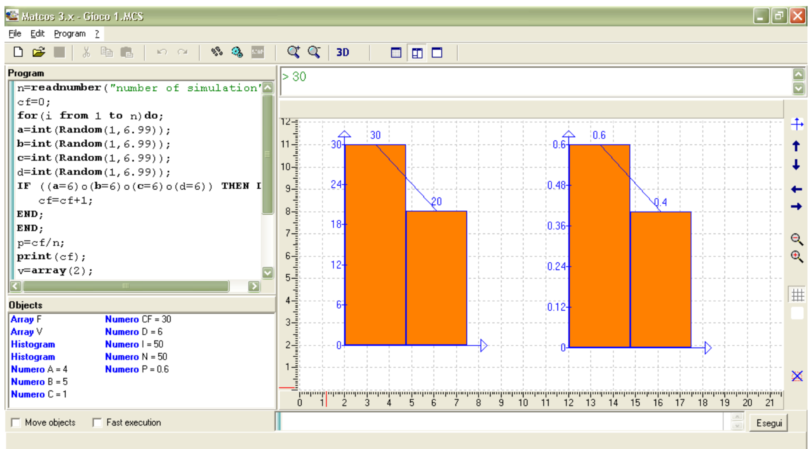


Figure 21: Output Game 1 with n = 50.

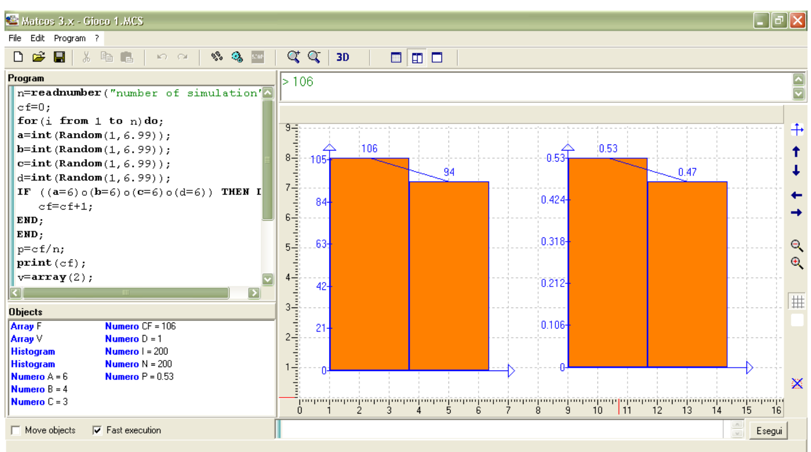


Figure 22: Output Game 1 with n = 200.

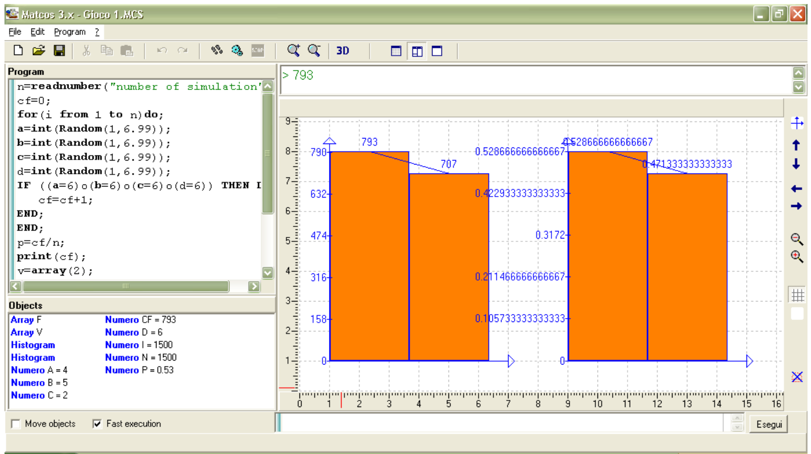


Figure 23: Output Game 1 with n = 1500.

The students, after virtually simulating the game for a number of launches increasingly high, verify that the probability of winning of the player is greater than the dealer.

Discussion of the results obtained and compared with the theoretical model

In this phase, you pass to the examination of the results obtained in output and the comparison with theoretical ones; i.e., students must proceed to the resolution of the game algebraic proposed and compare the results with the numerical values of the relative frequencies obtained in the virtual simulation.

In terms of probability, given the event:

A= "*Get at least a 6 throwing 4 dice*"

The complementary event is: A^C = "*Getting no 6 throwing 4 dice*"

The probability of the A^C is given by:

$$p(A^C) = \left(\frac{5}{6}\right)^4 = \frac{625}{1296} \approx 0{,}4823$$

Thus, the probability to the contrary is given by:

$$p(A) = 1 - p(A^C) = 1 - \left(\frac{5}{6}\right)^4 \approx 0{,}5177$$

Students they should find, therefore, as the theoretical results agree with the numerical value of the relative frequencies obtained in output. Furthermore, observe that increasing the number of launches the value of the relative frequency approaches the actual value of the probability; Ultimately, then the probability of winning of the player is greater than that of the bench.

The game continues…

The teacher, at this point, offers students the same game with some variations.

Game 2 – Launch of 2 dice

The gentleman at the counter of a gaming table throws two dice 24 times. The victory of the game if you get out at least one double six throwing two dice.

Drawing on its experience, the students begin to make predictions about the odds of the two players; conditioned, too, from the results obtained previously almost all agree that the more likely to win the player.

The teacher, for its part, continues to maintain a neutral position so as not to affect the forecasts expressed by the various groups that, this time, proceed immediately to the scheduling algorithm of the new game, so that you can implement to computer.

Algorithm of Game 2

The steps of the algorithm related to the simulation of Game 2 are:

1. Assignment: n (simulation number);

   v (array to record the results obtained).

2. Cycle (simulation of launch of 2 dice 24 times)

   - initialization: $cf = 0$ (counter for the number of favorable cases);
   - Cycle (simulation of launch of 2 dice)
     - Creating two variables a, b;

– Calculating the sum s of a e b.

3. Control action: if ($s = 12$) then do

- Increase counter cf;
- Control action: if ($cf \neq 0$) then do
  - filling of the array v.

4. Calculating the relative frequencies;
5. Print action: $f, n - f$;
6. Graphic rappresentation: histogram of the absolute and relative frequencies (Figure 24, Figure 25).

Even in this case, the algorithm is easily implementable in MatCos.

Code MCS2 - Game 2

```
n=readnumber("number of simulation");
v=array(n);
for(j from 1 to n)do;
 cf=0;
 for(i from 1 to 24)do;
  a=int(random(1,6.99));
  b=int(random(1,6.99));
  s=a+b;
  if (s=12) then do;
   cf=cf+1;
  end;
 end;
 if(cf<>0)then do;
  v(j)=1;
 end;
 f=0;
end;
for(k from 1 to n)do;
 f=f+v(k);
end;
p=f/n;
w=array(2);
w(1)=f; w(2)=n-f;
t=array(2);
```

```
t(1)=p; t(2)=1-p;
brushcolor(255,128,0);
histogram(w); histogram(t);
```

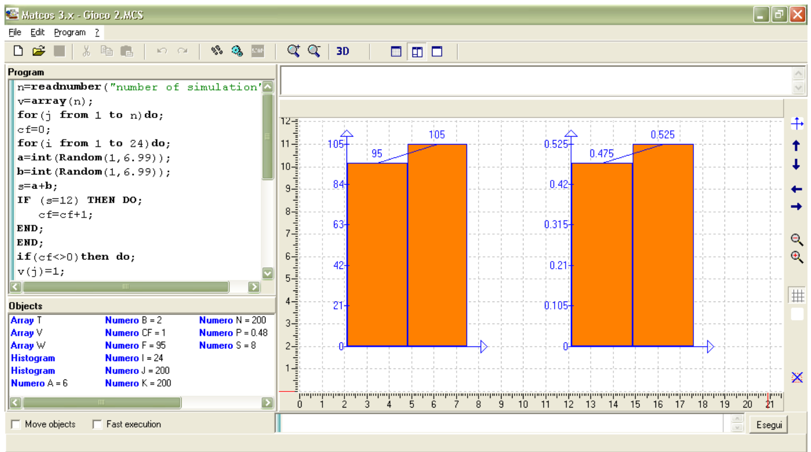


Figure 24: Output Game 2 with n = 200.

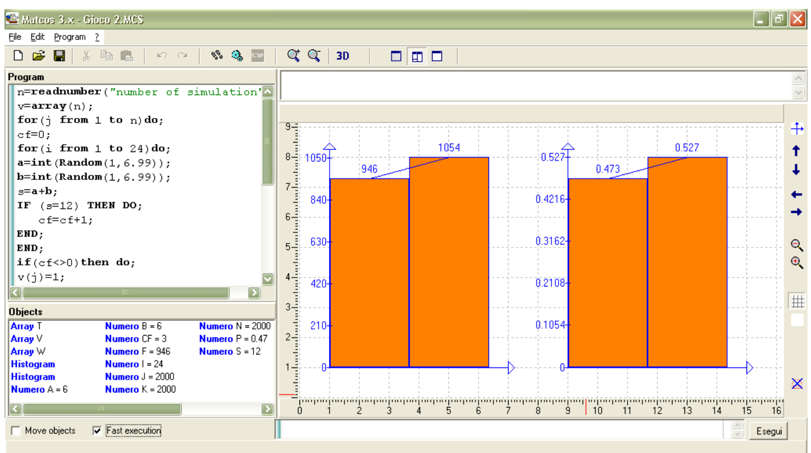


Figure 25: Output Game 2 with n = 2000.

The output of the virtual simulation returns discordant results with forecasts, which confirmed the victory of the player with a higher probability; this provides the ability to analyze the significance of the probability assigned to the event in question:

E = "*Get at least a dual 6 throwing 24 times a pair of dice*"

The complementary event is: E^C = "*Getting no double 6 throwing 24 times a pair of dice*"

The probability of E^C is:

$$p(E^C) = \left(\frac{35}{36}\right)^{24} \approx 0{,}5086$$

Thus, the probability contrary is: $p(E) = 1 - p(E^C) = 1 - \left(\frac{35}{36}\right)^{24} \approx 0{,}4914.$

A comparison with the theoretical model students will understand that the probability of a random phenomenon - for sure - is not to establish a priori whether the phenomenon will happen or not in the course of a single experiment.

However, when the experiment is repeated several times in terms of uniformity, the causes absolutely unpredictable that determine the different outcomes seem to bend as a rational order. In fact, the probability of a random phenomenon, calculated by the relative frequency, is getting closer to that calculated by the classical definition as the number of tests performed. The teacher will remark, further the fact that the empirical probability can be applied only to repeatable experiments under the same conditions for a relatively high number of times, sufficient to stabilize the relative frequency.

This doesn't always happen, such as in the result of a football match or in weather forecasts for the next day. In these cases, the evaluation of the probability of a certain event should be done with a frame of reference different from that of the idea of the empirical frequency; this can be the subject of study for the future.

4.6.4 Activity 4: Drawing from a pack of poker cards

Real simulation of random phenomena

The teacher divides the class into groups of 3-4 students as usual, grouping students so they are at similar levels in Math skills, and distributes a pack of poker cards to them.

Secondly, each group is asked to list the characteristics of a pack of such cards[61]; once the main features have been identified, the teacher delivers the first problematic situation.

<u>Game 3 - Extraction with replacement of two cards from a pack of poker cards</u>

Two extractions with replacement are performed from a pack of poker cards. The player wins if she or he extracts at least one hearts. What is the probability for the player to win?

As a result of the stimulus provided by the teacher, students begin to formulate some hypotheses. Common judgment goes as follows:

"In a pack of poker cards, there are 13 hearts on a total of 52 cards. If we extract a single card the probability of obtaining a heart is 13/52 = 1/4 = 0.25!"

It follows a brief discussion:

S1: *In a pack of poker cards, there are 13 hearts on a total of 52 cards. If we extract a single card, the probability of obtaining a heart is 13/52!*

S2: *We can think of two Extractions with replacement as two draws from two different packs of cards, that is, we take two packs of poker cards P1 and P2, and we draw a card from P1 and P2!*

S1: *Clever!*

S3: *Then I have two equal chances…*

S2: *Yes, you mean that the chances to extract one hearts from each of the two packs (which are identical) are the same!*

S3: *But we want to establish the probability of the event 'at least one hearts', how shall we do that?*

[61] Not all students have had experience with poker cards.

S1: *In my opinion, the probability is 1/4 ... the hearts are always 13 of 52 in both extractions, are they not?*

S3: *Yes, maybe you're right ...*

S2: *I'm not convinced of what you are saying!*

After a brief discussion, the students, prompted by the teacher, begin to simulate the proposed situation.

At the same time, the teacher guides them through a careful analysis of the text, pointing out the meaning of words, and invites them to register the results on a chart, in order to remember them later. This is the same procedure they followed earlier when they were introduced to probability, with the simulation of tossing a coin, and the roll of a dice. After 30 repeated tests, the groups compare the results obtained.

At this point a discrepancy of results will become evident; and here is where the teacher initiates a guided discussion, after which the need to repeat a large number of tests emerges to obtain significant results. An answer to this need is provided by the use of computers, used to simulate random events. The software used is MatCos, already used in class.

Computer simulation of random phenomena

The next step involves the computer simulation of the proposed situation with MatCos software. This phase has a strong educational value because students have a real opportunity to use and show what they learnt in the first phase. Also at this stage the writing of the algorithm gives rise to a constructive debate on the analysis of the event. The problem is here ‘dismembered’ and each part is a crucial step of the solving algorithm.

S4: *Let’s think of when the event occurs of which we want to calculate the probability!*

S2: *Well, you're absolutely right! As pointed out by the Teacher 'at least one' does not mean exactly one, but 'greater than or equal to one.' Then, the player wins if he or she gets one hearts from P1 and one non-hearts from P2, or if he or she draws one non-hearts from P1 and one hearts from P2, or if they draw one hearts from P1 and one hearts from P2!*

S3: *That's right, so we need two whole variables ranging from 1 to 52 including the extremes, to memorise the results obtained from the extraction of two cards from two packs!*

S4: *But how can we say if it is hearts or not?*

S3: *We can think of a correspondence between natural numbers from 1 to 13 with hearts and those from 14 to 52 with a different suit from hearts!*

S2: *Perfect! Now everything is easier, by entering the two variables into a cycle and the control with 'if' with the appropriate increase in the counter of favourable cases, we have done it!*

At the end of the discussion, the simulation concerning the proposed issue will follow, and the subsequent coding in the MatCos programming language.

Algorithm of Game 3

Constructing the algorithm is an important and delicate phase because the students have to design the 'finite sequence of steps' that enables the computer to get to the solution or to highlight the need for further study.

The steps of the algorithm of the two simulated extractions with replacement from a pack of poker cards are the following:

1. Assigning the number of simulations to be performed;
2. Initializing a counter to record the number of successes;

3. Cycle for the simulations:
    a. creating of two variables (to which the result of the extraction for each card is assigned);
    b. Control action to verify the drawing of at least one hearts and appropriate increase of the counter.
4. Calculation of relative frequency;
5. Printing of the relative frequency;
6. Representation of results obtained on a histogram.

The implementation of the algorithm in MatCos is as follows.

Code MCS3 – Game 3

```
n=readnumber; cf=0;
for(i from 1 to n)do;
 a=int(random(1,52.99));
 b=int(random(1,52.99));
 if(((a>=1) and (a<=13))or((b>=1) and (b<=13)))then do;
  cf=cf+1;
 end;
end;
p=cf/n;
print("in ", n ," extraction with replacement, we have obtained at least one hearts ", cf , " cases");
brushcolor(255,128,0);
histogram(cf,n-cf);histogram(p,1-p);
```

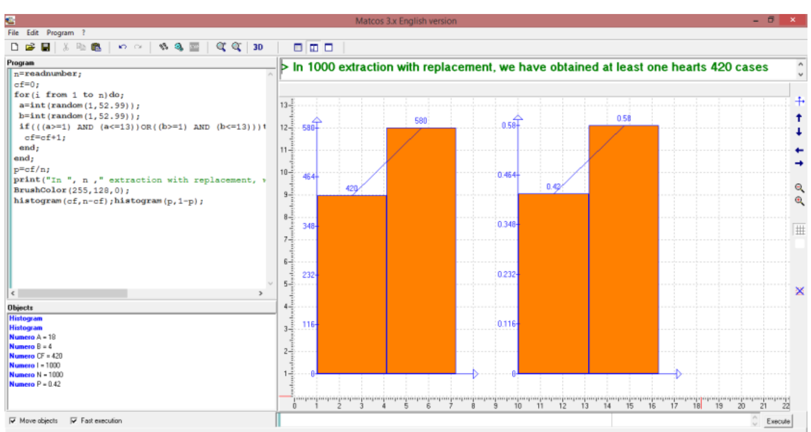


Figure 26: Output Game 3 with n = 1000.

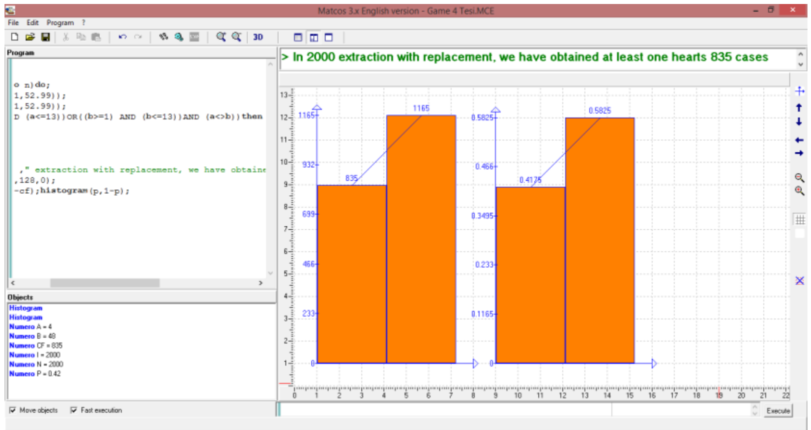


Figure 27: Output Game 3 with n = 2000.

The output of the virtual simulation yields results, which differ from the predictions, which confirmed a victory with a percentage of 25%; this offers the chance to analyse the meaning of the probability assigned to the event.

Discussion of the results and comparison with the theoretical model

Finally, we go on to examine the results obtained in the output and compare with them with the theory; i.e., students must use algebra to solve the proposed game, and compare the results with the numerical values of the relative frequencies obtained in the virtual simulation.

At this point, the teacher intervenes, going over some considerations made by the students; the event of which we want to calculate the probability is as follows:

E = "*Extraction of at least one hearts in two extractions with replacement*"

The event is realized if in one of the two extractions you get one hearts or if you get one hearts in both.

In order to summarize the situation, it seems appropriate to make use of the tree diagram (Figure 28):

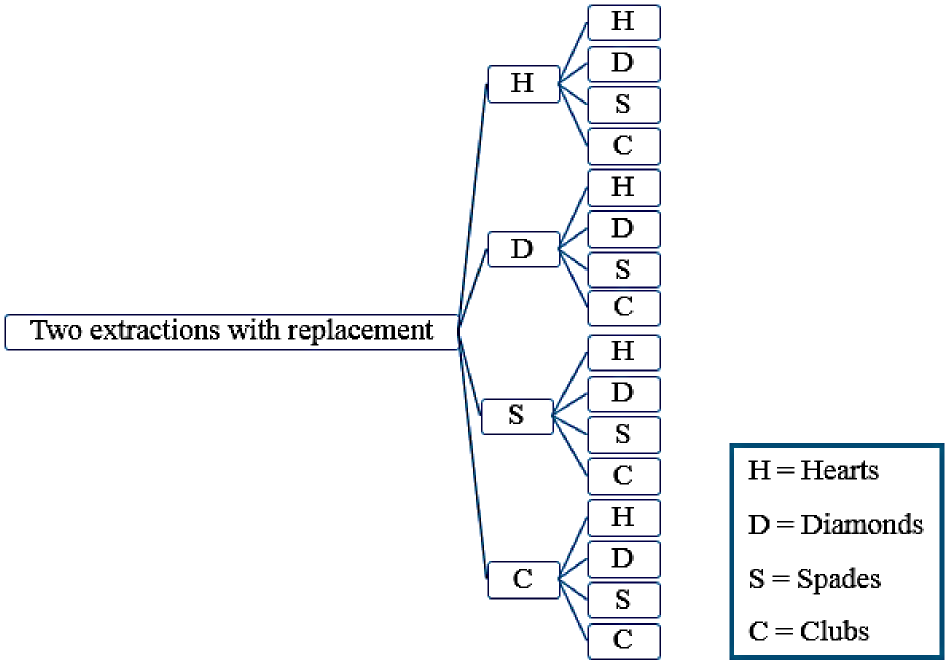


Figure 28: Tree diagram relating to the extraction with replacement of two poker cards.

The possible cases are in total 16, the favourable cases 7, and then the probability of E is:

$$p(E) = \frac{7}{16} = 0{,}43$$

As all branches are equally probable.

The students should find that the theoretical results agree with the numerical value of the relative frequencies obtained in output.

New proposal

At this point, the teacher proposes to the students the same problematic situation with some variations:

<u>Game 4 – Extraction without replacement of two cards from a pack of poker cards</u>

Two extractions without replacement from a pack of poker cards are done.

The player wins if at least one hearts is extracted. What is the probability for the player to win?

A small variation of the problematic situation becomes an opportunity for a new exploration: students, confident from previous experience, make some predictions about the probability of the player to win; influenced, too, by the results obtained previously almost all settle on a probability of 43%, because as some of them say:

"*The fact that we do not replace the first card extracted does not change the final result*".

The teacher for her part continues to maintain a neutral position so as not to affect the predictions expressed by the various groups, that this time proceed immediately to designing the algorithm of the new situation, so it can be implemented on the computer.

During the drafting of the algorithm, substantial differences with the previous algorithm begin to emerge:

S2: *This time we cannot expect to extract two cards from two different packs of cards. The first time we draw a card and the situation is very similar to the previous case, but in the second extraction, we find ourselves a pack of cards with one less card, 51 cards. Also, we do not know if there are 12 or 13 hearts, it depends on what we extracted earlier!*

S1: *Then we have to modify the algorithm …*

S3: *Sure, but I think it is enough to establish the condition that the two variables are different!*

S2: *Yes, perfect!*

After the discussion, the students agree that it is enough just to add in the control action the instruction '*a different from b*' to the previous algorithm, where *a* and *b* are the variables associated with the two cards extracted. Below is the program output.

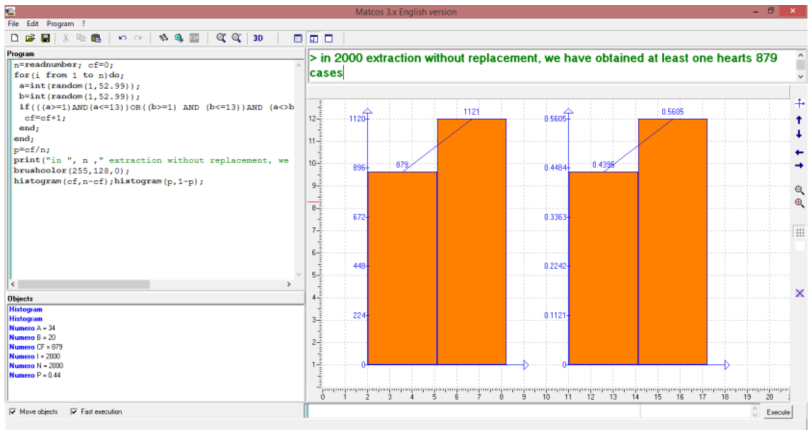


Figure 29: Output Game 4 with n = 2000.

At this point, the groups are able to operate a comparative analysis of the results obtained experimentally with the aid of the computer with the theoretical results, obtained also with a tree diagram, indicated with:

F = "*Extraction of at least one hearts in two extractions without replacement*"

and H, D, S, C extraction of one hearts, diamonds, spades and clubs, respectively, we have:

$$p(E) = p(HH) + p(HD) + p(HS) + p(HC) + p(DH) + p(SH) + p(CH) =$$

$$= \frac{1}{4}\cdot\frac{12}{51} + \frac{1}{4}\cdot\frac{13}{51} + \frac{1}{4}\cdot\frac{13}{51} + \frac{1}{4}\cdot\frac{13}{51} + \frac{1}{4}\cdot\frac{13}{51} + \frac{1}{4}\cdot\frac{13}{51} + \frac{1}{4}\cdot\frac{13}{51} = \frac{45}{106} \approx 0{,}4245 < 0{,}43$$

At this point, the teacher highlights what has already emerged during the discussion of the various groups: in the first problematic situation proposed the second extraction is not affected by the previous extraction as it occurs in the same situation. The second problematic situation is different, though. As a result of the first extraction, the pack of cards has now one less card. Also, the extraction of the second card is influenced by the earlier extraction. The fact that at first the students do not see a difference between the two situations highlights the usual resistance to accept that the probability of the occurrence of an event depends not only on the event itself, but also on the environment in which it occurs. After analysing the two problematic situations, the teacher further stimulates the students, asking them to indicate the differences between the two proposed situations. Here a new discussion begins.

S2: *It seems clear that the substantial difference between the two situations is that in the first the card is replaced, while in the second it is not!*

ALL: *Yes it does ...*

T: *No doubt, you are right, but what does this mean for the probability?*

S1: *There are different chances for the player to win, even though the event is the same!*

S4: *We have verified this on the computer*

S3: *The substantial difference lies in the fact that the second extraction occurs in different conditions!*

S4: *Yes, while in the first situation the first and the second extraction take place in the same conditions, in the second situation the second extraction is influenced by the earlier extraction!*

S3: *The first thing we can highlight is indeed the fact that there is one less card, and then if we extracted one hearts first, in the second extraction there are 12 chances out of 51 to extract another hearts; but if first we extracted a card of a different suit, there are 13 chances out of 51 to extract one hearts!*

Afterwards, the teacher introduces the definition of independent events:

"Let A and B be two events in a given experiment. It is said that A and B are independent if the probability of occurrence of A does not change as a result of the information that B occurred, and if the probability of occurrence of B does not change as a result of the information that A occurred". We also note that the definition of independence between the sets can be written in Mathematical terms as the following rule:

$$p(A \cup B) = p(A) \times p(B)$$

More examples and suggestions for exercises complete the experience, and it is through these that students would have the opportunity to consolidate what learnt in previous activities.

4.7 Final experimentation test

The adoption of teaching methods aimed at the development of critical thinking, in both the geometric and probabilistic field through the proper use of appropriate software is now well established. The activities described above go in that direction and foster in students the development of cognitive processes, such as the ability to adapt their knowledge to new situations and solve problems in new and unfamiliar contexts.

To evaluate the effectiveness of the experimental teaching [62] path a final test was administered to the students.

Given the structure of the experiment and related research issues, the final test was inspired by the conviction that learning does not take place through the passive repetition of unconscious actions but is gained gradually through an awareness process. As a result, the final test was structured in three parts of which the first (PART 0) was dedicated to the gathering of students' opinions regarding the educational activities they were involved in.

[62] Organized into a series of educational activities designed, implemented and developed according to the chosen theoretical framework.

That part was only administered to the sample of students who followed the trial supported by software.

The second and third part of the final test both consists of three questions.

In the first three questions, the students are asked to:

- Recognize plane algebraic curves in the graphical representations;
- Identify relationships between curves in the plane;
- Perform geometric constructions in the plane with ruler and compass in the environment "*pencil and paper*";
- Develop logical sequences of deductions.

The last three questions instead require the learners to:

- Identify appropriate strategies for solving problems in uncertain conditions;
- Classify the space of possible elementary events;
- Calculate the probability of events in finite equi-probable spaces.

Chapter 5: Results and discussion

In this chapter, we report and analyse the students' results in the initial and final tests. The drafting of statistical tables showing the students' initial level concludes the first phase of the study. Following an objective evaluation of the initial testing, laboratory activities were further calibrated so as to revisit the concepts that the students found difficult. This readjustment increased the control and flexibility of the activities for greater chances of success. The second phase of the trial ended with the completion of all the planned laboratory activities. The drafting of statistical tables relating to the final test ended the third and final stage of the study. Appropriate conclusions and hypotheses for future research close the chapter.

5.1 Data analysis

Based on the research hypotheses, for the elaboration of the answers given by the students in the two tests (initial and final) (Table 16). Specifically, the total sample of students was divided into two groups: the first consisted of 44 students who used the two software, and the second by 41 students who did not used software in the experiment. Within the two groups a distinction based on gender was not made as from an analysis of the data there were no substantial differences in responses to the test.

N° of students (without software)		**N° of students (with software)**	
III	**IV**	**III**	**IV**
24	17	21	23

Table 16: Distribution of students according to the class attended.

We report here the data concerning the initial test (Part A, Appendix A). To the question: *'Do you like Maths?'* 7 students responded ‘*No*’, 4 of whom belong to the sample of non-software users; Table 17 shows the reasons.

Question: ‘*'Do you like Mathematics?*’	
Reasons	**N° of students**
Because it is boring	3
Because I do not understand it	2
I am not fond of it	1
Not for me	1

Table 17: Distribution of students’ answers to question 3 of Part A.

The remaining students answered ‘*yes*’ for the following reasons (Table 18).

Question: ‘*'Do you like Mathematics?*’	
Reasons	**N° of students**
Because it is essential in everyday life	37
Because is it interesting	13
Because it trains the mind and helps to think	12
Because I understand it	10
Because I get good grades	6

Table 18: Distribution of students’ answers to question 2 of Part A.

To the question ‘*Have you ever used technological tools in Maths lessons?*’ all the students answered ‘*yes’*. Table 19 shows the different types of tools used.

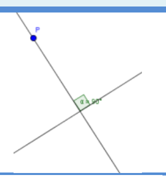
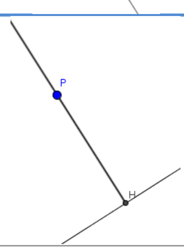
Question: *'Have you ever used technological tools in Maths lessons?'*	
Type	**N° of students**
Calculators and IWB	76
Spreadsheets	6
Dynamic geometry software	12
Computer Algebra System	0
Others (programming languages)	6

Table 19: Distribution of students' responses to question 5 of Part A.

Below are the answers provided by students to each question in the initial and final tests. The data is summarized in the table and processed using histograms, where the relative frequencies of each response are highlighted.

5.1.1 Initial test

Question One

Model students' answer		**N° of students (without software)**		**N° of students (with software)**	
		III	**IV**	**III**	**IV**
A		10	11	11	9
B		8	3	3	4

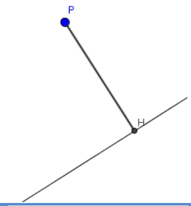
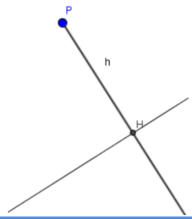
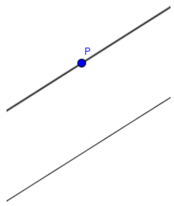
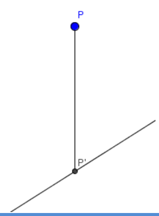
C		5	1	3	4
D		1	0	4	5
E		1	0	0	0
F		0	1	0	1

Table 20: Answers to Question 1 of the initial test.

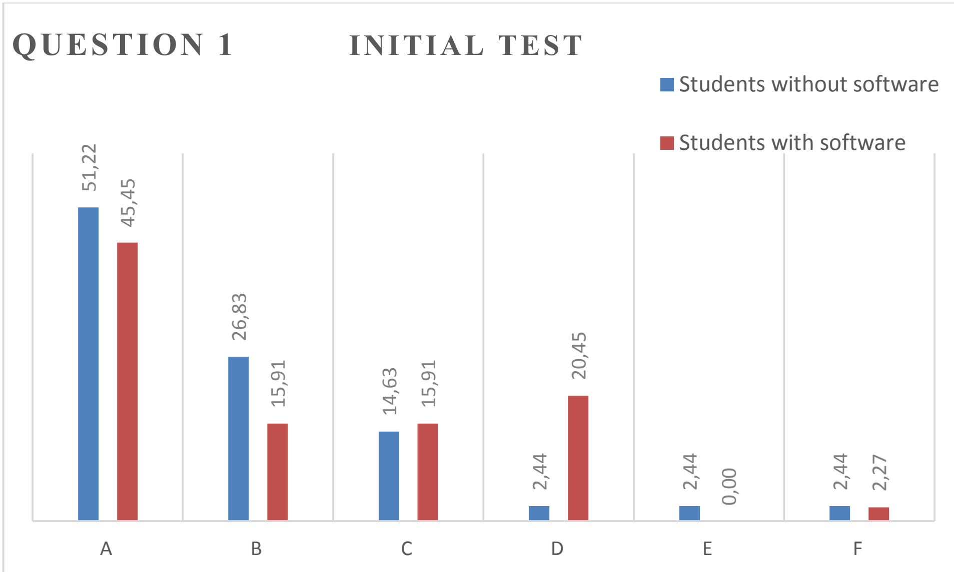


Figure 30: Responses to Question1 of the initial test.

It is apparent from Figure 30 that about half of the sample carries out the construction request correctly; however, the other half of the sample, instead of drawing a line perpendicular to the given line, draw:

- a segment the point and the foot of the perpendicular as extremes, and perpendicular to this line;
- a ray which has as its origin either the point or foot of the perpendicular and perpendicular to this line.

So, this second sample of students confuses the three geometric objects: line, ray and segment.

Question Two

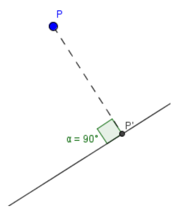
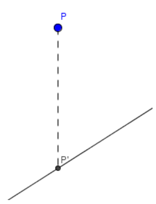
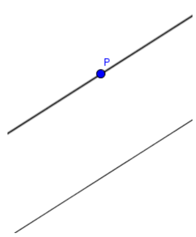
Students' model answer		N° students (without software)		N° of students (with software)	
		III	IV	III	IV
A		18	6	18	17
B		4	11	3	5
C		1	0	0	0
D	Does not answer	1	0	0	1

Table 21: Answers to Question 2 of the initial test.

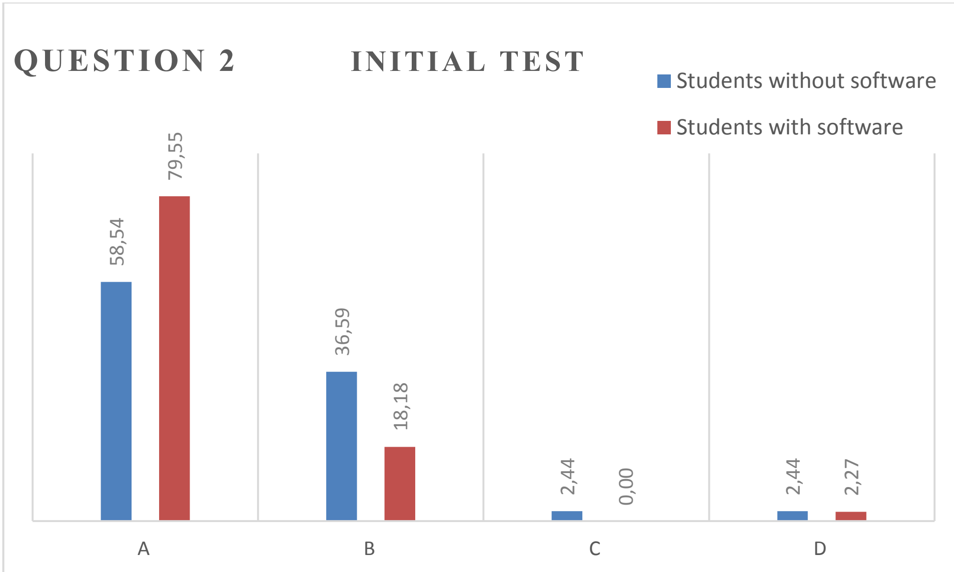


Figure 31: Responses to Question 2 of the initial test.

Figure 31 shows that about half of the students carry out the task properly. The request here is: '*Project a point on a given line*'. It is surprising that a high percentage of the sample ignore the meaning of projecting a point on a line. This question is closely related to the previous question, but analysis of the data shows that some students do not recognize the condition of verticality within the projection: they draw a vertical segment on the sheet of paper.

Question Three

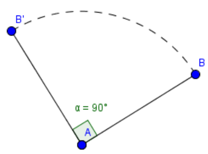
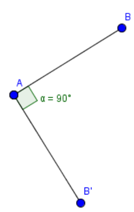
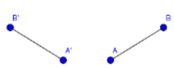
Students' model answer		N° of students (without software)		N° of students (with software)	
		III	**IV**	**III**	**IV**
A		23	14	20	21
B		1	2	1	2
C		0	1	0	0

Table 22: Answers to Question 3 of the initial test.

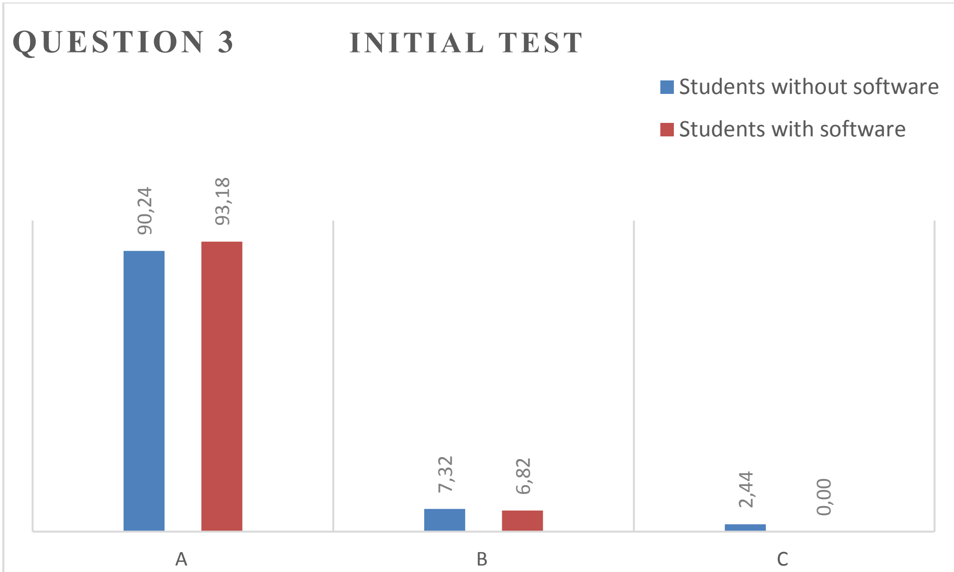
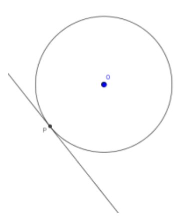


Figure 32: Responses to Question 3 of the initial test.

Figure 32 shows that almost all of the sample students responded correctly to the question; only a small part carries out the rotation in the clockwise rather than anticlockwise direction.

Question Four

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A		11	15	5	8

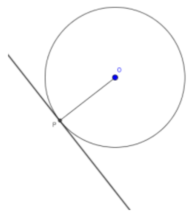
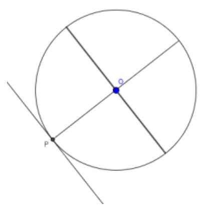
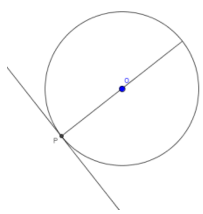
B		11	2	10	11
C		2	0	4	3
D		0	0	2	1

Table 23: Answers to Question 4 of the initial test.

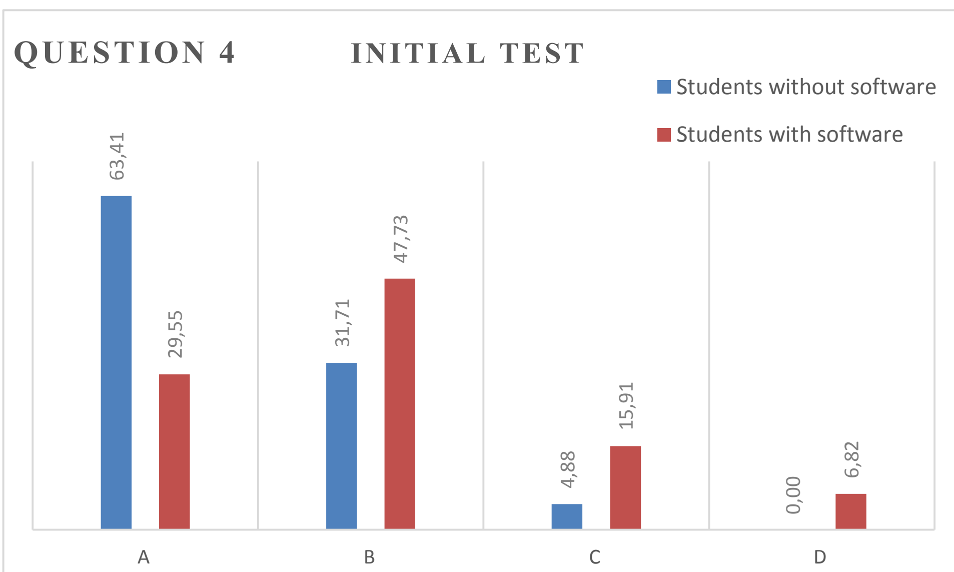


Figure 33: Answers to Question 4 of the initial test.

Figure 33 shows that almost all the students responded correctly to the question; however, we would like to point out that many of these students were helped in the construction by the relation satisfied by an arc tangent to a straight line, drawing the radius or diameter of the circle perpendicular to this line.

Question Five

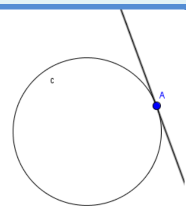
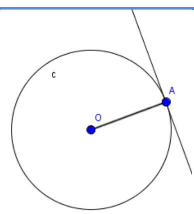
Students' model answer		**N° of students (without software)**		**N° of students (with software)**	
		III	**IV**	**III**	**IV**
A		23	16	21	23
B		1	1	0	0

Table 24: Answers to Question 5 of the initial test.

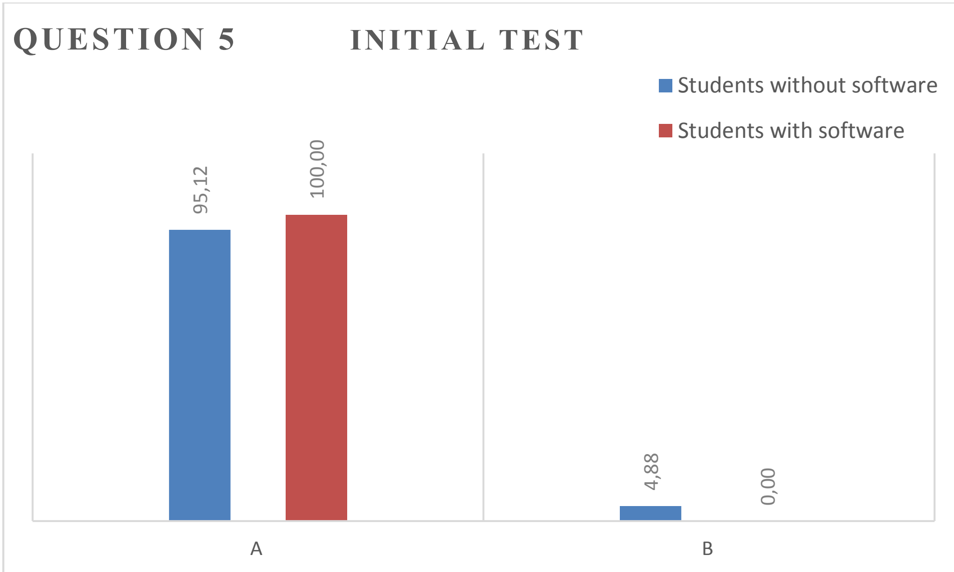


Figure 34: Answers to Question 5 of the initial test.

Figure 34 shows that all the students, with the exception of two, performed the required construction. This question is closely related to the previous one and students understood it, as shown by the results.

Question Six

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	$p = \frac{4}{6} = \frac{2}{3}$ because there are 4 numbers bigger than 2 in a dice with 6 sides	11	1	20	18
B	there are 4 numbers bigger than 2 in a dice with 6 sides $p = \frac{4 \cdot 100}{6} = 66{,}67\%$	9	16	0	5

C	4 because the numbers bigger than 2 are 3, 4, 5, 6 and there are 4	4	0	1	0

Table 25: Answers to Question 6 of the initial test.

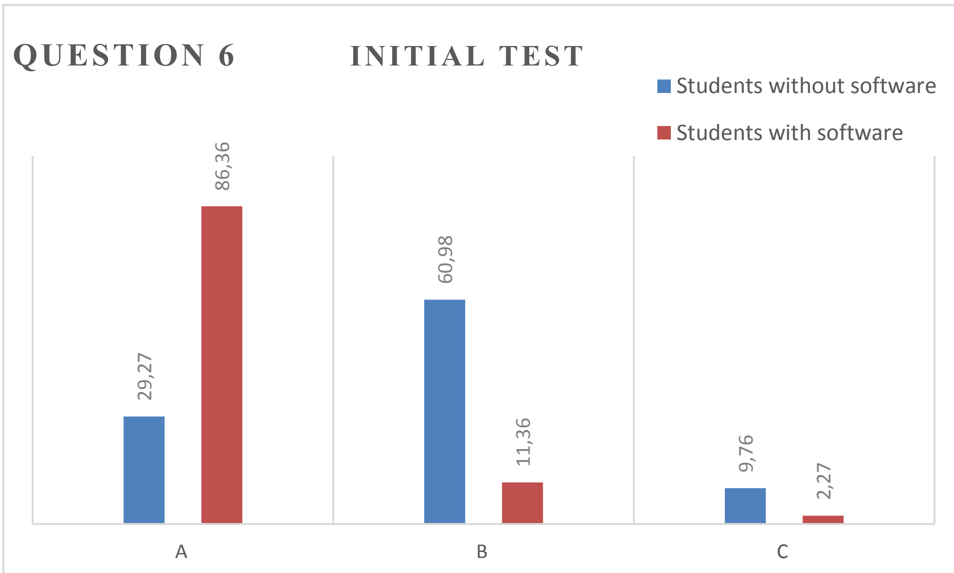


Figure 35: Answers to Question 6 of the initial test.

Figure 35 shows that almost the entire sample correctly describes the elementary random phenomenon and knows how to calculate its probability. One interesting fact that emerges from the analysis is that the sample of students in class IV of the group without software uses the percentage, so they associate the probability of an event to the relative frequency percentage.

Question Seven

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	$p = \frac{2}{4} = \frac{1}{2}$ because all the possible cases are TT, TC, CT, CC and those favourable are TC and CT	18	15	11	18
B	$p = \frac{1}{3}$ because all the possible cases are TT, TC and CC and only TC is favourable	3	2	5	2
C	$p = \frac{1}{4}$ because all the possible cases are TT, TC, CT, CC and the favourable one is TC	2	0	5	3
D	Does not answer	1	0	0	0

Table 26: Answers to Question 7 of the initial test.

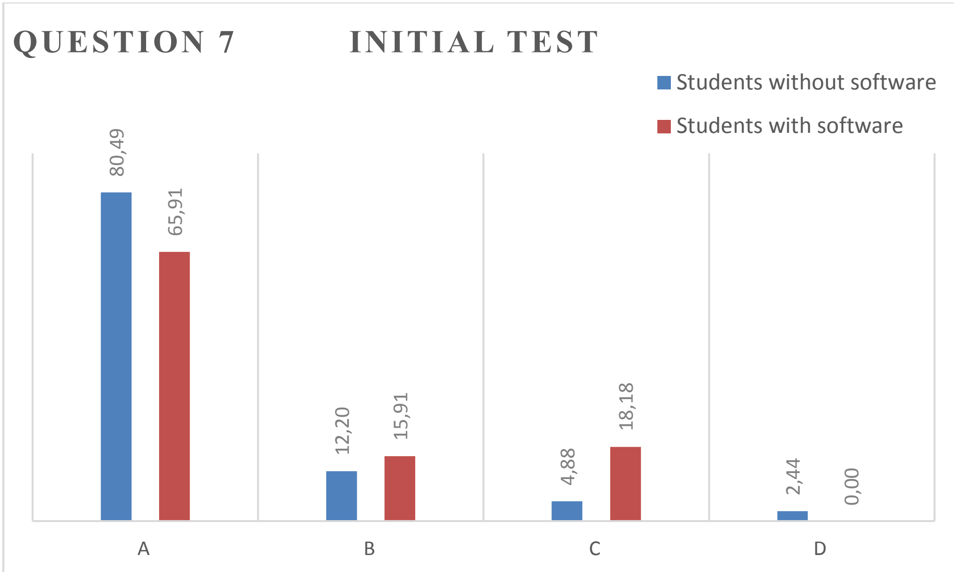


Figure 36: Answers to Question 7 of the initial test.

Question 7 has an uncertain situation a bit 'more complex’ than the one presented in the previous question; indeed, the question aims at identifying any students’ misconceptions in relation to non-elementary random phenomena. From Figure 36 it is apparent that a part of the sample identifies the possible cases and calculates the probability correctly while another part does not distinguish the cases tails-heads and heads-tails, which results in errors in the listing of both possible and favourable cases.

Question Eight

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	$p = \frac{3}{36} = \frac{1}{12}$ because throwing 2 dice we have 36 possible cases and the chances of getting two numbers whose sum is 10 are 3: 4+6, 5+5 e 6+4	3	0	5	2
B	$p = \frac{3}{12} = \frac{1}{4}$ because throwing 2 dice we have 12 possible cases and the chances of getting two numbers whose sum is 10 are 3 that is 4+6, 5+5 e 6+4	10	0	7	4
C	$p = \frac{2}{36} = \frac{1}{18}$ because throwing 2 dice we have 36 possible cases and the chances of getting two numbers whose sum is 10 are 2 that is 4+6 e 5+5	6	0	3	2
D	$p = \frac{4}{36} = \frac{1}{9}$ because throwing 2 dice we have 36 possible cases and the chances of getting two numbers whose sum is 10 are 4 that is 4+6 and 5+5 and the same changing the order of the addends	2	14	1	2
E	Does not answer	3	3	5	13

Table 27: Answers to Question 8 of the initial test.

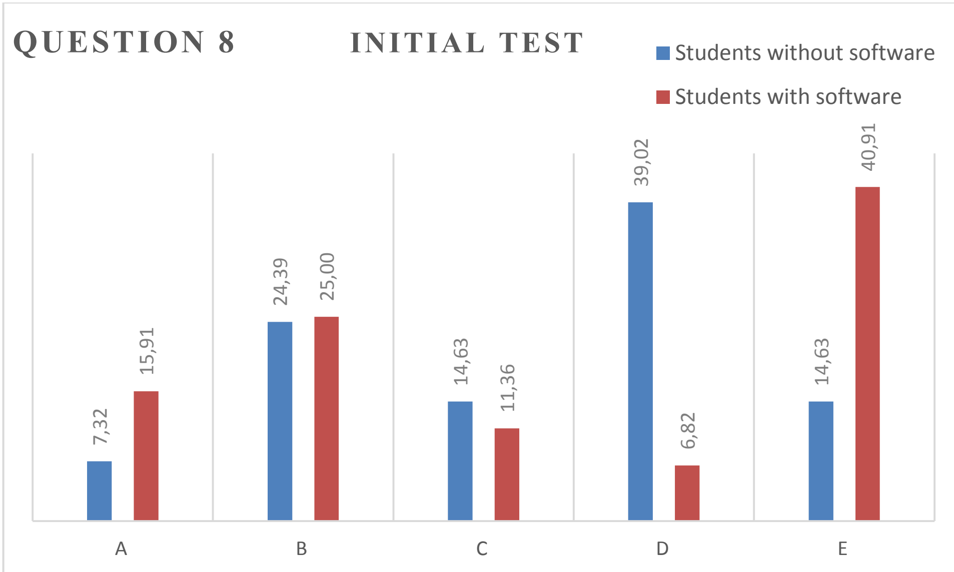


Figure 37: Answers to Question 8 of the initial test.

Question 8 is more difficult because instead of two coins there are two dice. Figure 37 shows that:

- A portion of the sample of students recognize only 12 (6 faces of a dice + 6 other faces) possible cases on the launch of two dice;
- A portion of the sample of students does not distinguish the cases (4,6) and (6,4) as favourable cases;
- A portion of the sample of students recognize the distinction between the cases (4,6) and (6,4), but considers the case (5,5) twice.

Question Nine

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	Giovanni and Francesco have the same chance of winning, that is, $\frac{3}{8}$	3	10	3	6
B	Francesco is more likely to win, because it is easier to get a heads than number two in the launch of 3 coins	11	1	6	8
C	Does not answer	10	6	12	9

Table 28: Answers to Question 9 of the initial test.

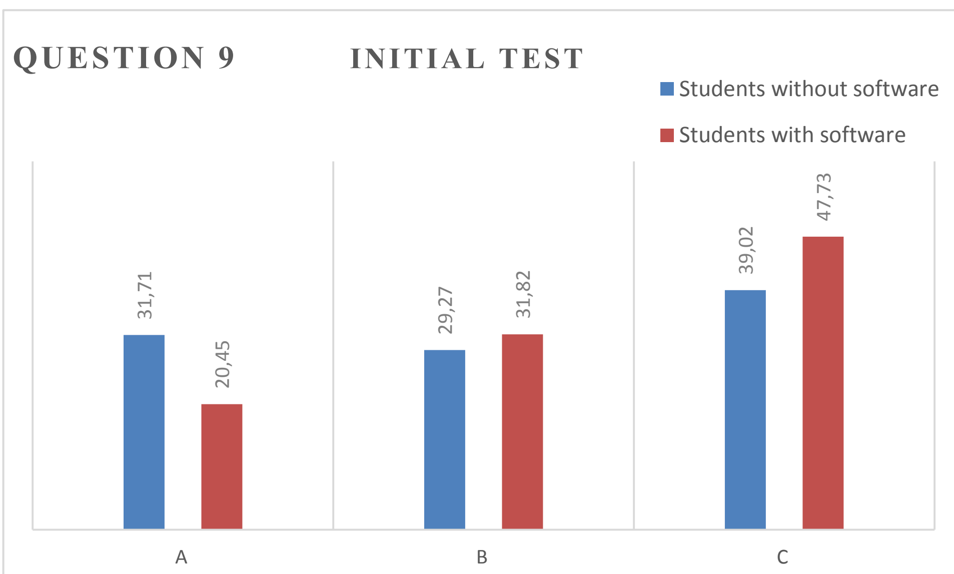


Figure 38: Answers to Question 9 of the initial test.

In question 9 the difficulty level further increases. The answers that students gave to this question show that they used the same mode of thinking already adopted in the preceding questions. This shows a strong misconception due to the influence of common sense: "*If I throw three coins it is easier to get a heads than with two coins*".

Question Ten

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	$p = \frac{2}{8} \cdot \frac{1}{7} = \frac{1}{28}$	2	2	3	2
B	$p = \frac{2}{8} \cdot \frac{2}{8} = \frac{1}{16}$	3	4	3	4
C	$p = \frac{2}{8}$	13	8	14	9
D	Does not answer	6	3	1	8

Table 29: Answers to Question 10 of the initial test.

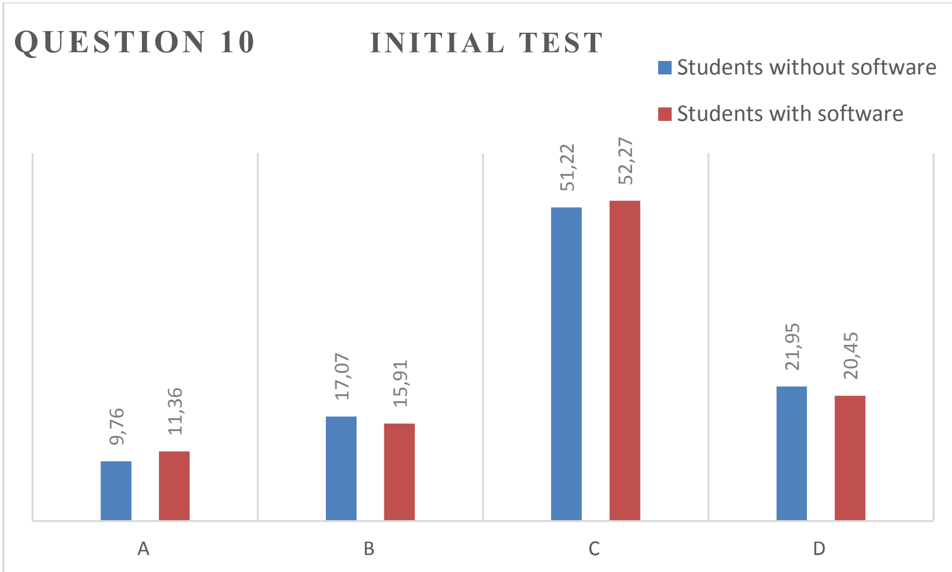


Figure 39: Answers to Question 10 of the initial test.

From Figure 39 it is clear that only a small number of students responded correctly. Analysing the responses of the sample that did not perform the task correctly it emerges that they considered the probability of getting a green ball from one extraction only, regardless of the actual task instructions. That is, the two groups of students making up the whole sample were not significantly different as far as their misconceptions were concerned. The type of misconceptions detected is at the basis of this review, as well as the implementation and development of laboratory activities. As for the details of the design activity, see chapter 4.

In the next paragraph, we report the results and analysis of the final test.

5.1.2 Final test

Question One

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	Tagents line in P point to a symmetrical curve in respect to the origin of the Cartesian axes	0	0	16	12
B	Tangent line in P point to a curve passing through the origin of the Cartesian axes	13	8	5	11
C	Two parables that meet in the origin of the Cartesian axes and a line tangent to one of them	9	0	0	0
D	Does not answer	2	9	0	0

Table 30: Answers to Question 1 of the final test.

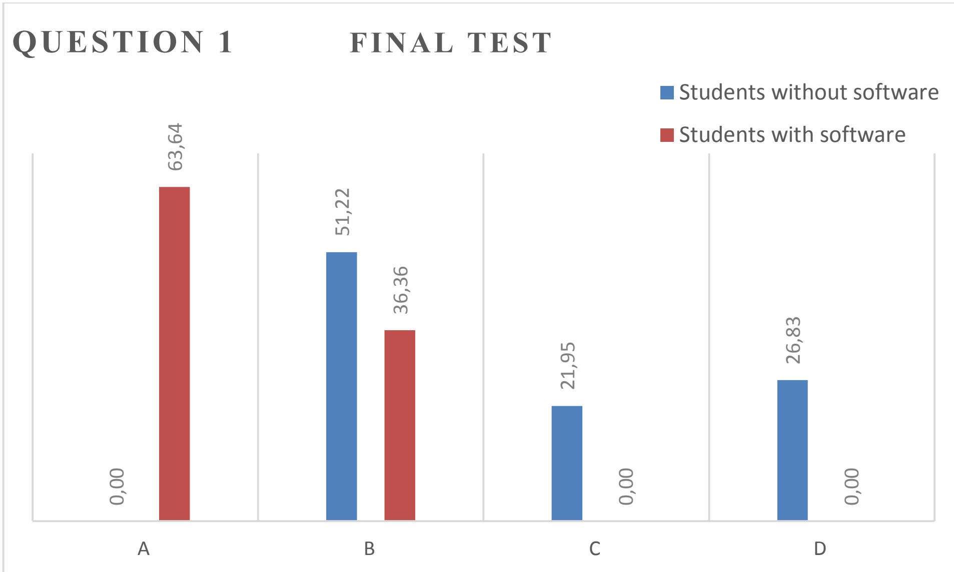


Figure 40: Answers to Question 1 of the final test.

Figure 40 shows that the group of students with software recognize relationships and properties among the depicted geometrical objects; instead, the second group, the students without software, confuses or even ignores them altogether. In fact, many students in the second group confuse the curve represented with two parables that meet in the origin of the Cartesian axes. An analysis of the students' responses already show substantial differences between the two groups.

Question Two

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	The tangent curve to each family member at least one point	16	16	19	21

B	Secant curve for each family member at a point	1	0	1	1
C	A sheaf of straight lines	5	0	0	0
D	A set of concentric circumferences	2	0	0	0
E	Does not answer	0	1	1	1

Table 31: Answers to Question 2 of the final test.

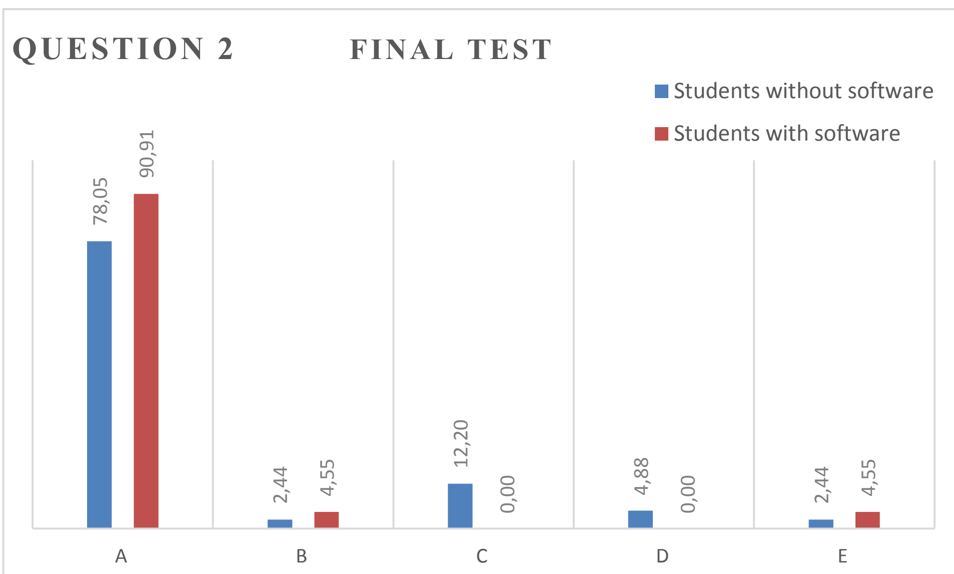
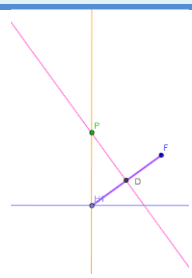
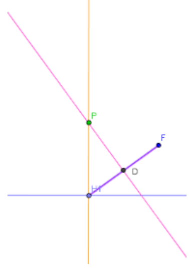
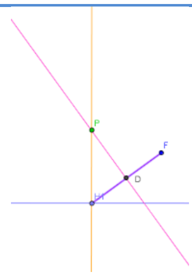
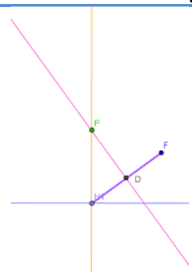


Figure 41: Answers to Question 2 of the final test.

From Figure 41 it shows that the two groups, in almost all, have responded correctly to Question theorist with item assigned.

Question Three

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	 P describes a parable because it is equidistant from F and the line d, by construction	1	0	14	16
B	 P describes a line	9	6	0	0
C	 P describes a triangle	2	0	0	0
D	 P describes a rectangle triangle	3	0	0	0

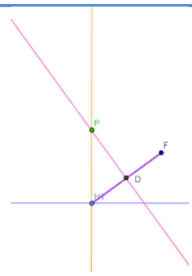
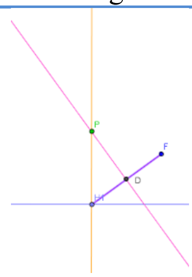
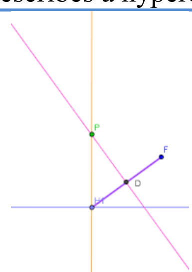
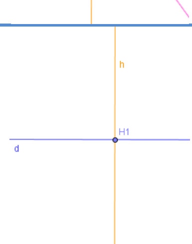
E	 P describes an isosceles triangle	1	0	0	0
F	 P describes a hyperbole	0	0	2	0
G		1	4	4	5
H		5	2	0	1
I	Does not answer	2	5	1	1

Table 32: Answers to Question 3 of the final test.

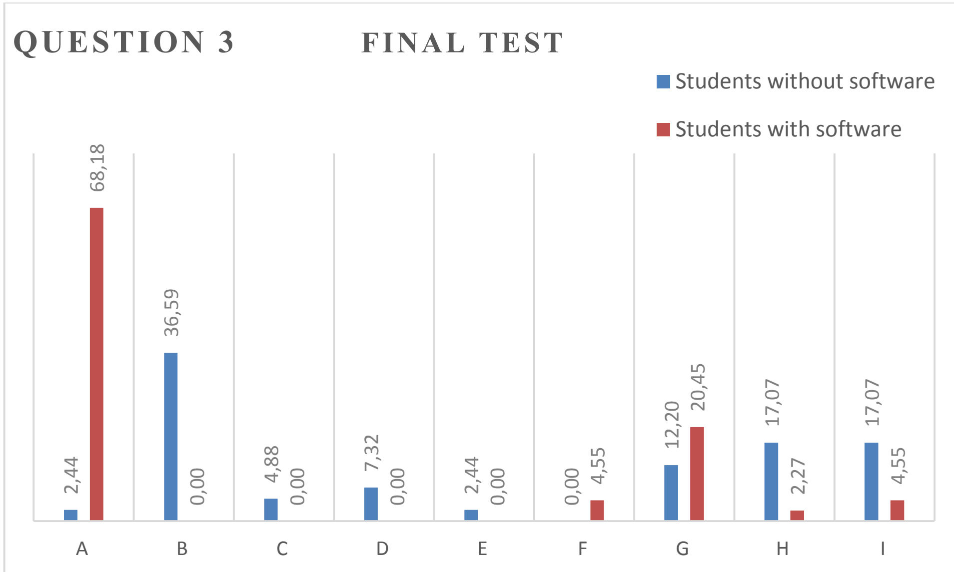


Figure 42: Answers to Question 3 of the final test.

Analysis of the results in Figure 42 show a clear difference between the two groups of students. Specifically, most of the students in the group that used the software are able to perform the geometric construction. For the other group of students it was difficult because they failed to complete the required geometric construction. In particular, the first group of students also solves the last part of the question that is determining the geometrical locus obtained from point P with varying of point H_1 on the directrix. So the last part of this question is intended to make students shift from the visualization level to that of the analysis: the students explore, observe and speculate.

Question Four

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	$p = \frac{7}{11} \cdot \frac{7}{11} \cdot \frac{7}{11}$	4	6	14	21
B	$p = \frac{7}{11}$	13	2	1	0
C	$p = \frac{7}{11} + \frac{7}{11} + \frac{7}{11}$	1	1	0	0
D	$p = 3 \cdot \frac{7}{11}$	4	0	0	0
E	Does not answer	2	8	6	2

Table 33: Answers to Question 4 of the final test.

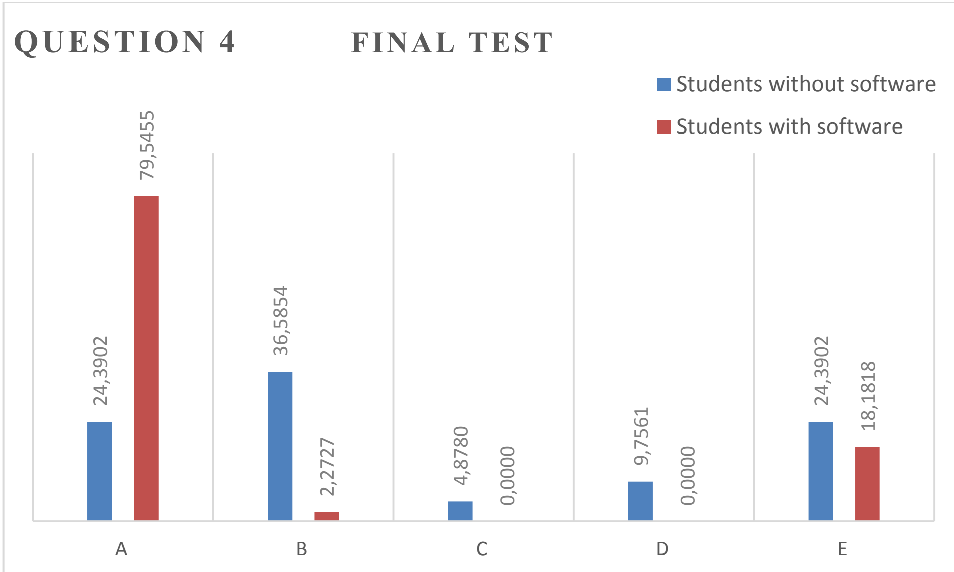


Figure 43: Answers to Question 4 of the final test.

Figure 43 shows that the majority of students with software had by now overcome their initial difficulties in the evaluation of the probability of occurrence of a composite random event. The other group of students, on the other hand, continued to experience difficulties, also in the understanding of the text, so much so that when asked to calculate the probability of getting 3 blue balls by three subsequent extractions with replacement, they responded by calculating the probability of getting a blue ball on a single extraction, mistaking the composite event for an elementary one. An analysis of this question shows how the simulation of random phenomena through the practice of programming helped students to focus their comments, to consciously control their own strategies and to reflect on the choices to be made for the resolution of the problem.

Question Five

Students' model answer		N° of students (without software)		N° of students (with software)	
		III	IV	III	IV
A	$p = \frac{7}{11} \cdot \frac{6}{10} \cdot \frac{5}{9}$	4	6	14	21
B	In the 1st extraction: $p = \frac{7}{11}$ In the II: $p = \frac{6}{10}$ In the III: $p = \frac{5}{9}$	5	2	0	0
C	$p = \frac{7}{11}$	2	0	1	0
D	$p = \frac{7}{11} + \frac{6}{10} + \frac{5}{9}$	1	1	0	0
E	$p = \frac{11-3}{11}$	5	0	0	0
F	Does not answer	7	8	6	2

Table 34: Answers to Question 5 of the final test.

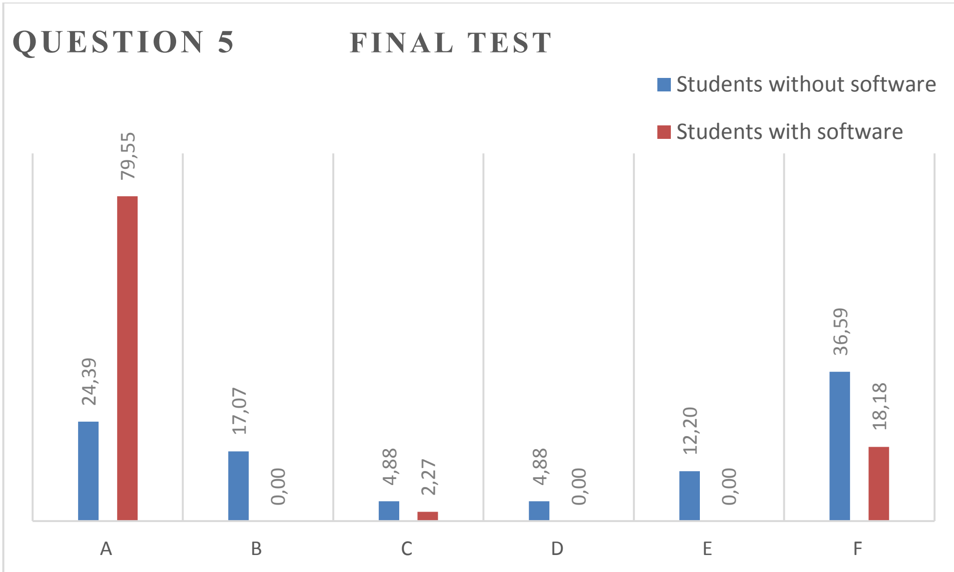


Figure 44: Answers to Question 5 of the final test.

The answers to this question also show that the student group with the software had overcome their initial difficulties in the evaluation of the probability of occurrence of a composite random event; moreover, many of them were using the tree diagram to simplify the proposed situation. On the other hand, the other group continued to show difficulties in understanding the text, so much so that the group calculated the probability of getting a blue ball from one single extraction, thus assimilating the composite event to an elementary event.

Question Six

Students' model answer		No of students (with software)		No of students (with software)	
		III	IV	III	IV
A	$p_1 > p_2$[63]	9	4	15	21
B	$p_1 < p_2$	11	4	0	0
C	Does not answer	4	9	6	2

Table 35: Answers to Question 6 of the final test.

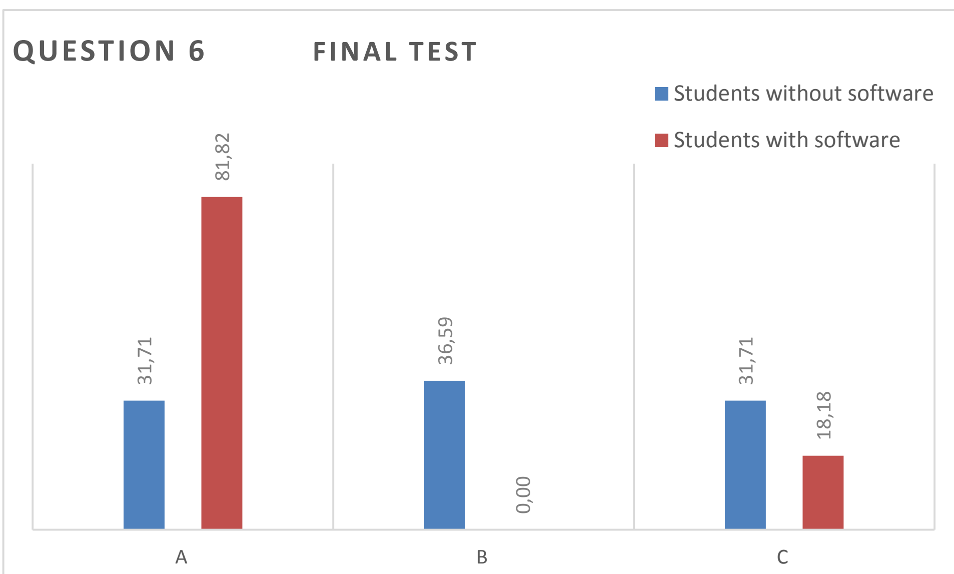


Figure 45: Answers to Question 6 of the final test.

[63] By $\boldsymbol{p_1}$ we mean the probability calculated in Question 4 and by $\boldsymbol{p_2}$ that calculated in Question 5.

The last question of the final test was designed to find out if the students were able to carry out a comparative study on the calculation of the probabilities encountered in the previous questions. Figure 45 shows once again that the majority of the students who used the software managed to carry out the required comparative examination, unlike the other group.

5.2 DISCUSSION

The purpose of the initial test was to establish the level of the students in relation to the topics chosen for the experimentation. Collecting initial information in order to ascertain any 'weaknesses and critical points' was important because it allowed the calibration of the planned workshop activities. At the same time the information represented a starting point for the creation of the final test, in accordance to the path developed for the sample of students.

From a detailed analysis of the tests results provided above, it can be said that the activities on plane algebraic curves the planning design - first in the traditional way and then on the computer – encouraged the students to think, to reason on some concepts which did not only involve Mathematics, and consequently to investigate the relationship between Mathematics, Design, and Computer Science.

In practice, the students after observing, intuiting, arguing and speculating had to design, formalize and display; through the use of simple instructions they simulated the shape of the curve investigating the relationships and the rapports between the elements involved, checking, therefore, the accuracy of the lines previously marked. This process has great educational value because it allows a more informed formalization where the learner can verify through experience the properties already studied in Geometry, exploiting their investigative potential and support to intuition. So the students had a real chance of working on the geometrical object in a constructive way, exploring properties, formulating conjectures and putting them to the test, also thanks to the instruments offered by the GeoGebra software. The introduction of the operational and representative dimension through a DSG such as GeoGebra in the workshops, led to a structuring of the actions and their effects into a descriptive framework in which the discussion and demonstrative value increases gradually. The use of the GeoGebra software facilitates representation and communication through the combination of visualization and construction in the design phases. In addition, the interactive experience with the model is essential to define a construction and make it productive from an educational point of view.

As regards the calculus of probability, the workshop activities carried out increased the degree of students' confidence in the effectiveness of statistical methods; at the same time the activities raised awareness of random events, thus preventing conceptual difficulties during the learning stages of Mathematics of the uncertain.

Virtual simulation, through the practice of programming, played a crucial role because it helped the student to develop good problem solving skills, in particular, it facilitated the understanding of the concept of probability of an event, by assigning to it the value of 'degree of reliability 'in the prediction of random phenomena.

The performed laboratory activities were specifically set up to provide an effective environment for probability learning which showed the close link between probability and statistics by means of a precise task (problem posing). We stress the role of intuition because, in most cases, students' probabilistic insights lead to wrong convictions and answers.

The experience gained by experimenting with the use of GeoGebra and the MatCos programming environment demonstrates which use of appropriate educational software in classroom practice can greatly help the teaching but especially the learning of Mathematics.

An analysis of the initial and final tests indicates a considerable learning gap between the students who used the two software in the educational activities on the one hand, and those that proceeded in the traditional way, without the use of software on the other. The first group showed that they had acquired skills for reasoning and abstraction that allowed them to imagine also what was not realized directly before their eyes, while the second group showed a static view and thinking closely tied to the visible and tangible, and were often unable to think critically.

So we can say that the final test results demonstrate that appropriate educational software in Mathematics teaching - learning represent a sort of "cognitive artefacts" , not just tools with which we work, but rather instruments with which one thinks.

Nevertheless, it is well known that the significance of a survey about knowledge and skills is affected by the climate in which it is being proposed. For this reason for both the final and initial tests, we created optimal conditions in order to encourage a conscious, motivated student participation so that the learners would give their best.

Finally, it must be stressed that students from both groups showed a greater degree of interest and curiosity towards the two tests compared to those normally offered during the school year.

5.2.1 The opinion of the students

A part of the final test (Part 0, Appendix B) was administered only to students who used the software in order to gather their impressions about the educational activities followed.

To the question, '*Are you satisfied with your participation in the experimental classes of Mathematics, with the aid of computers?*' 4 students answered '*No*' because they found it difficult to use the software, underlining their little inclination towards the use of technological tools. The remaining 81 students all answered '*Yes*', giving the following reasons:

Question: '*Are you satisfied with your participation in Mathematics lessons, with the aid of computers?*'	
Reasons	**N° of students**
Because they are more interesting and I learn more	23
Because the learning of Mathematics with computers is more fun	11
Because the topics are easier to understand	6

Table 36: Distribution of student answers to question 2 of part 0.

Below we report the positive aspects noted by the students in the educational activities carried out with the software:

- Interesting and simple lessons;
- The study of Mathematics is more fun;
- Practical applications of the subjects studied;
- The use of software makes studying easier and fun;
- Better understanding of the topics dealt with, with the aid of software;
- You learn while having fun and working together;
- The interaction with teachers.

The only negative aspect highlighted by the students was the small number of activities done, which they proposed to extend over the entire span of the school year.

5.3 CONCLUSIONS

In general, the proposed workshop activities challenge traditional views and contribute to project a different image of Mathematics as more interdisciplinary, and consequently more useful not only on a purely cognitive, but also informative level. Obviously, these activities can be expanded in detail, reconsidering the previous stages, further developing the interconnections of the various aspects. The computer, appropriately included in classroom practice, can address both contents and Maths topics in a more engaging way, while at the same time it enables the teacher to propose alternatives to traditional classroom teaching. Moreover, the active and informed use of technology stimulates imagination and creativity, encouraging the development of a scientific mind-set.

The computer provided the opportunity to explore a sort of analogy, which exists, between the way technology and the mind operate. The computer requires that each action and decision is made explicit. Indeed, it is as if the transition from intention to action materialized: the computer does not do anything that it is not instructed to do, and this helps the students to make the control component in cognitive functioning explicit.

In particular, the practice of programming allows the students to "create" a product with their skills, already formalized with the construction of the solving algorithm. It also allows them to break down complex problems into simpler ones and at the same time to fully understand constructed knowledge. The proposed activities provide the opportunity to compare a non-deterministic theoretical scheme with the system, real or artificial, of modelling. This has great importance from the didactic point of view: the class becomes a laboratory, the place where you practice an experimental form of Mathematics that may follow, or come before, the presentation of a more formalized model - in this case probabilistic - of reality.

The virtual simulation with the MatCos programming environment requires students to think actively, creatively and in complex ways, which include assessment, analysis, report preparation, synthesis, imagination, problem solving and decision-making.

The practice of programming strengthens the meaning of Mathematical concepts from the semantic point of view, adding to traditional knowledge the algorithmic and constructive aspects of Mathematics that are so often neglected in Italian schools.

Also, the use of a DGS such as GeoGebra, properly exploited in teaching practice for dealing with Geometry topics in general, favours the structuring of knowledge into meaningful networks, thus improving student skills. GeoGebra in the workshop activities is used as a simulation environment in order to allow the students to reproduce both the form and content of a curve.

Direct working experience of interacting with a model is essential to make a simulation productive from an educational point of view. That is why GeoGebra is first used at elementary level through the direct manipulation of the computational objects available with the interface, and later in a progressively more refined and dynamic way through the creation of one or more instruments. This gradual process brings to the fore the students' actual skills because it is based on the comparison and justification of the techniques involved. This leads to better conceptualisation. Last but not least is the synchronism between action and Mathematical representation; for example, in the case of Breguet's spiral students have a real opportunity to see the coils which gradually "come alive." This is a special occasion for reflecting on the inextricable links between Mathematics and Physics and the process of constructing a mathematical model from a real physical problem. Indeed, all the workshops were "spaces" where the students were able to build and expand their knowledge. In particular, it is evident that the meta-cognitive dimension of both software fostered a gradual increase of student autonomy in managing their own learning process, intellectual autonomy that allowed not only the application of those strategies, but also an understanding of the reasons and conditions of use regardless of the individual contexts, and so the ability to generalize and control them.

The heuristic dimension elicited by the two software in problem solving should also be mentioned.

In order to reach the solution of the problem the students had to follow necessarily the three basic routes:

- Trial and error;
- Reproductive;
- Productive.

In the last part of the experiment, the students favoured the productive path by thinking up unexpected or unusual solutions to the problem.

Once again, insight plays a role as the capacity to restructure the problem and discover new relationships between elements.

In conclusion, the results lead us to validate the research hypothesis:

3. Students who use the software in teaching achieve better learning results, than those who do not use them;
4. The use of educational software promotes the development of critical thinking and problem solving skills;

in addition, encourage the continuation and extension of the experiment to a larger sample of students belonging to different secondary schools.

5.4 FURTHER RESEARCH

The methodological choice of using a design-based research approach resulted in our collecting a very comprehensive set of data. It would be interesting to widen the student sample in future.

A larger sample could indeed lead to more accurate and complete results, while at the same time the investigation could be carried out in greater detail to further explore the research-questions.

With regard to the specific Mathematical contents, these open up vast fields of educational research. In particular, the calculus of probabilities - a topic often overlooked in classroom teaching practice - is fertile ground for study with the help of technology, as shown by research in this field. However, the total absence in the literature of a discussion with the practice of programming should be noted.

In a perspective broader than this thesis - which used computer programming to harmonize and synthesize the classical and frequency approach to probability - it would be desirable to continue making use of the pedagogical value of programming to study discrete and continuous random variables in order to cope with increasingly complex random situations.

References

Accomazzo, P., Beltramino, S., & Sargenti, A. (2013). *Esplorazioni matematiche con Geogebra*. A cura di O. Robutti. Ledizioni.

Ahlgren, A., & Garfield, J. (1991). Analysis of the probability curriculum. In *Chance encounters: probability in education* (pp. 107-134). Springer Netherlands.

Amodio, F., & Dapueto, C. (2014). La probabilità a scuola: il suo senso, la sua valenza educativa, gli strumenti per insegnarla. *L'insegnamento della matematica e delle scienze integrate*, 37, 471-487.

Aydin, E. (2005). The use of computers in mathematics education: a paradigm shift from "computer assisted instruction" towards "student programming". *The Turkish Online Journal of Educational Technology*, *4*(2), 27-34. http://www.tojet.net/volumes/v4i2.pdf

Artigue, M. (2002). Learning mathematics in a CAS environment: The genesis of a reflection about instrumentation and the dialectics between technical and conceptual work. *International Journal of Computers for Mathematical Learning*, *7*(3), 245-274.

Artigue, M., Drijvers, P., Lagrange, J.B., Mariotti, M.A., & Ruthven, K. (2009). Technologies Numériques dans l'enseignement des mathématiques, oùen est-on dans les recherches et dans leur intégration? In C. Ouvrier-Buffet, & M.J. Perrin-Glorian (Eds.), *Approches plurielles en didactique des mathématiques: Apprendre à faire des mathématiques du primaire au supérieur: quoi de neuf?* (pp.85-207). Paris: Université Paris Diderot Paris 7.

Arzarello, F., Ferrara, F., & Robutti, O. (2011). Mathematical modelling with technology: the role of dynamic representations. *Teaching Mathematics and its Applications*, *31*(1), 20-30.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practices in Cabri environments. *Zentralblatt fur Didaktik der Mathematic*, *34*(3), 66-72.

Baccaglini-Frank, A., & Mariotti, M.A. (2010) Generating conjectures in dynamic geometry: The maintaining dragging model. *International Journal of Computers for Mathematical Learning*, *15*(3), 225–53.

Bantchev, B. B. (2010). A brief tour to dynamic geometry software. *CaMSP February 2015 Newsletter*.

Barab, S.A., Cherkes-Julkowski, M., Swenson, R., Garret, S., Shaw, R.E., & Young, M. (1999). Principle of self-organization: Ecologizing the learner-facilitator system. *The journal of the Learning Science*, 8, 349-390.

Batanero, C., & Díaz, C. (2009). University students' knowledge and biases in conditional probability reasoning. *International Electronic Journal of Mathematics Education*, *4*(3), 131-162.

Batanero, C., Henry, M., & Parzysz, B. (2005). The nature of chance and probability. In G.A. Jones (Ed.), *Exploring probability in school: Challenges for teaching and learning* (pp. 15-37). Springer US.

Battista, M.T. (2008). Development of the Shape Makers Geometry Microworld. In M.K. Heid, & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 2. Research syntheses* (pp. 131-156). Charlotte, NC: Information Age.

Bell, M. (1993). Modeling and the uses of mathematics in the K-6 curriculum. Paper presented at the K-12 Modeling and Simulation Conference, Boston, MA.

Bell, S. (2013). Programming ability is the new digital divide: Berners-Lee. In Computerworld.

Beltrami, E. (1999). *What is random? Chance and order in mathematics and life*. New York: Copernicus/Springer-Verlang.

Bennet, D.J. (1998). *Randomness*. Cambrige, MA: Harvard University Press.

Biehler, R. (1991). Computers in probability education. In R. Kapadia, & M. Borovcnik (Eds.), *Chance Encounters - Probability in Education* (pp. 169-211). Dordrecht: Kluwer.

Biehler, R. (1993). Computer environments for the learning of mathematics. In R. Biehler, R. Scholtz, R.W. Sträßer, B. Winkelmann (Eds.), *Didactics of Mathematics as a Scientific Discipline - The State of the Art* (pp.189-199). Dordrecht: Kluwer.

Bishop-Clark, C. (1995). Cognitive style, personality, and computer programming. *Computers in Human Behavior*, *11*(2), 241-260.

Blum, W., Galbraith, P.L., Henn, H.W., & Niss, M. (Eds.) (2007). Modelling and Applications in Mathematics Education. The 14th ICMI Study. New York, NY: Springer Science + Business Media, LLC.

Blum, W., & Leiss, D. (2007). How do students and teachers deal with mathematical modelling problems? The example “Filling up”. In Haines et al. (Eds.), *Mathematical Modelling (ICTMA 12): Education, Engineering and Economics* (pp. 222–231). Chichester: Horwood Publishing.

Bork, A. (1981). *Learning with computers*. Digital Press, Bedford, Mass.

Borovcnik, M., Bentz, H.J., & Kapadia, R. (1991). A probabilistic perspective. In R. Kapadia, & M. Borovcnik (Eds.), *Chance encounters: Probability in education* (pp. 27-71). Springer Netherlands.

Brown, A., DeVries, D., Dubinsky, E., & Thomas, K. (1998). Learning binary operations, groups, and subgroups. *Journal of Mathematical Behavior*, 16, 187-239.

Bruner, J.S. (1974). *Il significato dell’educazione*. Armando, Roma.

Burrill, G., Allison, J., Breaux, G., Kastberg, S., Leatham, K., & Sanchez, W. (2002). Handheld graphing technology in secondary mathematics: Research findings and implications for classroom practice. Retrieved from the Texas Instruments website: http://education.ti.com/sites/UK/downloads/pdf/References/Done/Burrill,G.%20(2000).pdf

Castoldi, M. (Ed.). (2000). *Software per l'apprendimento. Come e perché scegliere e utilizzare prodotti multimediali nella scuola* (Vol. 14). FrancoAngeli.

Chazan, D. (1993). High school geometry students’ justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics*, 24, 359-387.

Chernoff, E.J. (2008). The state of probability measurement in mathematics education: a first approximation. *Philosophy of Mathematics Education Journal*, *23*, 19-29.

Ciarrapico, L. (2002). L'insegnamento della matematica dal passato recente all'attualita. *Archimede-Rivista di Matematiche Pure e Applicate*, *54*(3), 123-129.

Clements, D.H. (1990). Metacomponential development in a Logo programming environment. *Journal of Educational Psychology*, *82*(1), 141.

Clements, D.H., & Battista, M.T. (2000). Designing effective software. In A.E. Kelly, & R.A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 761-776). Mahwah, NJ: Erlbaum.

Clements, D.H., & Gullo, D.F. (1984). Effects of computer programming on young children's cognition. *Journal of Educational Psychology*, *76*(6), 1051.

Clements, D.H., & Meredith, J.S. (1993). Research on Logo: Effects and efficacy. *Journal of Computing in Childhood Education*, *4*(4), 263-290.

Clement, D., Sarama, J., Yelland, N.J., & Glass, B. (2008). Learning and teaching geometry with computers in the elementary and middle school. In M.K. Heid, & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 1. Research syntheses* (pp. 109-154). Charlotte, NC: Information Age.

Collins, A., & Brown, J.S. (1988). The Computer as tool of learning trough reflection. In H. Mandl, & A. Lesgold (Eds.), *Learning Issue for Intelligent Tutoring Systems* (pp.1-8). New York: Springer.

Connors, M.A. (1997). Technology in Mathematics Education: A Personal Perspective. *Journal of Computing in Higher Education*, *8*(2), 94-108.

Cosmides, L., & Tooby, J. (1996). Are humans good intuitive statisticians after all? Rethinking some conclusions from the literature on judgment under uncertainty. *Cognition*, *58*(1), 1-73.

Costabile, F.A., Serpe, A. (2012). MatCos: A Programming Environment for Mathematics Education. In G. Lee (Eds.), *Proceedings of the International Conference on Computer and Advanced Technology in Education* (Vol. 859971, pp.

25-30). Beijing, China, ICCATE 2011.New York: ASME Press: New York. http://asmedl.org/ebooks/asme/asme_press/859971

Costabile, A., & Serpe, A. (2013). Dal metodo simbolico-ricostruttivo al metodo esperienziale-simbolico-ricostruttivo. In F.A. Costabile (a cura di), *Aspetti pedagogici e cognitivi dell'apprendimento della Matematica con le tecnologie. La programmazione come metodologia. Il software MatCos* (pp. 39-110). Carocci Editore.

Cresci, L. (2006). *Le curve celebri: invito alla storia della matematica attraverso le curve piane più affascinanti*. Muzzio (IS).

Davenport, J. H., Siret, Y., & Tournier, É. (1988). *Computer algebra* (Vol. 5). London: Academic Press.

De Carle, D. (2014). *Orologiaio riparatore*. Milano, IT: Hoepli editore.

De Finetti, B. (1957). *Matematica logico intuitiva*. Edizioni cremonese.

De Finetti, B. (1967). *Il "saper vedere" in matematica.* Torino: Loescher.

De Finetti, B. (1981). *Scritti (1926-1930)*, con Prefazione di M. De Felice. CEDAM, Padova.

Dix, A., Finlay, J., Abowd, G.D., & Beale, R. (2004). *Human-Computer Interaction*. (3rd ed.). Harlow, England: Pearson Education.

Doerr, H.M. (1996 a). Integrating the study of trigonometry, vectors, and force through modeling. *School Science and Mathematics*, *96*(8), 407-418.

Doerr, H.M. (1996 b). Stella ten years later: A review of the literature. *International Journal of Computers for Mathematical Learning*, *1*(2), 201-224.

Doerr, H.M., & Pratt, D. (2008). The learning of mathematics and mathematical modeling. In M.K. Heid, & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 1. Research syntheses* (pp. 259-286). Charlotte, NC: Information Age.

Doerr, H.M., & Zangor R. (2000). Creating meaning for and with the graphing calculator. *Educational Studies in Mathematics*, *41*(2), 143–163.

Drijvers, P., Ball, L., Barzel, B., Heid, M.K., Cao, Y., & Maschietto, M. (2016). Uses of Technology in Lower Secondary Mathematics Education. In *Uses of Technology in Lower Secondary Mathematics Education* (pp. 1-34). Springer International Publishing.

Dubinsky, E., & Tall, D. (2002). Advanced mathematical thinking and the computer. In *Advanced mathematical thinking* (pp. 231-248). Springer Netherlands.

Dunham, P., & Hennessy, S. (2008). Equity and use of educational technology in mathematics. In M.K. Heid, & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 1. Research syntheses* (pp. 345-418). Charlotte, NC: Information Age.

Enriques, F. (1921). Insegnamento dinamico. Periodico di Matematica, Serie IV, vol.I, 6-16.

Edwards, D., & Hansom, M. (1989). *Guide to mathematical modeling*. Boca Raton, FL: CRC Press.

Everitt, B.S. (1999). *Chance rules: An informal guide to probability, risk and statistics*. New York: Copernicus/Springer-Verlang.

Falcade, R., Laborde, C., & Mariotti, M.A. (2007). Approccing function: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, *66*(3), 317-333.

Fandiño Pinilla, M.I. (2010). Probabilità, statistica, combinatoria: l'altra metà del cielo. *Vita Scolastica*, *65*(4), 14-16.

Feller, W. (1968). *An Intoduction To Probability Theory and its Applications*. John Wiley, New York.

Fessakis, G., Gouli, E., & Mavroudi, E. (2013). Problem solving by 5–6 years old kindergarten children in a computer programming environment: A case study. *Computers & Education*, *63*, 87-97.

Feurzeig, W., Papert, S., Bloom, M., Grant, R., & Solomon, C. (1969). Programming languages as a conceptual framework for teaching mathematics. *Report N. 1889*. Bolt Beranek & Newman Inc., Cambridge, MA.

Fierli, M. (2003). *Tecnologie per l'educazione*, Edizioni Laterza, Bari.

Fischbein, E. (1975). *The intuitive sources of probabilistic thinking in children*. Dordrecht, The Netherlands: Reidel.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*, 39-162.

Fischbein, E., & Gazit, A. (1984). Does the teaching of probability improve probabilistic intuitions?. *Educational studies in mathematics*, *15*(1), 1-24.

Fischbein, E., Nello, & M.S., Marino, M.S. (1991), Factors affecting probabilistic judgments in children and adolescents. *Educational Studies in Mathematics*, *22*, 523-549.

Franklin, C., Kader, G., Mewborn, D.S., Moreno, J., Peck, R., Perry, M., & Scheaffer, R. (2005). *A Curriculum Framework for K-12 Statistics Education. GAISE Report*. American Statistical Association.

Fuchs, K.J., & Blum, W. (2008). Selbständiges Lernen im Mathematikunterricht mit, beziehungsreichen' Aufgaben. In J. Thonhauser (Ed.), *Aufgaben als Katalysatoren von Lernprozessen* (pp. 135–147). Münster: Waxmann.

Gal, I. (2005). Towards "probability literacy" for all citizens: building blocks and instructional dilemas. In G. Jones (Ed.), *Exploring Probability in Schools: Challenges for Teaching and Learning* (pp. 39-63). Springer, New York.

Galbraith, P., Stillman, G., Brown, J., & Edwards, I. (2007). Facilitating middle secondary modelling competencies. In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling: Education, engineering and economics* (pp. 130-1409. Chichester: Horwood.

Gallegos, R.R., & Rivera, S.Q. (2015). Developing Modelling Competencies Through the Use of Technology. In G. A. Stillman, W. Blum, & M. S. Biembengut (Eds.), *Mathematical Modelling in Education Research and Practice* (pp. 443-452). Springer International Publishing.

Gallese, V., & Lakoff, G. (2005). The Brain's Concepts: The Role of the Sensory-Motor System in Reason and Language. *Cognitive Neuropsychology*, *22*, 455-479.

Gardner, H. (1987). *Formae mentis*. Feltrinelli, Milano.

Gigerenzer, G., Swijtink, Z., Porter, T., Daston, L., Beatty, J. & Kruger, L. (1989). *The empire of chance: how probability changed science and everyday life*. New York: Cambridge University Press.

Greer, B., & Mukhopadhyay, S. (2005). Teaching and learning the mathematization of uncertainty: Historical, cultural, social and political contexts. In G. Jones (Ed.), *Exploring Probability in School: Challenges for Teaching and Learning* (pp. 297-324). Springer, New York,.

Goos, M. (1998). Technology as a tool for transforming mathematical tasks. In P. Galbraith, W. Blum, G. Booker, & I. D. Huntley (Eds.), *Mathematical Modelling: Teaching and assessment in a technology-rich world* (pp. 103-113). Chichester: Horwood.

Goos, M., Galbraith, P., Renshaw, P., & Geiger, V. (2003). Perspectives on technology mediated learning in secondary school mathematics classrooms. *The Journal of Mathematical Behavior, 22*(1), 73-89.

Grimaldi, R. (2006). Disuguaglianze digitali nella scuola. *Gli usi didattici delle tecnologie dell'informazione e della comunicazione in Piemonte, Franco Angeli*.

Gürbüz, R. (2010). The effect of activity-based instruction on conceptual development of seventh grade students in probability. *International Journal of Mathematical Education in Science and Technology, 41*(6), 743-767.

Guzman, A., & Nussbaum, M. (2009). Teaching Competencies for Technology Integration in the classroom. *Journal of Computer Assisted Learning, 25*, 453-469.

Hannafin, R., Burruss, J., & Little, C. (2001). Learning with dynamic geometry programs: Perspectives of teachers and learners. *Journal of Educational Research, 94*, 132-44.

Hatfield, L.L. (1984). Toward comprehensive instructional computing in mathematics. *Computers in mathematics education*, 1-9.

Hawkins, D. (1990). Defining and bridging the gap. *Science education: A minds-on approach for the elementary years*, 97-139.

Heersmink, R. (2013). A taxonomy of cognitive artifacts: function, information, and categories. *Review of philosophy and psychology*, *4*(3), 465-481.

Heid M.K. (1988). Resequencing skills and concepts in applied calculus using the computer as a tool. *Journal for Research in Mathematics Education, 19*(1), 3-25.

Heid, M.K. (1997). The technological revolution and mathematics reform. *American Journal of Education*, *106*, 5-61.

Heid, M.K., & Blume, G.W. (2008). Technology and the development of algebraic understanding. In M.K. Heid, & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 1. Research syntheses* (pp. 55-108). Charlotte, NC: Information Age.

Heid, M.K., Sheets, C., & Matras, M.A. (1990). Computer-enhanced algebra: New roles and challenges for teachers and students. In T.J. Cooney (Ed.), *Teaching and learning mathematics in the 1990s* (1990 NCTM yearbook) (pp. 194-204). Reston, VA: National Council of Teachers of Mathematics.

Hennessy, S., Ruthven, K., & Brindley, S. (2005). Teacher perspectives on integrating ICT into subject teaching: commitment, constraints, caution, and change. *Journal of curriculum studies*, *37*(2), 155-192.

Hilbert, T. S., Renkl, A., Kessler, S., & Reiss, K. (2008). Learning to prove in geometry: Learning from heuristic examples and how it can be supported. *Learning and Instruction*, *18*(1), 54-65.

Hoffkamp, A. (2010). The use of interactive visualisations to foster the understanding of concepts of calculus-Design principles and empirical resuts. Presentation at the *I2GEO 2010 conference*. http://www.cermat.org/i3geo2010/downloads/files/I2GEO2010-Hoffkamp.pdf .

Hohenwarter, J., Hohenwarter, M., & Lavicza, Z. (2008). Introducing dynamic mathematics software to secondary school teachers: the case of GeoGebra. *Journal of Computers in Mathematics and Science Teaching*, *28*(2), 135-146.

Hollebrands, K. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education, 38*, 164-92.

Hollebrands, K., Laborde, C., & Sträβer, R. (2008). Technology and the learning of geometry at secondary level. In M.K. Heid, & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 2. Research syntheses* (pp. 31-52). Charlotte, NC: Information Age.

Hollebrands, K.F., Conner, A.M., & Smith, R.C. (2010). The nature of arguments provided by college geometry students with access to technology while solving problems. Journal for Research in Mathematics Education, *41*(4), 324–350.

Hölzl, R. (1996). How does 'dragging' affect the lerning of geometry. *International Journal of Computers for Mathematical Learning*, *1*(2), 169-187.

Howe, J.A.M., Ross, P.M., Johnson, K.R., Plane, F., & Inglis, R. (1982). Teaching mathematics through programming in the classroom. *Computers & Education*, *6*(1), 85-91.

Hudson B. (1996). Group work with Multimedia in Mathematics: Contrasting Patterns of Interaction. *British Journal of Educational Technology, 27(3)*, 172-190.

Jones, G. (2005). Introduction. In Jones, G., (Ed.), *Exploring Probability in School: Chellenges for Teaching and Learning* (pp. 1-12). Springer, New York.

Jones, K., Gutiérrez, A., & Mariotti, M.A. (2000). Proof in dynamic geometry environments. *Educational Studies in Mathematics*, *44*, 1-3.

Kelly, A.V. (1984). Microcomputers and the curriculum: Uses and Abuses. In A.V. Kelly. (Ed), *Microcomputers and the Curriculum*, (pp. 1-19). London: Harper and Row.

Kotenkamp, U., Blessing, A.M., Dohrmann, C., Kreis, Y., Libbrecht, P., & Mercat, C. (2010). Interoperable interactive geometry for Europe-First technological and educational results and future challeges of the Intergeo project. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *proceeding of the Sixty European*

Conference on Research on Mathematics Education (pp. 1150-1160). INRP. http://www.inrp.fr/editions/cerme6 .

Kuzler, B. (2000). The Algebraic Calculator as a Pedagogical Tool for Teaching Mathematics. *International Journal of Computer Algebra in Mathematics Education, 7*(1), 5-23.

Laborde, C. (2001). The use of new technologies as a vehicle for restructuring teachers' mathematics. In T. Conney, & F.L. Lin (Eds.), *Making sense of mathematics teachers education* (pp. 87-109). Dordrecht: Kluwer Academic Publishers.

Laborde, C. (2001b). Integration of technology in the design of geometry tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning*, *6*(3), 283–317.

Laborde, C., Kynigos, C. H. R. O. N. I. S., Hollebrands, K. A. R. E. N., & Strässer, R. U. D. O. L. F. (2006). Teaching and learning geometry with technology. *Handbook of research on the psychology of mathematics education: Past, present and future*, 275-304.

Laborde, C. (2007). The role and uses of technologies in mathematics classrooms: Between challenge and modus vivendi. *Canadian journal of science, mathematics & technology*, *7*(1), 68-92.

Laborde, C., & Laborde, J.M. (2008). The development of a dynamical geometry environment. In M. K. Heid, & G. W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 1. Research syntheses* (pp. 155-206). Charlotte, NC: Information Age.

Lagrange, J.B., Artigue, M., Laborde, C., & Trouche, L. (2003). Technology and mathematics education: A multidimensional study of the evolution of research and innovation. In A.J. Bishop, M.A. Clements, C.Keitel, J. Kilpatrick, & K.S. Leung (Eds.), *Second International Handbook of Mathematics Education* (pp.237-269). Dordrecht: Kluwer.

Lagrange, J.B., & Ozdemir Erdogan, E. (2009). Teachers' emergent goals in spreadsheet-based lessons: analyzing the complexity of technology integration. *Educational Studies in Mathematics*, *71*(1), 65-84.

Lavy, P. (1992). *Le tecnologie dell'intelligenza*. Synergon, Bologna.

Leikin, R., & Grossman, D. (2013). Teachers modify geometry problems: from proof to investigation. *Educational Studies in Mathematics*, *82*, 515-531.

Lesmeister, L.M. (1996). The effect of graphing calculators on secondary mathematics achievement. (Unpublished MS thesis, University of Houston). *Dissertation Abstracts International*, 35, 01.

Lesh, R., & Doerr, H.M. (2000). Symbolizing, communicating, and mathematizing: key components of models and modeling. In P. Cobb, E. Yackel, & K. McClain (Eds.) *Symbolizing and communicating in mathematics classrooms: Perspectives on discourse, tools, and instructinal design* (pp. 361-383). Mahwah, NJ: Erlbaum.

Lesh, R., Surber, D., & Zawojewski, J. (1983). Phases in modeling and phase-related processes. In J.C. Bergeron, & N. Herscovics (Eds.) *Proceedings of the Fifth Annual Meeting Psychology of Mathematics Education-North American Chapter* Vol. 2 (pp. 129-136). Montreal, Quebec, Canada.

Lester, F.K., & Kehle, P.E. (2003). From problem solving to modeling: The evolution of thinking about research on complex mathematical activity. In R. Lesh, & H.M. Doerr (Eds.) *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 501-518). Mahwath, NJ: Erlbaum.

Leung, A. (2003). Dynamic geometry and the theory of variation. In N. A. Pateman, B.J. Dougherty, & J.T. Zilliox (Eds.), *Proceedings of the 27th PME International Conference:Vol. 3* (pp. 197-204).

Lévy, P. (1992). *Le tecnologie dell'intelligenza*, Synergon, Bologna.

Liao, Y.K.C., & Bright, G.W. (1991). Effects of computer programming on cognitive outcomes: A meta-analysis. *Journal of Educational Computing Research*, *7*(3), 251-268.

Lye, S.Y., & Koh, J.H.L. (2014). Review on teaching and learning of computational thinking through programming: What is next for K-12?. *Computers in Human Behavior*, *41*, 51-61.

Maaß, K. (2005a). Modellieren im Mathematikunterricht der S I. *Journal für Mathematikdidaktik*, *26*(2), 114–142.

Mariotti, M.A. (1995). Le rappresentazioni grafiche e l'apprendimento della geometria. In B. D'Amore (Eds.), *Insegnare ad apprendere la matematica in Aula: situazioni e prospettive* (pp. 47-58). Bologna: Pitagora.

Mariotti, M.A. (1996). Costruzioni in geometria. *L'insegnamento della Matematica e delle Scienze Integrate B*, *19*, 261-88.

McLuhan, M. (1976). *La Galassia Gutenberg*. Armando, Roma.

Menghini, M. (2016). History of Mathematics Education-Italy. *arXiv preprint arXiv:1603.08499*.

Momin Fasiyoddin Inayat, Shaikh Naeem Hamid (2016). Integrating new technologies and tools in teaching and learning of mathematics: An overview. *Journal of Computer and Mathematical Sciences*, *7*(3), 122-129.

Monaghan, J. (2004). Teachers' activities in technology-based mathematics lessons. *International Journal of Computer for Mathematical Learning*, *9*, 327–357.

Mong-Chen Chiang, Yi-Chan Deng, Han-Zen Chang, Hui-Chun Liao, Chin-Wen Ho & Tak-Wai Chan (2005). EduBingo: A Bingo-like Game for Mathematics Skill Building. In: *Proceedings of the 2005 IEEE International Workshop on Wireless and Mobile Technologies in Education (WMTE'05)*, 93-95.

Mousley, J., Lambdin, D., & Koc, Y. (2003). Mathematics teacher education and technology. In *Second international handbook of mathematics education* (pp. 395-432). Springer Netherlands.

Mouza, C. (2009). Does Research-Based Professional Development Make a Difference? A Longitudinal Investigation of Teacher Learning in Technology Integration. *Teachers College Record*, *111*(5), 1195-1241.

Nicaise, M., & Barnes, D. (1996). The union of technology, constructivism, and teacher education. *Journal of Teacher Education*, *47*(3), 205-12.

Nickerson, R.S. (1982). Computer programming as a vehicle for teaching of thinking skills. *Thinking*, *4*, 42-48.

Olive, J. (1998). Opportunities to explore and integrate mathematics with the Geometer's Sketchpad. *Designing learning environments for developing understanding of geometry and space*, 395-418.

Olive, J., & Lobato, J. (2008). The learning of rational number concepts using technology. In M.K. Heid, & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 1. Research syntheses* (pp. 1-53). Charlotte, NC: Information Age.

Olson, D.R. (1979). *Linguaggi, media e processi educativi*. Loescher, Torino.

Oprea, J.M. (1988). Computer programming and mathematical thinking. *Journal of Mathematical Behavior*, *7*(2), 175-190.

Papert, S. (1980). *Mindstorms: Computers and Powerful Ideas*. New York: Basic Books.

Pea, R.D. (1985). Beyond amplification: Using the computer to reorganize mental functionning. *Educational Psychologist*, *20*, 167-182.

Pea, R.D., & Kurland, D.M. (1984). On the cognitive effects of learning computer programming. *New ideas in psychology*, *2*(2), 137-168.

Pratt, D. (2000). Making sense of the total of two dice. *Journal for Research in mathematics Education*, 602-625.

Pratt, D. (2005). How do teacher foster understanding of probability? In G.A. Jones (Ed.), *Exploring probability in school: Chellenges for teaching and learning* (pp.171-190). New York: Springer.

Quesada, A. R., & Maxwell, M.E. (1994). The effects of using graphing calculators to enhance college students' performance in precalculus. *Educational Studies in Mathematics*, *27*(2), 205-215.

Ritter, S., Haverty, L., & Koedinger, K. R. (2008). Technology and the learning of geometry at secondary level. In M. K. Heid, & G. W. Blume (Eds.), *Research on* *technology and the teaching and learning of mathematics:Vol. 2. Research syntheses* (pp. 157-181). Charlotte, NC: Information Age.

Romberg, T.A. (1994). Classroom instruction that fosters mathematical thinking and problem solving: Connections between theory and practice. In A.H. Schoenfeld (Ed.), *Mathematical thinking and problem solving* (pp. 287–304). Hillsdale, NJ: Erlbaum. Rushkoff, D. (2010). *Program or be programmed: Ten commands for a digital age*. New York, NY: O/R Books.

Roschelle J.,Pea R.,Hoadley C.,Gordin D., & Means B. (2000). Changing how and what children learn in school with computer-based technologies. *The Future of Children, 10*(2), 76-101.

Rossi, C. (1999). *La matematica dell'incertezza: didattica della probabilità e della statistica*. Zanichelli.

Russo, L. (2000). *Segmenti e bastoncini. Dove sta andando la scuola?*. Feltrinelli Editore.

Ruthven, K., & Hennessy S. (2002). A practitioner model of the use of computer-based tools and resources to support mathematics teaching and learning. *Educational Studies in Mathematics*, *49*(1), 47-88.

Ruthven, K., Hennessy, S., & Deaney, R. (2008). Constructions of dynamic geometry: A study of the interpretative flexibility of educational software in classroom practice. *Computers & Education*, *51*(1), 297-317.

Shaughnessy, J.M. (1992). Research in probability and statistics: Reflections and directions. In D. Grouws (Ed.), *Handbook of research on mathemat-ics teaching and learning, New York/Reston* (pp. 465-494) .VA: Macmillan/National Council of Teachers of Mathematics.

Sherman, M. (2014). The role of technology in supporting students' mathematical thinking: Extending the metaphors of amplifier and reorganizer. *Contemporary Issues in Technology and Teacher Education*, *14*(3).

Siller, H.S., & Greefrath, G. (2010). Mathematical modelling in class regarding to technology. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceeding of the Sixty European Conference on Research on Mathematics Education* (pp. 1150-1160). INRP. http://www.inrp.fr/editions/cerme6 .

Siniscalco, M. T., & Pedrizzi, T. (Eds.). (2005). *Ocse pisa 2003* (Vol. 46). FrancoAngeli.

Speiser, R., & Walter, C. (1998). Two dice, two sample spaces. In L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee, & W.K. Wong (Eds.), *Proceedings of the Fifth International Conference on Teaching Statistics, Vol. 1* (pp. 67-73). Voorburg, The Netherlands: International Statistical Institute Permanent Office.

Steinbring, H. (1991). The theoretical nature of probability in the classroom. In R. Kapadia, & M. Borovcnik (Eds.), *Chance encounter: Probability in education* (pp. 135-168). Cordrecht, The Netherlands: Kluwer.

Stillman, G.A., Kaiser, G., Blum, W., & Brow, J.P. (Eds.) (2013). *Teaching Mathematical Modelling: Connecting to Research and Practice*, International Perspectives on the Teaching and Learning of Mathematical Modelling. Springer Science + Business Media Dordrecht.

Stillman, G.A., Blum, W., & Salett Biembengut, M. (Eds.) (2015). *Mathematical Modelling in Education Research and Practice Cultural, Social and Cognitive Influences,* International Perspectives on the Teaching and Learning of Mathematical Modelling. Springer International Publishing Switzerland.

Stohl, H. (2005). Probability in teacher education and development. In *Exploring probability in school* (pp. 345-366). Springer US.

Straesser, R. (2002). Cabri-Geometre: Does dynamic geometry software (DGS) change geometry and its teaching and learning?. *International Journal of Computers for Mathematical Learning, 6*(3), 319-333.

Tall, D., Smith, D., & Piez, C. (2008). Technology and calculus. In M.K. Heid, & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 1. Research syntheses* (pp. 207-258). Charlotte, NC: Information Age.

Taylor, R. (1980). *The computer in the school: tutor, tool, tutee*. Teacher College Press, Columbia University, New York.

Tikhomirov, O.K. (1981). The psychological consequences of computerization. In J.V. Werthsch (Ed.), *The concept of activity in Soviet psychology* (pp. 256-278). Armonk: M. E. Sharpe.

Tomasi, L. (2012). La matematica nella Scuola secondaria di II grado, dalle sperimentazioni degli anni Ottanta al riordino del 2010. *Annali online della Didattica e della Formazione Docente*, *4*(4), 203-227.

Van Hiele, P.M. (1986). *Structure and insight*. Orlando, FL: Academic Press.

Vidakovic, D., Berenson, S., & Brandasma, J. (1998). Children's intuition of probabilistic concepts emerging from fair play. In L. Pereira-Mendoza, L. Seu Kea, T. Wee Kee, & W.K. Wong (Eds.), *Proceedings of the Fifth International Conference on Teaching Statistics, Vol. 1* (pp. 67-73). Voorburg, The Netherlands: International Statistical Institute Permanent Office.

Vighi, P. (2012). Introduzione al concetto di probabilità nella scuola secondaria superiore. *Annali online della Didattica e della Formazione Docente*, *4*(4), 117-129.

Vygotskij, L.S. (1990). *Mente e linguaggio*. Laterza, Roma-Bari.

Vygotskij, L.S. (1990). *Pensiero e linguaggio* (1934). Laterza, Roma-Bari.

Weber, T.E. (1998). Graphing technology and its effect on solving inequalities. (Unpublished doctoral dissertation, Wayne State University, Detroit, MI). *Dissertation Abstracts International*, *60*(01A), 88.

Wedelin, D., & Adawi, T. (2015). Applied Mathematical Problem Solving: Principles for Designing Small Realistic Problems. In G.A. Stillman, W. Blum, & M.S. Biembengut (Eds.), *Mathematical Modelling in Education Research and Practice* (pp. 417-427). Springer International Publishing.

Wu Xiuguo (2009). Discrete Mathematics Teaching Reformation: Adding Experiments. *First International Workshop on Education Technology and Computer Sci.*, *IEEE*, 566-569.

Yerushalmy, M. (1991).Student perceptions of aspects of algebraic function using multiple representation software. *Journal of Computer Assisted Learning*, *7*, 42-57.

Yuh-Ming Cheng and Peng-Fei Chen (2007). Applied Software Agents mechanism to Mathematics Learning in Elementary School. In: *Proceeding of Innovative Computing, Information and Control (ICICIC '07) Second International Conference*, 201-201.

Zbiek, R.M., & Hollebrands, K. (2008). A research-informed view of the process of incorporating mathematics technology into classroom practice by inservice and prospective teachers. In M.K. Heid, & G.W. Blume (Eds.), *Research on technology and the teaching and learning of mathematics:Vol. 1. Research syntheses* (pp. 287-344). Charlotte, NC: Information Age.

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July 22, 2008	Master degree in Mathematics "cum laude" at University of Calabria (Italy)
July 26, 2006	Bachelor degree in Mathematics "cum laude" at University of Calabria (Italy)
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Publication

Article

[1] Frassia, M.G. (2016). Il “saper vedere” in Matematica: un approccio alla probabilità nella scuola secondaria superiore. *Progetto Alice*, *17*(49), 119-136.

[2] Serpe, A., & Frassia, M.G. (2016). A Journey through the geometrical alchemies of art: an example of constructivist teaching. *Experiences of Teaching with Mathematics, Sciences and Technology*, *2*(1), 249-260.

[3] Serpe, A., & Frassia, M.G. (2015). The deltoid as envelope of line in high school: A constructive approach in the classroom. *Quaderni di ricerca in didattica (Mathematics)*, *2*(25), 525-535.

[4] Frassia, M. G. (2014). Esperienze di programmazione al computer: ‘punti di vista’ sul calcolo della probabilità. *Mondo Digitale*, *13*(51), 726-735.

In conference proceeding

[1] Frassia, M.G., & Serpe, A. (in press). PNSD on the road con GeoGebra. In O. Robutti (Ed.), *Proceedings of VI GEOGEBRA ITALIAN DAY – 2016 - L’attività dei docenti con GeoGebra nella formazione e nella sperimentazione*. Torino, Italy.

[2] Serpe, A., Frassia, M.G., & Caligiuri, S. (in press). L’approccio del Sig. Rossi alle funzioni goniometriche con l’uso di MatCos. In *Proceedings of VII Convegno nazionale di Didattica della Fisica e della Matematica – 2015 - Insegnare e imparare Matematica e Fisica: Insegnanti e studenti per una didattica inclusiva*. Torino, Italy.

[3] Serpe, A., & Frassia, M.G. (2017). Technology will solve student’s probability misconceptions: integrating simulation, algorithmics and programming.

Proceedings of CERME 10. Dublin, Ireland. https://keynote.conferenceservices.net/resources/444/5118/pdf/CERME10_0094.pdf

[4] Frassia, M.G., & Serpe, A. (2016). Arte e Matematica si incontrano in classe … . In B. D'Amore, & S. Sbaragli (Eds.), *Proceedings of Incontri con la Matematica n. 30. La Matematica e la sua didattica. Convegno trentennale* (pp. 133-134). Bologna, Italy: Pitagora.

[5] Serpe, A., & Frassia, M.G. (2016). Spirale di Breguet con GeoGebra. In O. Robutti (Ed.), *Proceedings of IV GeoGebra Italian Day – 2014 - La formazione docenti con GeoGebra* (pp. 141-151). Torino, Italy: Ledizioni.

[6] Frassia, M.G. (2016). Enhance learning of mathematics in secondary school with programming: some examples with environment MatCos. In *INTED2016 Proceedings. 10th International Technology, Education and Development Conference* (pp. 7835-7841). Valencia, Spain: IATED. doi: 10.21125/inted.2016.0844

[7] Frassia, M.G. (2015). Games design with the programming environment MatCos for enhanced learning of probability theory: a didactic example for high schools. In *EDULEARN15 Proceedings. International Conference on Education and New Learning Technologies* (pp. 1165-1172). Barcelona, Spain: IATED.

[8] Serpe, A., & Frassia, M.G. (2015). Un laboratorio interdisciplinare tra matematica, informatica e disegno: la voluta ionica del Vignola. In O. Robutti, & M. Mosca (Eds.), *Proceedings of VI Convegno Nazionale di Didattica della Fisica e della Matematica – 2013 – I docenti di matematica e di fisica di fronte ai mutamenti della scuola: concetti, processi, valutazione* (pp. 365-376). Torino, Italy: Ledizioni.

[9] Serpe, A., & Frassia, M.G. (2014). Simulazioni di fenomeni aleatori: un esempio di laboratorio dal reale al virtuale. In B. D'Amore, & S. Sbaragli (Eds.), *Proceedings of Incontri con la Matematica n.28. Parliamo tanto e spesso di didattica della Matematica* (pp. 175-178). Bologna, Italy: Pitagora.

[10] Frassia, M.G. (2014). Esperienze di programmazione al computer: 'punti di vista' sul calcolo della probabilità. In *Proceedings of DIDAMATICA 2014. Nuovi Processi e Paradigmi per la Didattica* (pp. 726-735). Napoli, Italy: AICA.

[11] Gattuso, C., Gattuso, P., Serpe, A., Piccioli, C., & Frassia, M.G. (2013). Methods, technologies and mathematical models for increasing the value of and enhancing cultural heritage through jewelry design. In *Proceedings of IVth Conference "Diagnosis, Conservation and Valorization of Cultural Heritage"* (pp.). Napoli, Italy: Ethos Edizioni.

Paper presented at conference

[1] Frassia, M.G., & Serpe, A. (2016, November). *Arte e Matematica si incontrano in classe …* . Paper presented at Incontri con la Matematica n. 30. La Matematica e la sua didattica. Convegno trentennale, Bologna, Italy.

[2] Frassia, M.G., & Serpe, A. (2016, October). *Matematica in classe senza sfuggire dalla realtà*. Paper presented at XXXIII Convegno UMI-CIIM. Criticità per l'insegnamento della matematica nella scuola di oggi, Pavia, Italy.

[3] Frassia, M.G., & Serpe, A. (2016, October). *PNSD on the road con GeoGebra*. Paper presented at VI GEOGEBRA ITALIAN DAY – 2016 - L'attività dei docenti con GeoGebra nella formazione e nella sperimentazione, Torino, Italy.

[4] Frassia, M.G., & Serpe, A. (2016, July). *Mathematical modeling and geometry teaching in a computer-based environment: a concrete example in secondary*

school. Paper presented at 13th International Congress on Mathematical Education, Hamburg, Germany.

[5] Serpe, A., & Frassia, M.G. (2016, July). *Mathematization of uncertainty with the aid of computers: A model of activity in high school*. Paper presented at 13th International Congress on Mathematical Education, Hamburg, Germany.

[6] Frassia, M.G. (2016, March). *Enhance learning of mathematics in secondary school with programming: some examples with environment MatCos*. Paper virtually presented at 10th International Technology, Education and Development Conference, Valencia, Spain.

[7] Frassia, M.G. (2016, January). *L'impatto dei software sulla didattica della matematica nella scuola secondaria di ii grado (I primi risultati)*. Presented at Seminario Nazionale di Ricerca in Didattica della Matematica - Sezione Giovani, Rimini, Italy.

[8] Serpe, A., Frassia, M.G., & Caligiuri, S. (2015, October). *L'approccio del Sig. Rossi alle funzioni goniometriche con l'uso di MatCos*. Paper presented at VII Convegno nazionale di Didattica della Fisica e della Matematica – 2015 - Insegnare e imparare Matematica e Fisica: Insegnanti e studenti per una didattica inclusiva, Torino, Italy.

[9] Serpe, A., & Frassia, M.G. (2015). *The deltoid as envelope of line in high school: A constructive approach in the classroom*. Paper presented at International Commissione for the Study and Improvement of Mathematics Education, Aosta, Italy.

[10] Frassia M.G. (2015, July). *Games design with the programming environment MatCos for enhanced learning of probability theory: a didactic example for high*

schools. Paper virtually presented at International Conference on Education and New Learning Technologies, Barcelona, Spain.

[11] Serpe, A., & Frassia, M.G. (2014, November). *Simulazioni di fenomeni aleatori: un esempio di laboratorio dal reale al virtuale*. Presented at Incontri con la Matematica n.28. Parliamo tanto e spesso di didattica della Matematica, Bologna, Italy.

[12] Serpe, A., Frassia, M.G., & Costabile, F.A. (2014, October). *Verso probabilità: un percorso adeguato ai tempi*. Presented at XXXII Convegno UMI-CIIM. Il Valore Formativo della Matematica nella Scuola di oggi, Livorno, Italy.

[13] Serpe, A., & Frassia, M.G. (2014, October). *Spirale di Breguet con GeoGebra*. Presented at IV GeoGebra Italian Day – 2014 - La formazione docenti con GoeGebra, Torino, Italy.

[14] Frassia, M.G. (2014, May). *Esperienze di programmazione al computer: 'punti di vista' sul calcolo della probabilità.* Presented at DIDAMATICA 2014. Nuovi Processi e Paradigmi per la Didattica, Napoli, Italy.

[15] Serpe, A., & Frassia, M.G. (2013, October). *Un laboratorio interdisciplinare tra matematica, informatica e disegno: la voluta ionica del Vignola*. Presented at VI Convegno Nazionale di Didattica della Fisica e della Matematica – 2013 – I docenti di matematica e di fisica di fronte ai mutamenti della scuola: concetti, processi, valutazione, Torino, Italy.

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This thesis was typed by Maria Giovanna Frassia.

Appendix A

Initial test

PARTE A

Nome e Cognome____

Classe: ***Sesso***:

☐ III ☐ F

☐ IV ☐ M

1. ***Ti piace la Matematica?***

   ☐ Sì

   ☐ No

2. ***Se hai risposto sì alla domanda n°1. Perché?***

   ☐ Perché la capisco

   ☐ Perché a scuola prendo sempre buoni voti in questa disciplina

   ☐ Perché la Matematica è indispensabile nella vita di tutti i giorni

   ☐ Altro ____________________________________

3. ***Se hai risposto no alla domanda n°1. Perché?***

   ☐ Perché è noiosa

   ☐ Perché è difficile

   ☐ Perché non la capisco

   ☐ Altro ____________________________________

4. ***Nelle lezioni di Matematica, hai mai usato strumenti tecnologici?***

   ☐ Sì

   ☐ No

5. ***Se hai risposto sì alla domanda n°4, quali? (È possibile dare più di una risposta)***

   ☐ Foglio elettronico, come ad esempio Excell

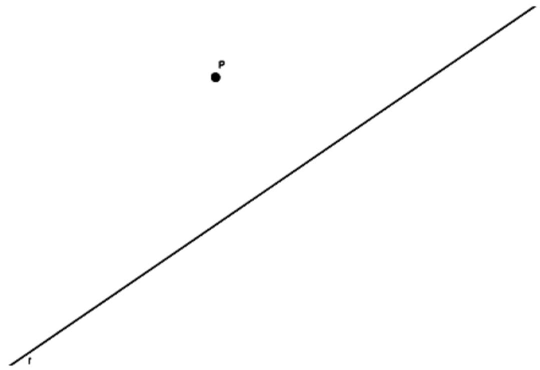
   ☐ Software di geometria dinamica, come ad esempio GeoGebra

   ☐ Computer Algebra System, come ad esempio Derive

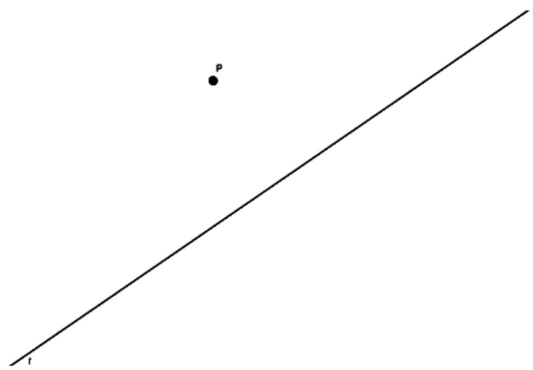
   ☐ Altro ____________________________________

PARTE B

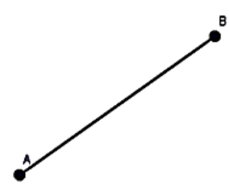
1. ***Disegnare la retta perpendicolare alla retta r e passante per il punto P.***



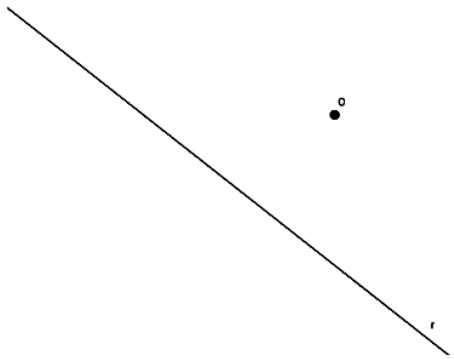
2. ***Proiettare il punto P sulla retta r.***



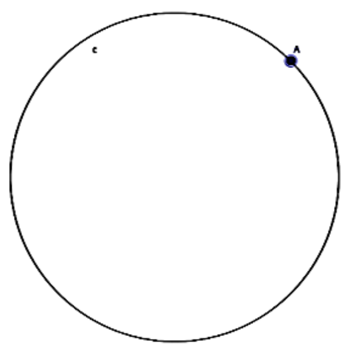
3. ***Disegnare il segmento ottenuto dalla rotazione del segmento $\overline{AB}$ rispetto al punto A di un angolo di 90°, in senso antiorario.***



4. Disegnare circonferenza con centro O e raggio pari alla distanza del punto O dalla retta r.



5. Disegnare la retta tangente alla circonferenza C nel punto A.



6. Nel lancio di un dado regolare a 6 facce, numerate da 1 a 6, qual è la probabilità di ottenere un numero maggiore di 2?

7. ***Qual è la probabilità di ottenere una testa e una croce, lanciando di due monete regolari?***

8. ***Nel lancio di due dadi regolari a 6 facce, numerate da 1 a 6, qual è la probabilità di ottenere come somma dei risultati ottenuti sulle due facce 10?***

9. ***Francesco e Giovanni giocano con tre monete regolari. Francesco vince se a seguito del lancio delle tre monete esce esattamente una croce; Giovanni vince se a seguito del lancio delle tre monete escono esattamente 2 croci. Chi dei due ha maggiori possibilità di vittoria?***

10. ***In un'urna ci sono 2 palline verdi e 6 azzurre. Vengono estratte contemporaneamente due palline dall'urna, qual è la probabilità di estrarre due palline verdi?***

Appendix B

Final test

PARTE 0 (Solo per gli alunni coinvolti nella sperimentazione)

Nome e Cognome____

Classe:
- ☐ III
- ☐ IV

Sesso:
- ☐ F
- ☐ M

1. ***Sei soddisfatto della tua partecipazione alle lezioni sperimentali di Matematica, con l'ausilio del computer?***

   ☐ Sì

   ☐ No

2. ***Se hai risposto sì alla domanda n°1. Perché?***

   ☐ Perché l'apprendimento della Matematica con l'uso del computer risulta più divertente

   ☐ Perché è più facile comprendere gli argomenti studiati

   ☐ Perché è più interessante e imparo di più

   ☐ Altro ____

3. ***Se hai risposto no alla domanda n°1. Perché?***

   ☐ Perché non so usare il computer

   ☐ Perché è più complesso

   ☐ Perché mi annoia

   ☐ Altro ____

4. ***Riporta di seguito gli aspetti positivi delle lezioni che hai seguito con la sperimentazione***

____

____

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____

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5. ***Riporta di seguito gli aspetti negativi delle lezioni che hai seguito con la sperimentazione***

____

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____

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PARTE 1A

Nome e Cognome____

Classe: ***Sesso***:

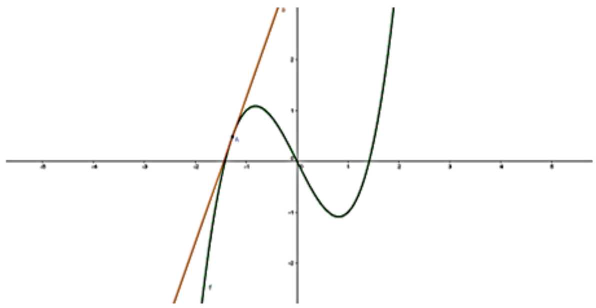
☐ III ☐ F

☐ IV ☐ M

Ho seguito le lezioni del:

☐ 1 aprile ☐ 11 aprile ☐ 22 aprile

1. ***Descrivi il grafico in figura.***



2. ***L'inviluppo di una famiglia o di un insieme di curve piane è:***

☐ la curva tangente a ciascun membro della famiglia in almeno un punto

☐ la curva secante a ciascun membro della famiglia in un punto

☐ un fascio di rette

☐ un insieme di circonferenze concentriche

3. ***Eseguire la seguente costruzione:***

- Disegnare nel piano una retta (d);
- Considerare un punto H_1 di questa retta;
- Tracciare la perpendicolare h alla retta d dal punto;
- Tracciare il segmento FH_1, dove F è un punto del piano;
- Tracciare la retta a asse del segmento FH_1;
- Indicare con P il punto di intersezione dell'asse a con la perpendicolare h;

Al variare di H_1 sulla direttrice il punto P descrive un luogo, quale? Motiva la risposta.

PARTE 1B

Nome e Cognome____

4. ***Un'urna contiene 11 palline, 4 delle quali sono rosse, le rimanenti sono blu. Dall'urna vengono estratte tre palline una dopo l'altra con reintroduzione. Qual è la probabilità di ottenere tre palline blu? Motiva la risposta.***

5. ***Un'urna contiene 11 palline, 4 delle quali sono rosse, le rimanenti sono blu. Dall'urna vengono estratte tre palline una dopo l'altra senza reintroduzione. Qual è la probabilità di ottenere tre palline blu? Motiva la risposta.***

6. ***Valutare le differenze tra i quesiti n°4 e n° 5 in termini di probabilità.***