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The optimal retraction problem and solutions of integrodifferential systems

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Abstract

In this Ph.D thesis we study the following topics related to fixed point theory:

(A) Existence of solutions for integro-differential systems with or without impulses.

(B) The optimal retraction problem and minimal displacement problem.

The contributions to the first topics appear in the following papers:

- G. Marino, P. Pietramala, L. Muglia, *Impulsive neutral integrodifferential equations on unbounded intervals*, Mediterr. J. Math. 1 1 (2004) 93–108

- G. Marino, P. Pietramala, L. Muglia, *Impulsive neutral semilinear equations on unbounded intervals*, Nonlinear Funct. Anal. Appl. 9 4 (2004) 527–543

- G. Marino, V. Colao, L. Muglia, *A note on weakly isotone maps and common solutions for differential systems*, Acta Math. Sin. (English series), 22 4 (2006) 1171-1174

The contribution to the second topics appear in the following paper:

- G. Goebel, G. Marino, L. Muglia, R. Volpe, *The retraction constant and the minimal displacement characteristic of some Banach spaces*, to appear in Nonlinear Analysis TMA

Contents

Introduction

To introduce fixed point theory usually one distinguishes two principal branches: *topological theory* (TFPt) and *metric theory* (MFPt).

The first includes those topics which join topology and functional analysis (e.g. those related to Leray-Schauder theory) while, MFPt, includes methods and results that usually involve proprieties of an isometric nature.

In consequence of this we say that a set K has the *topological fixed point proprieties* (TFPp) if, for any continuous function T from K into K, there exists $x \in K$ such that $Tx = x$.

On the other hand we'll write of *metric fixed point proprieties* (MFPp) if *metric type conditions* on the map imply the existence of fixed points.

Obviously there does not exist a clean separation for this branches because metric type conditions are often used to prove theorems which are non-metric and vice-versa.

In particular, in the second chapter of this thesis, we will point out that a well-known theorem of TFPt leads a number of metric considerations on the existence of lipschitzian retractions of the ball into the spheres.

This work is set up following a tree-scheme.

We have two chapters in which we present the topics related to the fixed point theory that we studied during the Ph.D. time. Every chapter starts with an historical hint joined with the presentation of objects, technics and results which involve the topics.

The last section of any chapters contains the original results that we achieve about some problem.

The first chapter concerns the application of fixed point theory to ordinary differential equations (ODEs).

In particular we put our attention on specifies classes of equations: neutral differential equations and impulsive differential equations.

We study these classes for two reasons. Firstly because this equations are proper instruments to represent mathematical models of real phenomema like economics, physics phenomena, engineering problem et al.. Secondly because there are not many results in literature especially about strong solutions and solutions defined on unbounded intervals.

Our main results are two existence theorems for strong solutions of semilinear neutral impulsive equations and integrodifferential neutral impulsive equations, in both cases defined on unbounded intervals.

At the end of chapter 1 we insert as an appendix some results on common fixed point theory and its applications to differential systems.

Like for ODEs, searching for common solutions of differential systems can be interpreted as a search for common fixed points of opportune maps. Many authors prove by metric type conditions, similar to "contractibility" hypotheses, fixed points for condensing mapping.

In some cases these hypotheses have a consequence: the maps coincide on all of their domain. In other cases the hypotheses seem too strong to obtain results.

Starting from a recent interesting result of Dhage, we prove that, having hypotheses on the Cauchy condition of the problems joined with the weak isotonie hypothesis on the functions, gives exactly one common solution and we are able to exhibit it.

In the second chapter, citing Goebel, we present some metric consequence of topological fixed point theory. In particular we study the *minimal displacement problem* introduced by Goebel in 1973 and the *optimal retraction problem* which has its roots in the famous *Scottish Book*, problem 36 given by S. Ulam.

In the first question Goebel, by the minimal displacement constant, measures how lacks to a $k-$lipschitzian map to have fixed points. Later he defines the function $\varphi_X(k)$ to describe the sup-value of the displacements for any $k-$lipschitzian map defined from X to X, and the function $\psi_X(k)$ to describe the sup-value of the displacements for any $k-$lipschitzian map defined from the ball $B \subset X$ into itself.

For every Banach space it is known that $\psi_X(k) \leq \varphi_X(k) \leq 1-\frac{1}{k}$. If $\psi_X(k) = 1-\frac{1}{k}$ we say the space *extremal*.

It is an open problem (except for extremal spaces) to find a closed formula for $\psi_X(k)$ or $\varphi_X(k)$ for any Banach space.

In the optimal retraction problem one asks what is the minimal lipschitzian constant in a Banach space (that we denote by $k_0(X)$) for which there exists a $k_0(X)$-lipschitzian retraction of the ball into the sphere.

For all Banach spaces (included extremal spaces) *an exact value for $k_0(X)$ it is unknown*. However many authors have given some upper and lower bounds in significant spaces.

Our contribution is on this second question. We prove that for every Banach space with uniform norm that is "cut invariant" (for example $BC(I)$ with I possibly unbounded, or the space of sequences convergent to zero c_0 et al.) we have $k_0(X) \leq 23.31$. Moreover for the space of bounded functions vanishing in a point $z \in Q$ (Q a connected metric space) we obtain $k_0(BC_z(Q)) \leq 12$. In the end, following a result of Annoni and Casini ($k_0(l_1) \leq 8$) we prove that $k_0(X) \leq 8$ for the spaces $L_1[0,1]$, $AC_0[0,1]$ and $BV[0,1] \cap C_0[0,1]$.

Chapter 1

Fixed point theory and applications to the differential equations

1.1 Preface

Drawing on Hale's words, even if fixed point theory cannot be considered an essential instrument for the development of the theory of ODEs, it is indeed a very used tool.

Take the first order Cauchy problem:

$$\begin{cases} x' = f(t,x), & t \in J \\ x(t_0) = x_0 \end{cases} \tag{1.1.1}$$

where $J \subset \mathbb{R}$ is an interval (possibly unbounded)and $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous.

The problem to search for existence results of differential equations like (1.1.1) can be interpreted as to show that an opportune map S has fixed points.

Many authors observe (for example let's cite Cecchi, Furi and Marini [20]), in order to apply fixed point theory, one finds mainly two kinds of difficulties: i) algebraic-type difficulties due to the necessity of finding appropriate *a priori* estimates; ii) topological difficulties due to the fact that to apply fixed point theorems one needs of a suitable topology in which S is, at least, continuous.

For example, if J is a compact interval, we can associate to (1.1.1), a continuous and compact map S defined on the whole $C(J, \mathbb{R}^n)$ into itself. Hence we don't face topological problems.

However there are many interesting problems, especially those arising in applied sciences, where the above situation does not occur.

Some of most used fixed point theorems in the theory of ordinary differential equations are roughly divided into two categories: the first based on "compactness" hypotheses, the second based on "contractibility"-type hypotheses [1].

Of the following, theorem (1.1.1) and theorem (1.1.2) belong to the first class while theorem (1.1.3) to the second.

Theorem 1.1.1 (Schauder (1930)) *Compact convex subsets K of an infinite dimensional Banach spaces X have the TFPp, i.e. every continuous mapping $S : K \to K$ has, at least, a fixed point.*

Proof. See [65] or [34] □

Theorem 1.1.2 (Schaefer (1955)) *Let X be a normed linear space. Let $S : X \to X$ be a completely continuous operator, that is, X is continuous and the image of any bounded set is contained in a compact set. Let*

$$\zeta(S) := \{x \in X : x = \lambda Sx \text{ for some } 0 < \lambda < 1\}.$$

Then either $\zeta(S)$ is unbounded or S has a fixed point.

Proof. See the original proof [63] or [65] □

Theorem 1.1.3 (Banach-Caccioppoli) *Let (X, d) be a complete metric space and $S : X \to X$ be a contraction. Then S has a unique fixed point x_0 and $S^n(x) \to x_0$ for each $x \in X$.*

In both theorems (1.1.1) and (1.1.2), the ability to characterize the relatively compact subsets of the spaces considered, is needed.

Usually the spaces in which one searches for solutions for (1.1.1) are $C[a, b]$, $BC(Q)$ the space of continuous and bounded functions on a topological space, $L^p[a, b]$ $(1 \leq p < \infty)$, Sobolev's spaces, Orlicz's spaces, et al..

In the case of continuous functions, the well known Ascoli-Arzelá theorem, to characterize the relatively compact sets.

Theorem 1.1.4 (Ascoli-Arzelá (1883-1885)) *Let $K \subset C([a, b])$ equicontinuous and totally bounded. Then K is relatively compact.*

However even if we find some compactness criteria on mentioned spaces, not necessarily these criteria are sufficiently handle to be applied for our scopes.

In next section we focus our attention on some compactness criteria on the space BC(Q). Later, we will examine results on particular classes of differential equations like neutral differential equations and impulsive differential equations.

Our contribution to the topics will be an existence results for impulsive neutral semilinear equations and integrodifferential equations.

[1] TFP theorems lies in the first class while, MFP theorems lies in the second.

1.2 Compactness in $BC(Q)$

Let Q be a topological space and let $BC(Q)$ be the space of bounded and continuous real functions with domain Q endowed with the sup-norm.

To our knowledge the first compactness criterion on $BC(Q)$ is due to Bartle [6].

The paper contains even a wider quantity of results in different Banach spaces as in terms of compactness criteria as in terms of weakly-compactness criteria.

Firstly by definition of *universal net* [2] and by its proprieties, Bartle proves a compactness criterion for the space $C(K)$, with K a compact space.

Theorem 1.2.1 *[6] Let K be a compact topological space, let F be a bouded set in $C(K)$, and D be dense in K. Then the following statements are equivalent:*

1. *F is relatively compact in the topology of uniform convergence;*
2. *Given $\varepsilon > 0$ there is a finite partition $(A_i)_{i=1,\dots,n}$ of K such that if x, y belong to the same A_i, then*
$$|f(x) - f(y)| < \varepsilon, \qquad f \in F;$$
3. *F is equicontinuous on K;*
4. *If F_0 is a denumerable subset of F, $x_0 \in K$ and (x_n) is a sequence in D for which $f(x_n) \to f(x_0)$, $f \in F_0$, then the convergence is uniform on F_0;*
5. *From every sequence in F one can extract a subsequence which converges uniformly on K.*

Afterwards, denoting with $\beta(Q)$ the Stone-Ĉech compactification of Q [3], the author observes that the same Q can be mapped on a dense subset of the compact space $\beta(Q)$ and this mapping is continuous and open. Therefore, the spaces $BC(Q)$ and $C(\beta(Q))$ are isometrically isomorphic and the following holds

Theorem 1.2.2 *[6] The following statements are equivalent for a bounded subset $F \subseteq BC(Q)$:*

1. *F is relatively compact;*
2. *F is equicontinuous on Q;*
3. *If F_0 is a denumerable subset of F and (q_n) is a sequence in Q for which $(f(q_n))_{n\in\mathbb{N}}$ converges for each $f \in F_0$, then the convergence is uniform on F_0;*

[2]For details see J. K. Kelley, *Convergence in topology*, Duke Math. J. 17 (1950) 277-283

[3]See M. H. Stone, *On the compactification of topological space*, Ann. Soc. Polon. de Math. 21 (1948) 153-160

4. *For any positive ε there is a partition $Q = \cup_{i=1}^{n} A_i$ such that if q', q'' belong to the same A_i then*
$$|f(q') - f(q'')| < \varepsilon, \qquad \forall f \in F.$$

This theorem represents the state of the art until 2002 when De Pascale, Lewicki and Marino in [24] proved a more handle compactness criteria for the space $BC(Q)$, for $BC(Q, \mathbb{R}^n)$ and for operators mapping $BC(Q, \mathbb{R}^n)$ onto itself.

Theorem 1.2.3 *[24] Let F be a bounded subset of $BC(Q)$. Let $\phi_1, \dots, \phi_k$ be k bounded functions for which*
$$|f(t) - f(s)| \leq \sum_{j=1}^{k} |\phi_j(t) - \phi_j(s)|, \tag{1.2.1}$$
for all $f \in F$ and $t, s \in Q$. Then F is relatively compact in $BC(Q)$.

Remark 1.2.4 *In roughly terms, theorem (1.2.3) states that if it is possible to control the oscillations for all functions of a bounded set F by a finite number of bounded functions (***not necessarily continuous***) then the closure of F is compact.*

The authors of [24] ask the following question: supposing that F satisfies the hypothesis of theorem (1.2.3) and said k the number of "control" functions, may $(k-1)$ functions be enough too?

De Pascale et al. give only a partial answer; they state that if the "control" functions ϕ_i are continuous the answer is negative.

Example 1.2.5 *[24]*

Let us consider the functions $f(t) = \frac{t}{t+1}$ and $g(t) = \sin t$ on the straight line $[0, +\infty)$. The set $F = \{f, g\}$ is clearly compact. Suppose that there exists a **continuous and bounded** *function ϕ that bounds the oscillation of f and g, this function must be necessarily injective.*

Since the set $A = \sin^{-1}\left[0, \frac{1}{2}\right]$ is an infinite union of separated intervals with Lebesgue's measure greater or equal to $\frac{1}{2}$ then our function ϕ will be necessarily unbounded.

The next theorems are the $\mathbb{R}^n$-version of theorem (1.2.3). The usual $\mathbb{R}^n-$norm will be indicate also with $|\cdot|$.

Theorem 1.2.6 *[24] Let $F \subset BC(Q, \mathbb{R}^n)$ be a bounded set. Let $\phi_1, \dots, \phi_k : Q \to \mathbb{R}^n$ be k bounded functions such that:*
$$|f(t) - f(s)| \leq \sum_{j=1}^{k} |\phi_j(t) - \phi_j(s)|,$$
for $t, s \in Q$ and $f \in F$. Then F is relatively compact in $BC(Q, \mathbb{R}^n)$.

To the end of paper [24] we find a compactness criterion for continuous nonlinear operator defined on $BC(Q,\mathbb{R}^n)$. This criterion appears very useful when one searches solutions of differential equations by fixed point methods (for example by means of Schauder theorem).

Theorem 1.2.7 *[24] Let $T : BC(Q,\mathbb{R}^n) \to BC(Q,\mathbb{R}^n)$ be a continuous operator. Suppose that for any bounded set $F \subset BC(Q,\mathbb{R}^n)$, $T(F)$ is a bounded set and there exist k bounded functions $\phi_1,\dots,\phi_k : Q \to \mathbb{R}^n$ such that:*

$$|(Tf)(t) - (Tf)(s)| \leq \sum_{j=1}^{k} |\phi_j(t) - \phi_j(s)|,$$

for all $t, s \in Q$ and for all $f \in F$. Then T is a compact operator.

Remark 1.2.8 *The meaning of theorem (1.2.7) is similar to theorem (1.2.3). To prove that an operator is compact is enough to be able to control the oscillations of every function in the image of a bounded set F by a finite number of bounded (***not necessarily continuous***) functions ϕ_i.*

Some other questions in [24]: can the boundedness of ϕ_j be dropped? Is the "control" condition (1.2.1) necessary for the compactness? Does there exist a similar condition that is also necessary?

The answer to the first question is negative and the authors give an example.

Example 1.2.9 *[24]*

Let us consider the sequence $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$f_n(t) = \begin{cases} 0 & t = 0 \\ \dfrac{1}{n}\,t & t \in [0,n] \\ 1 & t \geq n. \end{cases}$$

Any function of the set $F = (f_n)_{n\in\mathbb{N}}$ is an equilipschitzian mappings with constant equal to 1 and so the condition (1.2.1) is satisfied when $\phi(t) = t$. However F is not compact.

The answer to the second question is false and again in [24] we find an example.

We shall need of the following interesting combinatorial lemma:

Lemma 1.2.1 *[24] Let $\phi_1,\dots,\phi_k$ be k bounded functions on $[0,+\infty)$. Then for every sufficient large $\mu \in \mathbb{N}$ there exist two real number $s_1, s_2 \in [\mu,\mu+1)$ such that $|s_1 - s_2| \geq \frac{1}{\mu}$ and $\displaystyle\sum_{j=1}^{k} |\phi_j(s_1) - \phi_j(s_2)| < \frac{1}{\log \mu}$.*

Example 1.2.10 *([24])*

Let $BC([0,\infty))$ and F be a subset of $BC([0,\infty))$ for which:

- $Im(f) \subset [0,2]$;
- *for any $\nu \in \mathbb{N}, \nu \geq 2, |f(t)-1| \leq \frac{1}{\log \nu}$ for $t \geq \nu - 1$ and $|f(t)-f(s)| \leq \nu|t-s|$ for $t,s \leq \nu - 1$.*

The set F, which is equilipschitzian on all bounded subset of $[0,+\infty)$, is compact as in the uniform convergence topology on compact subset of $\mathbb{R}^+$, as in the sup *−norm topology (this last needed to prove).*

If there exist $\phi_1,\ldots,\phi_k$ bounded functions for which (1.2.1) holds, one can construct a function $g \in F$ such that g does not satisfy (1.2.1) for $s = s_1$ and $t = s_2$ when in previous lemma we choose $\mu \geq 3$ for any $f \in F$. This is a contradiction.

Regarding the third question the authors show that:

Theorem 1.2.11 *[24] Let $F \subset BC(Q)$ be a bounded set. Then F is relatively compact if and only if there exists a sequence of bounded functions $(\phi_j)_{j\in\mathbb{N}}$ such that:*

$$|f(t)-f(s)| \leq \sum_{j=1}^{\infty} |\phi_j(t)-\phi_j(s)|,$$

for $t,s \in Q,\ f \in F$ and the series is uniformly convergent.

Let us conclude this subsection mentioning that there are some generalizations of the previous theorem.

In 2002, Caponetti, Lewicki and Trombetta in [17] introduce the parameter ω_∞ in the function space L_0 [4]. This parameter measures the lack of equimeasurability in a similar way as in condition (1.2.1) for evey $f \in L_0$. Estimating the lack of compactness of any subset $A \subset L_0$ in terms of ω_∞ the authors are able to characterize their totally boundedness.

Here we do not examine the general case but we restrict our attention on a particular situation.

Definition 1.2.12 *[17] Let Ω be a topological space, (M,d) a pseudometric space, $\eta_\infty : \mathcal{P}(\Omega) \to [0,+\infty]$ a sub-measure defined by:*

$$\eta_\infty(G) = 0 \text{ if } \quad G = \emptyset,$$
$$\eta_\infty(G) = +\infty \text{ if } \quad G \neq \emptyset,$$

[4] for details on this space see [67]

and $F = \{f : \Omega \to M\}$. Let ρ_∞ be the pseudo-metric on F defined by:

$$\begin{aligned}\rho_\infty(f,g) &= \sup\{d(f(x),g(x)) : x \in \Omega\} \\ &= \inf\left\{a > 0 : \eta_\infty\{x \in \Omega : d(f(x),g(x)) \geq a\} \leq a\right\}.\end{aligned}$$

If $\mathcal{A}$ is an algebra on $\mathcal{P}(\Omega)$ we indicate with $\mathfrak{B}(\mathcal{A},\Omega,M) = \overline{\{f : f \text{ is } \mathcal{A}\text{-simple}\}}^{(F,\rho_\infty)}$

For any subset A in $\mathfrak{B}$, the parameter $\omega_\infty(A)$ can be written:

$$\omega_\infty(A) = \inf \Big\{ \varepsilon > 0 : \exists\, (\phi_j)_{j\in\mathbb{N}} \subset \mathfrak{B} | \\ \forall f \in A, \forall t,s \in \Omega,\ d(f(s),f(t)) \leq \varepsilon + \sum_{j=1}^{\infty} d(\phi_j(s),\phi_j(t)) \Big\}.$$

where the series converges uniformly in $\Omega \times \Omega$.

Let us denote by $BTC(\Omega, M)$ the subspace of $\mathfrak{B}$ containing all continuous functions for which $f(\Omega)$ is totally bounded. The authors show that:

Theorem 1.2.13 *[17] Let $A \subseteq BTC(\Omega,M)$ such that for any $f \in A$, $f(\Omega) \subset M_0$ with M_0 relatively compact. Then A is totally bounded if and only if there exists a sequence (ϕ_j) in $\mathfrak{B}$ such that:*

$$d(f(t),f(s)) \leq \sum_{j=1}^{\infty} d(\phi_j(t) - \phi_j(s)),$$

for any $t,s \in \Omega$, $f \in A$ and the series is uniformly convergent in $\Omega \times \Omega$.

It is natural to observe that if $M = \mathbb{R}$ then $BTC(\Omega,\mathbb{R}) = BC(\Omega)$ and so the previous generalize theorem (1.2.11) in [24].

1.3 Neutral and impulsive functional differential equations

To introduce the class of neutral functional differential equations and some related results we need of well-known facts of semigroups theory and fixed point theory.

The existence results following in this chapter rely on Schaefer theorem (1.1.2). We cite the definition in Hale and Verduyn Lunel:

Definition 1.3.1 *[43] An equation that depend on past and present values but that involve derivatives with delay as well as the function itself is called neutral functional differential equation. This class of differential equations is expressed by:*

$$\frac{d}{dt}[x(t) - F(t,x_t)] = G(t,x(t),x_t),$$

where x_t is the function defined on [-r,0], $r \geq 0$, by $x_t(\tau) = x(t+\tau)$ and F, G are appropriate functions.

Neutral differential equation theory has been extensively developed in literature. On this theory we recall Hale and Verduyn Lunel's book [43] and the reference therein, some papers of Dauer and Balachandran, Marino, Pietramala and Xu, and Hernández and Henríquez [44].

In some of these recent results, the authors obtained existence results for mild solutions on closed intervals; in some others the authors investigate existence of strong solutions on unbounded interval.

Definition 1.3.2 *[59] Let X a Banach space. A one parameter family $T(t)$, $0 \leq t < +\infty$ of bounded linear operator from X into X is a semigroup if:*

(i) $T(0) = Id$ *(the identity operator on X);*

(ii) $T(s+t) = T(s)T(t)$ *for all $t, s \geq 0$.*

Definition 1.3.3 *[59] The linear operator A defined by:*

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t}, \tag{1.3.1}$$

on

$$D(A) = \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists} \right\},$$

is the infinitesimal generator of the semigroup and $D(A)$ is its domain.

Definition 1.3.4 *[59] Let X be a Banach space. A semigroup $T(t)$, $0 \leq t < +\infty$, of bounded linear operators on X is a strongly continuous semigroup of bounded linear operators (in the sequel C_0-semigroup) if*

$$\lim_{t \downarrow 0} T(t)x = x, \qquad \forall x \in X. \tag{1.3.2}$$

Example 1.3.5 *[59]*
Let X the Banach space of bounded uniformly continuous functions on $\mathbb{R}$ with the supremum norm. For $f \in X$ we define:

$$(T(t)f)(s) = f(t+s).$$

It is very easy to verify (i) and (ii) of definition (1.3.2) and that $T(t)$ is a C_0-semigroup. The infinitesimal generator for $T(t)$ is defined on $D(A) = \{f \in X : \exists f' \in X\}$ and $(Af)(s) = f'(s)$ when $f \in D(A)$.

Definition 1.3.6 *[59] A C_0-semigroup $T(t)$ is called compact for $t > t_0$, if for every $t > t_0$, T(t) is a compact operator. $T(t)$ is called compact if it is compact for $t > 0$.*

Definition 1.3.7 *[59] Let $\Delta = \{z : \theta_1 < arg\, z < \theta_2, \theta_1 < 0 < \theta_2\}$ and for $z \in \Delta$, let $T(z)$ be a bounded linerar operator. The family $T(z)$, $z \in \Delta$, is an analytic semigroup in Δ if:*

(i) $z \to T(z)$ is analytic in Δ;

(ii) $T(0) = I$ and $\lim_{z\to 0} T(z)x = x$ for all $x \in X$;

(iii) $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$.

A semigroup $T(t)$ will be called analytic if it is analytic in some sector Δ containing the nonnegative real axis.

Afterward denoting with X a Banach space we restrict our considerations on the neutral integrodifferential equations:

$$\begin{cases} \dfrac{d}{dt}[x(t) - g(t, x_t)] = Ax(t) + f\left(t, x_t, \displaystyle\int_0^t h(t, s, x_s)ds\right), \qquad t \in [0, b] := J \\ \\ x|_{[-r,0]} = \phi \end{cases} \tag{1.3.3}$$

where A is the infinitesimal generator of a C_0-semigroup in a Banach space X, $h : J\times J\times C([-r, b], X) \to X$, $f : J\times C([-r, b], X)\times X \to X$ and $g : J\times C([-r, b], X) \to X$ are continuous functions.

All proofs in this thesis use fixed point methods; for the reader's convenience we summarize the steps of the method in the next sequential scheme:

0. *To define a nonlinear operator S;*
1. *To show that the fixed point of S are solution of the problem;*
2. *To prove that S is a continuous operator;*
3. *To prove that S is a compact operator;*
4. *To verify the Schaefer's alternative.*

Dauer and Balachandran in [23] prove an existence theorem for mild solutions of (1.3.3) and they apply it to prove the existence of solution for partial integrodifferential equations. They conclude with an application of the main theorem to the controllability problems for neutral systems.

Let us consider the integral equations:

$$\begin{aligned} x(t) = T(t)[\phi(0) - g(0,\phi)] + g(t,x_t) + \int_0^t AT(t-s)g(s,x_s)ds \\ + \int_0^t T(t-s)f\left(s, x_s, \int_0^s h(s,\tau,x_\tau)d\tau\right) ds \qquad t \in [0,b] = J. \end{aligned} \tag{1.3.4}$$

Definition 1.3.8 *[23] A function $x : [-r,b] \to X$, $b > 0$, is called mild solution of (1.3.3) if $x|_{[-r,0]} = \phi$, $x|_{[0,b)}$ is continuous, and for each $0 \le t < b$ the function $AT(t-s)g(s,x_s)$, $s \in [0,t)$, is integrable, and if the integral equation (1.3.4) is satisfies.*

Let us assume the following:

(i) A is the infinitesimal generator of a compact semigroup $T(t)$ in X such that

$$|T(t)| \le M_1 \quad (M_1 \ge 1) \text{ and } |AT(t)| \le M_2 \quad (M_2 > 0).$$

(ii) For each $(t,s) \in J \times J$, the function $h(t,s,\cdot) : C([-r,b],X) \to X$ is continuous and for each $x \in C$ the function $h(\cdot,\cdot,y) : J \times J \to X$ is strongly measurable.

(iii) For each $t \in J$ the function $f(t,\cdot,\cdot) : C([-r,b],X) \times X \to X$ is continuous, and for each $(x,y) \in C([-r,b],X) \times X$ the function $f(\cdot,x,y) : J \to X$ is strongly measurable.

(iv) For every positive integer k there exists $\alpha_k \in L^1(0,b)$ such that

$$\sup_{\|x\|,|y| \le k} |f(t,x,y)| \le \alpha_k(t), \qquad t \in J \text{ a.e.}.$$

(v) The function $g : J \times C \to X$ is completely continuous and, for any bounded set Q in $C([-r,b],X)$, the set $\{t \to g(t,x_t) : x \in Q\}$ is equicontinuous in $C([0,b],X)$.

(vi) There exist constants $c_1 < 1$ and $c_2 > 0$ such that

$$|g(t,\phi)| \le c_1\|\phi\| + c_2, \qquad t \in J, \phi \in C.$$

(vii) There exists an integrable function $m : [0,b] \to [0,\infty)$ and a constant $\alpha > 0$ such that

$$|h(t,s,x)| \le \alpha m(s)\Omega_0(\|x\|), \qquad 0 \le s < t \le b, \quad x \in C([-r,b],X),$$

where $\Omega_0[0,\infty) \to (0,\infty)$ is a continuous nondecreasing function.

(viii) There exists an integrable function $p : [0, b] \to [0, \infty)$ such that

$$|f(t, x, y)| \leq p(t)\Omega(\|x\| + |y|), \qquad 0 \leq t \leq b, x \in C([-r, b], X), y \in X,$$

where $\Omega : [0, \infty) \to (0, \infty)$ is a continuous nondecreasing function.

(ix)
$$\int_0^b \widehat{m}(s)ds < \int_c^\infty \frac{ds}{s + \Omega(s) + \Omega_0(s)},$$

where $c = \frac{1}{1 - c_1}[M_1((1 + c_1)\|\phi\| + c_2) + c_2 + M_2 c_2 b]$ and

$$\widehat{m}(t) = \max\left\{\frac{M_2 c_1}{1 - c_1}, \frac{M_1 p(t)}{1 - c_1}, \alpha m(t)\right\}.$$

In [23] the authors proved that:

Theorem 1.3.9 *[23] If the assumption $(i) - (ix)$ are satisfies, then the problem (1.3.3) has a mild solution on [-r,b].*

Example 1.3.10 *[23] Consider the partial integrodifferential equation:*

$$\begin{aligned} &\frac{\partial}{\partial t}[z(y, t) - p(t, z(y, t - r))] \\ &\quad = \frac{\partial^2}{\partial y^2} z(y, t) + q\left(t, z(y, (t - r)), \int_0^t k(t, s, z(y, s - r))ds\right), \end{aligned} \tag{1.3.5}$$

with $y \in [0, \pi], t \in [0, b]$ and

$$\begin{aligned} z(0, t) = z(\pi, t) = 0 \qquad & t \geq 0, \\ z(t, y) = \phi(y, t) \qquad & -r \leq t \leq 0, \end{aligned}$$

where ϕ is continuous, p, q, k are continuous and satisfy certain smoothness conditions.

Let $g(t, w_t)(y) = p(t, w(t-y))$, $h(t, s, w_s)(y) = k(t, s, w(s-y))$ and $f(t, w_t, v)(y) = q(t, w(t - y), v(y))$ when $y \in [0, \pi]$.

Let $X = L^2[0, \pi]$ and define $A : X \to X$ by $Aw = w''$ with domain $D(A) = \{w \in X : w, w'$ are absolutely continuous, $w'' \in X, w(0) = w(\pi) = 0\}$.

Then

$$Aw = \sum_{n=1}^{\infty} n^2 < w, w_n > w_n \qquad w \in D(A),$$

where $w_n(y) = \sqrt{2/\pi}\sin(ny)$, $n \in \mathbb{N}$ is the orthogonal set of eigenvectors of A.

One can verify that A is the infinitesimal generator of an analytic semigroup $T(t), t \geq 0$ given by:

$$T(t)w = \sum_{n=1}^{\infty} e^{-n^2 t} < w, w_n > w_n$$

Since every analytic semigroup $T(t)$ is compact, there exist a constant $N \geq 1$ and $N_1 > 0$ such that $|T(t)| \leq N$ and $|AT(t)| \leq N_1$. The function $p : [0,b] \times [0,\pi] \to [0,\pi]$ is completely continuous and uniformly bounded so there exists a constant $n_1 > 0$ for which $\|p(t, w(t-y))\| \leq n_1$.

The functions $k : [0,b] \times [0,b] \times [0,\pi]$ and $q : [0,b] \times [0,\pi] \times [0,\pi]$ are measurable and there exist two integrable functions $l_1, l_2 : [0,b] \to [0,\infty)$ and a constant $n_2 > 0$ such that $\|k(t,s,w)\| \leq \alpha l_1(s)\Omega_0(\|w\|)$ and $\|q(t,v,w)\| \leq l_2(t)\Omega(\|v\| + \|w\|)$ where $\Omega_0, \Omega[0,\infty) \to (0,\infty)$ are continuous and nondecreasing, and

$$\int_0^b \hat{n}(s)ds < \int_c^\infty \frac{ds}{s + \Omega_0(s) + \Omega(s)},$$

where $c = N(\|\phi\| + n_1) + n_1 + Nn_1 b$ and $\hat{n}(t) = \max\{Nl_2(t), n_2 l_1(t)\}$.

Since all the conditions of theorem (1.3.9) are satisfies, the equation (1.3.5) has a mild solution on $[-r, b]$.

Marino, Pietramala and Xu in [52] extend the previous results in a more general setting. First of all they consider an operator $A(t)$ time-dependent. Moreover they search for strong solutions defined on unbounded interval. This solutions belongs to $BC[-r,\infty)$.

Marino et al. consider the problem:

$$\begin{cases} \dfrac{d}{dt}[x(t) - g(t, x_t)] = A(t)x(t) + f\left(t, x_t, \displaystyle\int_0^t h(t,s,x_s)ds\right), \qquad t \geq 0 \\ \\ Lx = H(x) \end{cases} \tag{1.3.6}$$

where L is a bounded linear operator and H is a continuous bounded (in general nonlinear) operator.

The initial conditions considered in (1.3.6) permit to consider a large type of equations, for example boundary conditions, Cauchy problems (and so (1.3.2)), limit problems.

Let $\mathcal{A}$ is the algebra of real $n \times n$ matrices M with norm $|M|$= sup { $|My|$: $|y| = 1$}.

The following lemma in Conti [22] permits to characterize the solutions of a linear homogeneous differential system and the evolution operator $E(t,s)$ of $A(t)$.

Lemma 1.3.11 *Let $A : [0,+\infty) \to \mathcal{A}$, $t \mapsto A(t)$ be a bounded, integrable, continuous function. Then:*

(i) The linear homogeneous differential system

$$\dot{y} = A(t)y, \quad t \geq 0, \tag{1.3.7}$$

is uniformly stable; i.e., any solution of (1.3.7) *is in* $BC([0,+\infty);\mathbb{R}^n)$ *and is uniformly stable.*

(ii) There exists a fundamental matrix $X(t)$ *of solutions of* (1.3.7) *such that:*

$$\|X(t)X^{-1}(s)\| \leq M_2, \quad 0 \leq s \leq t < +\infty, \tag{1.3.8}$$

for some constant $M_2 > 0$ *and* $E(t,s) := X(t)X^{-1}(s)$ *is the evolution operator of* $A(t)$*. One has:*

$$E(t,s) = I + \int_s^t A(\tau)E(\tau,s)d\tau, \tag{1.3.9}$$

$$E(t,t) = I, \ \ E(t,s)E(s,r) = E(t,r), \quad t,s,r \in [0,+\infty) \tag{1.3.10}$$

$$\frac{\partial E}{\partial t} = A(t)E(t,s), \quad \frac{\partial E}{\partial s} = -E(t,s)A(s), \ \ \textit{a.e. } t,s \in [0,+\infty) \tag{1.3.11}$$

The main theorem contained in [52] states that:

Theorem 1.3.12 *Assume that the following hypotheses hold:*

(h_1) $A : [0,+\infty) \to \mathcal{A}$, $t \to A(t)$, *is a bounded* ($\|A(t)\| \leq M_1$, $\forall t \geq 0$)*, integrable continuous function for which*

- *(i) there exists a continuous nondecreasing function* $\gamma_1 : [0,+\infty) \to [0,+\infty)$ *such that*

$$\left| \int_0^t X^{-1}(s)A(s)g(s,y_s)ds \right| \leq \gamma_1(\|y\|_\infty)$$

- *(ii) there exists a continuous nondecreasing function* $\gamma_2 : [0,+\infty) \to [0,+\infty)$ *such that*

$$\left| \int_0^t X^{-1}(s)f\left(s, y_s, \int_0^s h(s,\tau,u_\tau)d\tau\right) ds \right| \leq \gamma_2(\|y\|_\infty).$$

(h_2) $g : [0,+\infty) \times C([-r,0),\mathbb{R}^n) \to \mathbb{R}^n$ *is a continuous function such that:*

- *(i)* $g(t,y_y) := G_y(t)$ *is a differentiable function for any* $y \in BC([-r,+\infty),\mathbb{R}^n)$*;*
- *(ii)* $|g(t,\psi)| \leq c_1\|\psi\|_\infty + c_2$ *with* $c_1 < 1$*;*
- *(iii)* $\|G_y(t) - G_x(t)\|_\infty \leq G\|x-y\|_\infty$ *for a certain constant* $G > 0$ *and for all* $x, y \in BC([-r,+\infty),\mathbb{R}^n)$*;*
- *(iv) for all* j *there exists* $\overline{G}_j : [0,+\infty) \to [0,+\infty)$ *a bounded function such that:*

$$|G_y(t_1) - G_y(t_2)| \leq |\overline{G}_j(t_1) - \overline{G}_j(t_2)|, \qquad \forall t_1, t_2 \geq 0, \|y\|_\infty \leq j.$$

(h_3) $f : [0,+\infty) \times C([-r,0],\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function such that:*

(i) there exists an integrable function $p : \mathbb{R}^+ \to \mathbb{R}^+$ *and a continuous nondecreasing function* $\Omega : [0, +\infty) \to [0, +\infty)$ *for which:*

$$|f(t, x, y)| \le p(t)\Omega(\|x\|_\infty + |y|), \quad t \ge 0, x \in C([-r, 0], \mathbb{R}^n), y \in \mathbb{R}^n;$$

(ii) for every positive integer j *there exists an integrable function* $\alpha_j : [0, +\infty) \to [0, +\infty)$ *such that:*

$$\sup_{\|x\|_\infty, |y| \le j} |f(t, x, y)| \le \alpha_j(t), \quad t \ge 0 \; a.e..$$

(h_4) $h : [0, +\infty) \times [0, +\infty) \times C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ *is a continuous function such that:*

(i) there exist an integrable function $m : [0, +\infty) \to [0, +\infty)$ *and a continuous function* $\Omega_0 : [0, +\infty) =\!\!\to [0, +\infty)$ *for which:*

$$|h(t, s, x)| \le m(s)\Omega_0(\|x\|_\infty), \quad 0 \le s \le t < +\infty, x \in C([-r, 0], \mathbb{R}^n);$$

(ii)

$$\int_0^\infty \frac{ds}{s + \Omega(s) + \Omega_0(s)} = +\infty.$$

(h_5) $L : BC([-r, +\infty), \mathbb{R}^n) \to C([-r, 0], \mathbb{R}^n)$ *is a bounded linear operator such that:*

(i) $\|L\| \le 1$;

(ii) If $u, v \in BC([-r, +\infty), \mathbb{R}^n)$ *are such that* $u|_{[-r,0]} = v|_{[-r,0]}$ *then* $Lu = Lv$.

(h_6) $H : BC([-r, +\infty), \mathbb{R}^n) \to C([-r, 0], \mathbb{R}^n)$ *is a continuous operator such that*

$$\|Hu\|_\infty \le M_3,$$

for a certain constant $M_3 > 1$.

(h_7) There exists a linear continuous operator $\tilde{K} : \mathbb{R}^n \to KerD$, *where* $D = \frac{d}{dt} - A(t)$ *such that:*

(i) $(\tilde{K}v)(0) = v$ *for all* $v \in \mathbb{R}^n$;

(ii) for every positive integer j *there exists a bounded function* $V_j : [0, +\infty) \to [0, +\infty)$ *for which:*

$$|(\tilde{K}(H(y))(0))(t_1) - (\tilde{K}(H(y))(0))(t_2)| \le |V_j(t_1) - V_j(t_2)|,$$

for all $y \in BC([-r, \infty), \mathbb{R}^n)$ *and* $t_1, t_2 \ge 0$;

(iii) define $K : C([-r,0],\mathbb{R}^n) \to BC([-r,+\infty],\mathbb{R}^n)$ *by:*

$$(Ku)(t) := \begin{cases} u(t) & -r \le t \le 0 \\ (\tilde{K}(u(0)))(t) & t \ge 0 \end{cases}$$

(Note that K *is a bounded linear operator). We suppose that*

$$H(u) = LKH(u), \qquad \forall u \in BC([-r,\infty),\mathbb{R}^n).$$

Then the problem (1.3.6) has at least one solution.

Proof.[Hint] To apply Schaefer theorem and so to prove theorem (1.3.12), Marino et al. follow the scheme in page 12:

0. The authors define a selfmapping nonlinear operator S on $BC([-r,\infty),\mathbb{R}^n)$.

1. Marino et al. show that the fixed points of S are solutions for (1.3.6).

2. They prove that S is a continuous operator.

3. To prove that S is compact, the authors use the compactness criterion (1.2.7) showing that it is possible to control the oscillations of the image of S by a finite number of bounded functions.

4. Marino et al. verify the Schaefer's alternative following [23].

□

Let us conclude this section introducing another class of differential equations that find remarkable applications in physics and engineering problems: the *impulsive differential equations*.

The interest in systems with discontinuous trajectories have grown in recent years because of the needs of modern technology where impulsive automatic control systems are intensively developing, broadening the scope of their applications in technical problems, heterogeneous by their physical nature and functional purpose.

The possibility of wide practical applications of impulsive differential equations (suitable source of mathematical models to simulate processes of this type) have become a very active area of research. We refer to the monographs by Bainov and Simeonov [3] and Samoilenko and Perestyuk [62] where extensive bibliographies are given.

To the best of our knowledge, we should note that, in spite of large number of investigations of impulsive differential equations, no many existence results have so far been established to solve the general problem.

In particular Benchohra, Henderson ad Ntouyas [7] recently obtain many results for first order impulsive functional differential equation on finite intervals:

$$\begin{cases} \dfrac{d}{dt}[x(t) - g(t, x_t)] = A(t)x(t) + f(t, x_t), \quad t \in [0, b], t \neq t_k, k = 1, \dots, l \\ \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \qquad k = 1 \dots, l \\ \\ x|_{[-r,0]} = \phi \end{cases} \tag{1.3.12}$$

and for second order impulsive neutral differential equation:

$$\begin{cases} \dfrac{d}{dt}[x'(t) - g(t, x_t)] = A(t)x(t) + f(t, x_t), \quad t \in [0, b], t \neq t_k, k = 1, \dots, l \\ \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), \qquad k = 1 \dots, l \\ \\ \Delta x'|_{t=t_k} = \bar{I}_k(x(t_k^-)), \qquad k = 1 \dots, l \\ \\ x|_{[-r,0]} = \phi \text{ and } y'(0) = \eta \end{cases} \tag{1.3.13}$$

always by fixed point methods.

To define the concept of mild solution for (1.3.12) let us consider the following space:

$$\begin{aligned} \Omega := \quad & \Big\{ y : [-r, b] \to X : y_k \in C([t_k, t_{k+1}], X), k = 0, \dots, l, \\ & \exists\, y(t_k^+), y(t_k^-) = y(t_k), k = 1, \dots, l, \ y|_{[-r,0]} = \phi \Big\}, \end{aligned}$$

which is a Banach space with the norm:

$$\|y\|_\Omega = \max\left\{ \|y_k\|_{[t_k,t_{k+1}]}, k = 1, \dots, l \right\}.$$

Definition 1.3.13 *[7] A function $y \in C([-r, b], X)$ is a mild solution for (1.3.12) if $y|_{[-r,0]} = \phi$, $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1 \dots, l$, the function $AT(t-s)g(s, y_s)$ is integrable for any $s \in [0, t)$ and*

$$\begin{aligned} y(t) = T(t)[\phi(0) - g(0, \phi)] + g(t, y_t) + \int_0^t AT(t-s)g(s, y_s)ds \\ + \int_0^t T(t-s)f(s, y_s)ds + \sum_{0<t_k<t} I_k(y(t_k)) \qquad t \in [0, b] \end{aligned} \tag{1.3.14}$$

Theorem 1.3.14 *[7] Supposing the following hypotheses:*

(H1) A is the infinitesimal generator for compact semigroup of bounded operators $T(t)$ in X such that:

$$|T(t)| \leq M_1, \qquad \text{for some } M_1 \geq 1$$
$$|AT(t)| \leq M_2, \quad \text{and } \; M_2 \geq 0 \; t \in [0, b].$$

(H2) There exist constant $0 \leq c_1 \leq 1$ and $c_2 \geq 0$ such that $|g(t,u)| \leq c_1\|u\| + c_2$, $t \in [0,b]$ and $u \in C([-r,0],X)$.

(H3) There exist constants d_k such that $|I_k(y)| \leq d_k$, $k = 1,\ldots,l$ for any $y \in X$.

(H4) $|f(t,u)| \leq p(t)\phi(\|u\|)$ for almost all $t \in [0,b]$ and all $u \in C([-r,0],X)$ where $p \in L_1([0,b],\mathbb{R}^+)$ and $\phi : \mathbb{R}^+ \to (0,+\infty)$ is continuous and increasing with:

$$\int_0^b \hat{m}(s)ds < \int_c^\infty \frac{d\tau}{\tau + \phi(\tau)},$$

where

$$c = \frac{1}{1-c_1}\left\{ M_1(\|\phi\| + c_1\|\phi\| + c_2) + c_2M_2b + c_2 + \sum_{k=1}^{l} d_k \right\},$$

and

$$\hat{m}(t) = \frac{1}{c_1}\{M_2c_1, M_1p(t)\}.$$

(H5) The function g is completely continuous and for any bounded set $D \subset \Omega$ the set $\{t \to g(t,y_t) : y \in D\}$ is equicontinuous in Ω.

Then (1.3.12) has at least one mild solution on $[-r,b]$.

In a similar way Benchohra et al. obtain an existence theorem for second order impulsive differential equation.

Let us consider the problem (1.3.13).

Definition 1.3.15 *[40] We say that a family $\{C(t) : t \in \mathbb{R}\}$ of bounded operator is a strongly continuous cosine family if:*

1. *$C(0) = Id$, i.e the identity operator;*
2. *$C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t,s \in \mathbb{R}$;*
3. *the map $t \to C(t)y$ is strongly continuous for each $y \in X$.*

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$ associated to the given strongly continuous cosine family is defined by:

$$S(t)y = \int_0^t C(s)yds, \quad y \in X, t \in \mathbb{R}.$$

The infinitesimal generator $A : X \to X$ of a cosine family is defined by:

$$Ay = \frac{d^2}{dt^2}C(t)y\Big|_{t=0}$$

Definition 1.3.16 *[7] A function $y \in C([-r, b], X)$ is a mild solution for (1.3.13) if $y|_{[-r,0]} = \phi$, $y'(0) = \eta$, $\Delta y|_{t=t_k} = I_k(y(t_k^-))$, $k = 1\dots, l$, $\Delta y'|_{t=t_k} = \bar{I}_k(y(t_k^-))$, $k = 1\dots, l$ and:*

$$\begin{aligned} y(t) = C(t)\phi(0) + S(t)[\eta - g(0, \phi)] + \int_0^t C(t-s)g(s, y_s)ds \\ + \int_0^t S(t-s)f(s, y_s)ds + \sum_{0<t_k<t} [I_k(y(t_k)) + (t - t_k)\bar{I}_k(y(t_k))] \qquad t \in [0, b]. \end{aligned} \tag{1.3.15}$$

It is not difficult to verify that (1.3.13) has solution in Ω if and only if y is solution of (1.3.15).

Thus we conclude with the following existence theorem:

Theorem 1.3.17 *Assume that:*

(A1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$ of bounded linear operators on X.

(A2) There exist constant c_1 and c_2 such that $|f(t, u)| \le c_1\|u\| + c_2$, $t \in [0, b]$ and $u \in C([-r, 0], X)$.

(A3) There exist constants $d_k, \bar{d}_k$ such that $|I_k(y)| \le d_k$, $|\bar{I}_k(y)| \le \bar{d}_k$, $k = 1, \dots, l$ for any $y \in X$.

(A4) $|f(t, u)| \le p(t)\psi(\|u\|)$ for almost all $t \in [0, b]$ and all $u \in C([-r, 0], X)$ where $p \in L_1([0, b], \mathbb{R}^+)$ and $\psi : \mathbb{R}^+ \to (0, +\infty)$ is continuous and increasing with:

$$\int_0^b \hat{m}(s)ds < \int_c^\infty \frac{d\tau}{\tau + \phi(\tau)},$$

where

$$c = \frac{1}{1 - c_1}\left\{M\|\psi\| + Mb[|\eta| + c_1\|\phi\| + c_2] + Mc_2b + \sum_{k=1}^{l}[d_k + (b - t_k)\bar{d}_k]\right\},$$

with $M = \sup\{|C(t)| : t \in \mathbb{R}\}$ *and*

$$\hat{m}(t) = \max\{Mc_1, Mp(t)\}.$$

(A5) The function g *is completely continuous and for any bounded set* $D \subset C([-r, 0], X)$ *the set* $\{t \to g(t, y_t) : y \in D\}$ *is equicontinuous in* $C([0, b], X)$*.*

(A6) $C(t)$*,* $t \in \mathbb{R}$ *is completely continuous.*

Then the problem (1.3.13) has al least one solution on $[-r, b]$*.*

1.4 Contributions to the problem

1.4.1 Introduction and notations

To conclude this chapter we present our contributions to the topics of neutral impulsive differential equation. This contributions consists in two existence theorems respectively to semilinear equation and integrodifferential equation.

Both improve the results obtained by Benchohra, Henderson and Ntouyas in [7] since the operator A is time-dependent, our solutions are strong solutions and these solutions are defined on a infinite intervals.

Since the methods used to obtain these results are similar we prefer only to present the most general situation: the integro-differential equation.

Thus, we are concerned with the existence of strong solutions of nonlinear integrodifferential equations with impulsive effects of the form:

$$\begin{aligned}
&\frac{d}{dt}[x(t) - g(t, x_t)] = A(t)x(t) + f\Big(t, x_t, \int_0^t h(t, s, x_s)ds\Big),\ t \in [0, +\infty),\\
&t \neq t_k,\ k = 1, \ldots, l\\
&x(t_k{}^+) - x(t_k{}^-) = I_k(x(t_k{}^-)), k = 1, \ldots, l
\end{aligned} \tag{1.4.1}$$

with the boundary condition:

$$Lx = H(x). \tag{1.4.2}$$

In literature the previous boundary condition are often a Cauchy condition of the form $x_0 = \phi$ with ϕ fixed.

The following notations will be used throughout the remainder of this section:

- $\mathbb{R}^n$ is the space of real n-vectors y with norm $|y|$ (not necessarily the Euclidean norm);

- $\mathcal{A}$ is the algebra of real $n \times n$ matrices M with norm $|M|$= sup { $|My|$: $|y| = 1\}$;

- $t \to A(t)$ is a function from $[0, +\infty)$ into $\mathcal{A}$;
- $0 < t_1 < \ldots < t_l$ are given points in $(0, +\infty)$;
- Q denotes the subset of $\mathbb{R}$, $Q = [-r, \infty) \setminus \{t_1, \ldots, t_l\}$;
- if y is a function defined on $[-r, +\infty)$ into $\mathbb{R}^n$, $y(t_k^-)$ and $y(t_k^+)$ denote the left and right limits of y at $t = t_k$ respectively;
- $BC(X, \mathbb{R}^n)$ is the Banach space of all continuous bounded functions y from a topological space X into $\mathbb{R}^n$, endowed with the norm $\|y\|_\infty := \sup\{|y(t)|,\ t \in X\}$;
- Ω is the space of all bounded functions $y : [-r, \infty) \to \mathbb{R}^n$ such that y is continuous on all $t \neq t_k,\ k = 1, \ldots l$, $y(t_k^+)$ and $y(t_k^-)$ exist and $y(t_k^-) = y(t_k)$ for $k = 1, \ldots, l$. Ω is a Banach space endowed with the norm $\| \cdot \|_\infty$;
- Λ denotes the space of all functions $y : [-r, 0] \to \mathbb{R}^n$ such that y is continuous everywhere except at a finite number of points, $y(\bar{t}^{\,+})$ and $y(\bar{t}^{\,-})$ exist for all $\bar{t} \in [-r, 0]$ and $y(\bar{t}^{\,-}) = y(\bar{t})$. Λ is a normed space endowed with the norm $\| \cdot \|_\infty$;
- for any function $y \in \Omega$ and any $t \geq 0$, y_t denotes the function in Λ defined by $y_t(\theta) := y(t + \theta)$, $\theta \in [-r, 0]$;
- $B(X)$ is the Banach space of all bounded linear operators T on the Banach space X with norm $\|T\| := \sup\{\|Tx\|,\ x \in X,\ \|x\| = 1\}$.

1.4.2 Preliminaries

The following lemma together with Schaefer theorem and lemma (1.3.11) will be crucial in the proof of the main result of this section.

Lemma 1.4.1 *Let $T : \Omega \to \Omega$ be a continuous operator. Suppose that for any bounded set $F \subset \Omega$, $T(F)$ is a bounded set and there exist ν bounded functions $\varphi_j : Q \to \mathbb{R}^{n_j}$, $j = 1, 2, \ldots, \nu$ such that, for all $t, s \in [-r, +\infty)$ and for all $y \in F$*

$$|(Ty)(t) - (Ty)(s)| \leq \sum_{j=1}^{\nu} |\varphi_j(t) - \varphi_j(s)|.$$

Then T is a completely continuous operator.

Proof. The Banach space Ω is isometric to the Banach space:

$$\widetilde{\Omega} = \{y \in BC(Q, \mathbb{R}^n)\text{: there exist } y(t_k^+) \text{ and } y(t_k^-) \text{ for all } k = 1, \ldots, l\}.$$

Of course $\widetilde{\Omega}$ is closed in $BC(Q, \mathbb{R}^n)$. So the thesis follows by proposition 4 in [24]. □

From (1.3.9) in lemma (1.3.11) follows that the solution y of (1.3.7) such that $y(0) = v$ is $y(t) = E(t, 0)v$, $t \geq 0$. Of course, the operator $E(\cdot, 0) : \mathbb{R}^n \to ker(D)$, where $D = \frac{d}{dt} - A(t)$, defined by

$$(E(\cdot, 0)v)(t) = E(t, 0)v$$

is a linear continuous compact operator.

In the sequel we denote by $\widetilde{E} : C([-r, 0]; \mathbb{R}^n) \to BC([-r, +\infty); \mathbb{R}^n)$ the linear continuous operator defined by

$$(\widetilde{E}u)(t) = \begin{cases} u(t), & -r \leq t \leq 0 \\ \\ E(t, 0)u(0), & t \geq 0 \end{cases}$$

1.4.3 Main result

Let us consider the problem: find $x \in \Omega$ such that:

$$\begin{cases} \frac{d}{dt}[x(t) - g(t, x_t)] = A(t)x(t) + f(t, x_t, \int_0^t h(t, s, x_s)ds), \\ \qquad\qquad t \geq 0,\ t \neq t_k,\ k = 1, \ldots, l \\ x({t_k}^+) - x({t_k}^-) = I_k(x({t_k}^-))\ \ k = 1, \ldots, l \\ Lx = H(x) \end{cases} \tag{1.4.3}$$

Theorem 1.4.2 *Assume that the following hypotheses hold:*

(h_1) $g : [0, +\infty) \times \Lambda \to \mathbb{R}^n$ *is a continuous function such that:*

(i) $g(t, y_t)$ *is a differentiable function with respect to variable t for any* $y \in \Omega$ *and* $t \neq t_1, \ldots, t_l$.

(ii) *There exist two constants* c_1, c_2*, with* $c_1 < 1$*, such that:*

$$|g(t, \psi)| \leq c_1 \|\psi\|_\infty + c_2.$$

(iii) $|g(t, y_t) - g(t, x_t)| \leq G\|x - y\|_\infty$ *for a certain constant* $G > 0$ *and for all* $x, y \in \Omega$*,* $t \in [0, +\infty)$*.*

(iv) $\forall j \in \mathbb{N}\ \exists\, \overline{G}_j : [0, +\infty) \to [0, +\infty)$ *a bounded continuous function such that:*

$$|g(s_1, y_{s_1}) - g(s_2, y_{s_2})| \leq |\overline{G}_j(s_1) - \overline{G}_j(s_2)|,$$

$\forall\, s_1, s_2 \geq 0$ *and* $\|y\|_\infty \leq j$*.*

(h_2) $f : [0, +\infty) \times \Lambda \times \mathbb{R}^n \to \mathbb{R}^n$ *is a continuous function such that there exist a continuous integrable function* $p : [0, +\infty) \to [0, +\infty)$ *and a continuous nondecreasing function* $\Psi : [0, +\infty) \to [1, +\infty)$ *for which*

$$|f(t, x, y)| \leq p(t)\Psi(\|x\|_\infty + |y|),\ t \geq 0,\ x \in \Lambda,\ y \in \mathbb{R}^n.$$

(h_3) $h : [0, +\infty] \times [0, +\infty) \times \Lambda \to \mathbb{R}^n$ *is a continuous function such that:*

(i) *there exist a continuous function* $\alpha : [0, +\infty] \to [0, +\infty)$ *and a continuous nondecreasing function* $\Psi_0 : [0, +\infty) \to [0, +\infty)$ *for which*

$$|h(t, s, x)| \leq \alpha(s)\Psi_0(\|x\|_\infty),\ 0 \leq s \leq t < +\infty,\ x \in \Lambda$$

(ii)

$$\int_0^\infty \frac{ds}{s + \Psi(s) + \Psi_0(s)} = +\infty.$$

(h_4) $A : [0, +\infty) \to \mathcal{A},\ t \mapsto A(t)$ *is a bounded* $(\|A(t)\| \leq M_1, \forall t \geq 0)$ *integrable continuous function for which:*

(i) *there exists a continuous nondecreasing function* $\gamma_1 : [0, +\infty) \to [0, +\infty)$ *such that*

$$\Big|\int_0^t X^{-1}(s)A(s)g(s, y_s)ds\Big| \leq \gamma_1(\|y\|_\infty);$$

(ii) *there exists a continuous nondecreasing function* $\gamma_2 : [0, +\infty) \to [0, +\infty)$ *such that*

$$\Big|\int_0^t X^{-1}(s)f(s, y_s, \textstyle\int_0^s h(s, \tau, y_\tau)d\tau)ds\Big| \leq \gamma_2(\|y\|_\infty).$$

(h_5) $L : \Omega \to C([-r, 0], \mathbb{R}^n)$ *is a bounded linear operator for which if* $u, v \in \Omega$ *are such that* $u|_{[-r,0]} = v|_{[-r,0]}$ *then* $Lu = Lv$.

$H : \Omega \to C([-r, 0]; \mathbb{R}^n)$ *is a continuous compact operator such that*

$$\|H(u)\|_\infty \leq M_3,$$

for a certain constant M_3 *and* $H(u) = (L\widetilde{E}H)(u)$ *for* $u \in \Omega$.

(h_6) The functions $I_k : \mathbb{R}^n \to \mathbb{R}^n$, $k = 1, \ldots, l$ *are continuous and bounded,* $|I_k(v)| \leq D$, $\forall\, k = 1, \ldots, l$, $\forall v \in \mathbb{R}^n$.

Then the problem (1.4.1) has at least one solution.

To prove Theorem (1.4.2), first we suitably extend to $[-r, 0]$ the functions useful for us, putting, for $u \in \Omega$:

$$\overline{g}(t, u_t) := \begin{cases} g(t, u_t) & t \geq 0 \\ g(0, u_0) & -r \leq t \leq 0 \end{cases}$$

$$\overline{E}(t, 0)g(0, u_0) := \begin{cases} E(t, 0)g(0, u_0) & t \geq 0 \\ g(0, u_0) & -r \leq t \leq 0 \end{cases}$$

and define, for $u \in \Omega$,

$$w_u(t) := \begin{cases} \int_0^t E(t, s)A(s)g(s, u_s)ds & t \geq 0 \\ 0 & -r \leq t \leq 0 \end{cases}$$

$$z_u(t) := \begin{cases} \int_0^t E(t, s)f(s, u_s, \int_0^s h(s, \tau, u_\tau)d\tau)ds & t \geq 0 \\ 0 & -r \leq t \leq 0 \end{cases}$$

$$(Iu)(t) := \begin{cases} \sum_{t_k<t} E(t,t_k)I_k(u(t_k)) & t > t_1 \\ \\ 0 & -r \le t \le t_1 \end{cases}$$

Lemma 1.4.3 *For any $u \in \Omega$ let Su be the function defined by*

$$Su = \widetilde{E}H(u) + \overline{g}(\cdot, u.) - \overline{E}(\cdot, 0)g(0, u_0) + w_u + z_u + Iu.$$

Then $Su \in \Omega$ and $(Su)(t_k^+) - (Su)(t_k^-) = I_k(u(t_k))$.

Proof. We need to show that for $k = 1, \dots, l$, there exist $(Su)(t_k^+)$ and $(Su)(t_k^-) = (Su)(t_k)$.
First of all, $\widetilde{E}H(u)$ and $\overline{E}(\cdot, 0)g(0, u_0)$ are continuous functions on $[-r, +\infty)$, so:

$$(\widetilde{E}H(u))(t_k^+) = (\widetilde{E}H(u))(t_k^-) = (\widetilde{E}H(u))(t_k),$$

$$\overline{E}(t_k^-, 0)g(0, u_0) = \overline{E}(t_k^+, 0)g(0, u_0) = \overline{E}(t_k, 0)g(0, u_0).$$

Besides:

- $$\lim_{t \to t_k} g(t, u_t) = g(t_k, u_{t_k}).$$

  Indeed

  $$|g(t, u_t) - g(t_k, u_{t_k})| \le |\overline{G}_j(t) - \overline{G}_j(t_k)| \text{ for } j \ge \|u\|_\infty$$

  so the statement follows by $(h_1)(iv)$.

- $$\lim_{t \to t_k} w_u(t) = w_u(t_k).$$

Indeed

$$\begin{aligned} & |w_u(t) - w_u(t_k)| \\ = & \Big|\int_0^t E(t,s)A(s)g(s,u_s)ds - \int_0^{t_k} E(t_k,s)A(s)g(s,u_s)ds\Big| \\ \leq & \int_0^t \|E(t,s) - E(t_k,s)\| \cdot \|A(s)\| \cdot |g(s,u_s)|ds \\ & + \Big|\int_t^{t_k} \|E(t_k,s)\| \cdot \|A(s)\| \cdot |g(s,u_s)|ds\Big| \\ & \qquad\qquad \text{(by } (h_4), (h_1)\text{(ii), lemma (1.3.11)(ii))} \\ \leq & M_1(c_1\|u\|_\infty + c_2)\int_0^t \|E(t,s) - E(t_k,s)\|ds \\ & + M_1M_2(c_1\|u\|_\infty + c_2)|t_k - t| \to 0 \end{aligned}$$

for $t \to t_k$.

- $$\lim_{t\to t_k} z_u(t) = z_u(t_k).$$

  Indeed

$$\begin{aligned} & |z_u(t) - z_u(t_k)| \\ \leq & \int_0^t \|E(t,s) - E(t_k,s)\| \cdot \Big|f\Big(s, u_s, \int_0^s h(s,\tau,u_\tau)d\tau\Big)\Big|ds \\ & + \Big|\int_t^{t_k} \|E(t_k,s)\|\Big|f(s,u_s,\textstyle\int_0^s h(s,\tau,u_\tau)d\tau)\Big|ds\Big| \text{ (by } (h_2) \text{ and } (h_3)\text{(i))} \\ \leq & \int_0^t \|E(t,s) - E(t_k,s)\| \cdot \Psi(\|u\|_\infty + \textstyle\int_0^\infty \alpha(\tau)\Psi_0(\|u\|_\infty)d\tau)p(s)ds \\ & + M_2\Psi\Big(\|u\|_\infty + \int_0^\infty \alpha(\tau)\Psi_0(\|u\|_\infty)d\tau\Big)\Big|\int_t^{t_k} p(s)ds\Big| \to 0 \text{ for } t \to t_k. \end{aligned}$$

- $$(Iu)(t_k^-) = (Iu)(t_k) = \sum_{j=1}^{k-1} E(t_k,t_j)I_j(u(t_j)),$$

  and

$$(Iu)(t_k^+) = \sum_{j=1}^{k} E(t_k,t_j)I_j(u(t_j)).$$

Indeed let $t_{k-1} < t < t_k$. Then:

$$\begin{aligned} & |(Iu)(t) - (Iu)(t_k)| \\ = & \Big|\sum_{t_j<t} E(t,t_j)I_j(u(t_j)) - \sum_{t_j<t_k} E(t_k,t_j)I_j(u(t_j))\Big| \\ \leq & \text{ (by } (h_6)) \leq D\sum_{j=1}^{k-1} \|E(t,t_j) - E(t_k,t_j)\| \to 0. \end{aligned}$$

Analogously one can see that:

$$(Iu)(t_k^+) = \sum_{j=1}^{k} E(t_k,t_j)I_j(u(t_j)),$$

in such a way that:

$$(Iu)(t_k^+) - (Iu)(t_k^-) = I_k(u(t_k)).$$

Finally, by the previous steps, it follows that

$$(Su)(t_k^+) - (Su)(t_k^-) = I_k(u(t_k)).$$

□

Proof of Theorem 3.1 . Consider the operator $S : \Omega \to \Omega$ defined in lemma (1.4.3). We will show that the fixed points of S are solutions of problem (1.4.3) and that S is a continuous compact operator for which the set $\zeta(S)$ is bounded. This is sufficient, by the Schaefer's theorem, to conclude that S has fixed points.

STEP 1. *The fixed points of S are solutions of problem (1.4.3).*

Proof of Step 1 . Let $u \in \Omega$. Then, by the linearity of L,

$$\begin{aligned} L(S(u)) &= L(\widetilde{E}H(u)) + L(\overline{g}(\cdot,u.)) - L(\overline{E}(\cdot,0)g(0,u_0)) \\ &\quad +L(w_u) + L(z_u) + L(Iu) \end{aligned}$$

Now, since

$$w_u|_{[-r,0]} = z_u|_{[-r,0]} = Iu|_{[-r,0]} = 0,$$
$$\overline{g}(\cdot,u.)|_{[-r,0]} = \overline{E}(\cdot,0)g(0,u_0)|_{[-r,0]} = g(0,u_0)$$

we obtain:

$$L(Su) = L(\widetilde{E}H(u)),$$

and (h_5) yields

$$L(Su) = H(u). \tag{1.4.4}$$

Moreover, lemma (1.4.3) ensures that

$$(Su)(t_k^+) - (Su)(t_k^-) = I_k(u(t_k)). \tag{1.4.5}$$

Finally, for $t \geq 0$, and $t \neq t_1, \dots, t_l$,

$$\begin{aligned} \frac{d}{dt}(Su)(t) \quad &= \quad A(t)(\widetilde{E}(H(u)))(t) + \frac{d}{dt}g(t, u_t) - A(t)E(t,0)g(0, u_0) \\ &\quad +A(t)g(t, u_t) + A(t)w_u(t) + f(t, u_t, \textstyle\int_0^t h(t, s, u_s)ds) \\ &\quad +A(t)z_u(t) + A(t)(Iu)(t), \end{aligned}$$

so:

$$\frac{d}{dt}[(Su)(t) - g(t, u_t)] = A(t)(Su)(t) + f(t, u_t, \textstyle\int_0^t h(t, s, u_s)ds). \tag{1.4.6}$$

If x is a fixed point of S, $x = Sx$, (1.4.4), (1.4.5), (1.4.6) ensure that problem (1.4.3) is solved by such a fixed point.

STEP 2. *S is a continuous operator.*

Proof of Step 2. Let $\{u_m\} \subset \Omega$ be such that $u_m \stackrel{\|\cdot\|_\infty}{\longrightarrow} u$.
We show that $Su_m \stackrel{\|\cdot\|_\infty}{\longrightarrow} Su$. By definition of S, it is sufficient to show that:

$$\widetilde{E}(H(u_m)) \stackrel{\|\cdot\|_\infty}{\longrightarrow} \widetilde{E}(H(u)), \tag{1.4.7}$$

$$\overline{g}(\cdot, (u_m).) \stackrel{\|\cdot\|_\infty}{\longrightarrow} \overline{g}(\cdot, u.), \tag{1.4.8}$$

$$\overline{E}(\cdot, 0)g(0, (u_m)_0) \stackrel{\|\cdot\|_\infty}{\longrightarrow} \overline{E}(\cdot, 0)g(0, u_0), \tag{1.4.9}$$

$$w_{u_m} \stackrel{\|\cdot\|_\infty}{\longrightarrow} w_u, \tag{1.4.10}$$

$$z_{u_m} \stackrel{\|\cdot\|_\infty}{\longrightarrow} z_u, \tag{1.4.11}$$

$$Iu_m \stackrel{\|\cdot\|_\infty}{\longrightarrow} Iu. \tag{1.4.12}$$

Now, (1.4.7) is an immediate consequence of the continuity of the operators $\widetilde{E}$ and H.
To obtain (1.4.8), we note, by $(h_1)(iii)$, that

$$|\overline{g}(t, (u_m)_t) - \overline{g}(t, u_t)| \leq G\|u_m - u\|_\infty \to 0.$$

To obtain (1.4.9), we note that:

$$\|\overline{E}(\cdot,0)g(0,(u_m)_0)-\overline{E}(\cdot,0)g(0,u_0)\|_\infty=\|E(\cdot,0)[g(0,(u_m)_0)-g(0,u_0)]\|_\infty$$

$$\leq \text{(by lemma (1.3.11)(ii))} \quad \leq \quad M_2\cdot|g(0,(u_m)_0)-g(0,u_0)|\to 0 \;\; \text{(by } (h_1)\text{)}.$$

To obtain (1.4.10), using lemma (1.3.11) (ii) and $(h_1)(iii)$ we note that:

$$\begin{aligned}\|w_{u_m}-w_u\|_\infty &= \sup_{-r\leq t<\infty}|w_{u_m}(t)-w_u(t)|=\sup_{0\leq t<\infty}|w_{u_m}(t)-w_u(t)| \\ &= \Big\|\int_0^t E(t,s)A(s)[g(s,(u_m)_s)-g(s,u_s)]ds\Big\|_\infty \\ &\leq M_2G\|u_m-u\|_\infty\int_0^\infty\|A(s)\|ds\to 0 \text{ (by } (h_1)(iii)).\end{aligned}$$

To obtain (1.4.11), we note first that by the continuity of f and h it follows that

$$\xi_m(s)=f(s,(u_m)_s,\textstyle\int_0^s h(s,\tau,(u_m)_\tau)d\tau)-f(s,u_s,\int_0^s h(s,\tau,u_\tau)d\tau)\to 0,$$

pointwise. Moreover, since $\|u_m-u\|_\infty\to 0$, there exists $j\in\mathbb{N}$ such that $\|u_m\|_\infty$, $\|u\|_\infty\leq j$. So, by $(h_3)(i)$,

$$\Big|\int_0^s h(s,\tau,(u_m)_\tau)d\tau\Big|\leq\int_0^s\alpha(\tau)\Psi_0(\|u_m\|_\infty)d\tau\leq\Psi_0(j)\int_0^\infty\alpha(\tau)d\tau,$$

$$\Big|\int_0^s h(s,\tau,u_\tau)d\tau\Big|\leq\Psi_0(j)\int_0^\infty\alpha(\tau)d\tau$$

Put:

$$j':=\max\Big\{j,\Big[\Psi_0(j)\int_0^\infty\alpha(\tau)d\tau\Big]+1\Big\}, \tag{1.4.13}$$

where we use $[r]$ to denote the integer part of a positive real number r. Then by (h_2) we see that:

$$\begin{aligned}|\xi_m(s)| &\leq \Big|f\Big(s,(u_m)_s,\int_0^s h(s,\tau,(u_m)_\tau)d\tau\Big)\Big|+\Big|f\Big(s,u_s,\int_0^s h(s,\tau,u_\tau)d\tau\Big)\Big| \\ &\leq 2p(s)\Psi(2j')\end{aligned}$$

i.e. the sequence $\{\xi_m\}$ converges pointwise to 0 and it is dominated by a summable function. Hence, by lemma (1.3.11) (ii),

$$\begin{aligned}\|z_{u_m}-z_u\|_\infty &= \sup_{-r\leq t<\infty}|z_{u_m}(t)-z_u(t)|=\sup_{0\leq t<\infty}|z_{u_m}(t)-z_u(t)| \\ &= \Big\|\int_0^t E(t,s)\xi_m(s)ds\Big\|_\infty\leq M_2\int_0^\infty|\xi_m(s)|ds\to 0,\end{aligned}$$

by the dominated convergence theorem.
Finally, to obtain (1.4.12), we note that

$$\begin{aligned}
\|Iu_m - Iu\|_\infty &= \sup_{-r\le t<\infty} |(Iu_m)(t) - (Iu)(t)| \\
&= \sup_{t_1<t<\infty} |(Iu_m)(t) - (Iu)(t)| \\
&= \sup_{t_1<t<\infty} \Big| \sum_{t_k<t} E(t,t_k) I_k(u_m(t_k)) - \sum_{t_k<t} E(t,t_k) I_k(u(t_k)) \Big| \\
&\le \sup_{t_1<t<\infty} \sum_{t_k<t} \Big| E(t,t_k)[I_k(u_m(t_k)) - I_k(u(t_k))] \Big| \\
&\le M_2 \sum_{k=1}^{l} |I_k(u_m(t_k)) - I_k(u(t_k))| \to 0,
\end{aligned}$$

by (h_6).

STEP 3. *S is a compact operator.*

Proof of Step 3 . Let $T = S - \widetilde{E}H$. Since $\widetilde{E}H$ is a compact operator, it is enough to prove that T is a compact operator. Let

$$B_j = \{u \in \Omega \; : \; \|u\|_\infty \le j\}.$$

Thanks to lemma (1.4.1), it is enough to show that $T(B_j)$ is bounded set and that it is possible to control the oscillations of each function in $T(B_j)$ by means of a finite number of bounded functions. The boundedness of $T(B_j)$ follows from the inequalities:

$$|\overline{g}(t,u_t)| \le c_1 j + c_2, \qquad \text{(by (h}_1\text{)(ii))}$$

$$|\overline{E}(t,0) g(0,u_0)| \le M_2(c_1 j + c_2),$$

$$\|w_u\|_\infty \le M_2(c_1 j + c_2) \int_0^\infty \|A(s)\| ds, \qquad \text{(by (h}_4\text{))}$$

$$\|z_u\|_\infty \le M_2 \int_0^\infty p(t)\Psi(2j')dt \qquad \text{(by (h}_2\text{) and lemma(1.3.11)(ii))}$$

where j' is defined in (1.4.13),

$$\|Iu\|_\infty \le M_2 \sum_{k=1}^{l} |I_k(u(t_k))| \le M_2 l D \qquad \text{(by (h}_6\text{))} \quad .$$

To control the oscillations of $T(B_j)$ we need to distinguish three cases: $0 \le \tau_1 < \tau_2$, $\tau_1 < 0 < \tau_2$ and $\tau_1 < \tau_2 \le 0$.

We consider here only the case $0 \leq \tau_1 < \tau_2$, the others being simpler.
Let us assume $0 \leq \tau_1 < \tau_2$ and $u \in B_j$. Then:

$$\begin{aligned}
\Big|(T(u))(\tau_1) - (T(u))(\tau_2)\Big| &\leq \Big|g(\tau_1, u_{\tau_1}) - g(\tau_2, u_{\tau_2})\Big| \\
&+ |[E(\tau_1, 0) - E(\tau_2, 0)]g(0, u_0)| \\
&+ \Big|\int_0^{\tau_1} [E(\tau_1, s) - E(\tau_2, s)]A(s)g(s, u_s)ds\Big| \\
&+ \Big|\int_{\tau_1}^{\tau_2} E(\tau_2, s)A(s)g(s, u_s)\Big| \\
&+ \Big|\int_0^{\tau_1} [E(\tau_1, s) - E(\tau_2, s)] \cdot \\
&\quad \cdot f(s, u_s, \textstyle\int_0^s h(s, \tau, u_\tau)d\tau)ds\Big| \\
&+ \Big|\int_{\tau_1}^{\tau_2} E(\tau_2, s)f(s, u_s, \textstyle\int_0^s h(s, \tau, u_\tau)d\tau)ds\Big| \\
&+ \Big|(Iu)(\tau_1) - (Iu)(\tau_2)\Big|.
\end{aligned}$$

Now:

$$|g(\tau_1, u_{\tau_1}) - g(\tau_2, u_{\tau_2})| \leq |\overline{G}_j(\tau_1) - \overline{G}_j(\tau_2)| \quad \text{(by } (h_1)(iv)\text{)},$$

$$\Big|[E(\tau_1, 0) - E(\tau_2, 0)]g(0, u_0)\Big| \leq \|E(\tau_1, 0) - E(\tau_2, 0)\|(c_1 j + c_2),$$

$$\Big|\int_0^{\tau_1} [E(\tau_1, s) - E(\tau_2, s)]A(s)g(s, u_s)ds\Big| \leq \|X(\tau_1) - X(\tau_2)\|\gamma_1(j) \quad \text{(by } (h_4)(i)\text{)},$$

$$\Big|\int_{\tau_1}^{\tau_2} E(\tau_2, s)A(s)g(s, u_s)ds\Big| \leq M_2(c_1 j + c_2)\Big[\int_0^{\tau_2} \|A(s)\|ds - \int_0^{\tau_1} \|A(s)\|ds\Big],$$

$$\Big|\int_0^{\tau_1} [E(\tau_1, s) - E(\tau_2, s)]f(s, u_s, \textstyle\int_0^s h(s, \tau, u_\tau)d\tau)ds\Big| \leq \|X(\tau_1) - X(\tau_2)\|\gamma_2(j) \quad \text{(by } (h_4)(ii)\text{)},$$

$$\Big|\int_{\tau_1}^{\tau_2} E(\tau_2, s)f(s, u_s, \textstyle\int_0^s h(s, \tau, u_\tau)d\tau)ds\Big| \leq M_2\Psi(2j')\Big[\int_0^{\tau_2} p(s)ds - \int_0^{\tau_1} p(s)ds\Big].$$

To control the oscillations of $|(Iu)(\tau_1) - (Iu)(\tau_2)|$, we consider the following sub-cases involving only the first point of jump:

1. $0 \le \tau_1 < \tau_2 < t_1$. In this case we have $(Iu)(\tau_1) = (Iu)(\tau_2) = 0$.

2. $0 < \tau_1 \le t_1 < \tau_2$. By lemma (1.3.11) and (h_6) we obtain

$$|(Iu)(\tau_2) - (Iu)(\tau_1)| = \Big|\sum_{t_k<\tau_2} E(t,t_k)I_k(u(t_k))\Big| \le D\sum_{t_k<\tau_2} M_2.$$

3. $0 < t_1 < \tau_1 < \tau_2$. Then

$$\begin{aligned}\big|(Iu)(\tau_1) - (Iu)(\tau_2)\big| &= \Big|\sum_{t_k<\tau_1} E(\tau_1,t_k)I_k(u(t_k)) \\ &\quad - \sum_{t_k<\tau_2} E(\tau_2,t_k)I_k(u(t_k))\Big| \\ &\le \Big|\sum_{t_k<\tau_1} [E(\tau_1,t_k) - E(\tau_2,t_k)]I_k(u(t_k))\Big| \\ &\quad + \Big|\sum_{\tau_1\le t_k<\tau_2} E(\tau_2,t_k)I_k(u(t_k))\Big| \\ &\le D\sum_{k=1}^{l} \|E(\tau_1,t_k) - E(\tau_2,t_k)\| \\ &\quad + D\Big[\sum_{t_k<\tau_2} M_2 - \sum_{t_k<\tau_1} M_2\Big].\end{aligned}$$

So, if we define:

$$\phi_1(t) = \overline{G}_j(t), \quad \phi_2(t) = (c_1 j + c_2)E(t,0), \quad \phi_3(t) = \gamma_1(j)X(t),$$

$$\phi_4(t) = M_2(c_1 j + c_2)\int_0^t \|A(s)\|ds, \quad \phi_5(t) = \gamma_2(j)X(t),$$

$$\phi_6(t) = M_2\Psi(2j')\int_0^t p(s)ds, \quad \phi_7(t) = DE(t,t_1), \quad \phi_8(t) = DE(t,t_2), \dots$$

$$\dots, \phi_{l+6}(t) = DE(t,t_l), \quad \phi_{l+7}(t) = \begin{cases} D\sum_{t_k<t} M_2, & t > t_1 \\ 0, & t \le t_1 \end{cases}$$

we obtain that:

$$|(Tu)(\tau_1) - (Tu)(\tau_2)| \le \sum_{k=1}^{l+7} |\phi_k(\tau_1) - \phi_k(\tau_2)| \text{ for all } u \in B_j$$

in such a way that the thesis follows from lemma (1.4.1).

STEP 4. *The set*

$$\zeta(S) := \{u \in \Omega : u = \lambda Su \text{ for some } 0 < \lambda < 1\}$$

is bounded.

Proof of Step 4 . This is based on the idea in [23].
First of all, if $-r \leq t < 0$, then $(Su)(t) = (H(u))(t)$. So, by (h_5) it follows that $|(Su)(t)| \leq M_3$ for each $u \in \Omega$. Hence we only consider $t \geq 0$. Let $u(t) = \lambda(Su)(t)$. Thus:

$$\begin{aligned}|u(t)| = |\lambda||(Su)(t)| \leq |(Su)(t)| \leq |(\widetilde{E}H(u))(t)| + |g(t, u_t)| \\ +|E(t,0)g(0,u_0)| + \Big|\int_0^t E(t,s)A(s)g(s,u_s)ds\Big| \\ +\Big|\int_0^t E(t,s)f(s,u_s,\textstyle\int_0^s h(s,\tau,u_\tau)d\tau)ds\Big| + \Big|\displaystyle\sum_{t_k<t} E(t,t_k)I_k(u(t_k))\Big|.\end{aligned}$$

But, $\|u_0\|_\infty = \sup_{-r\leq\theta\leq 0}|u(\theta)|$ and $|u(\theta)| = |\lambda||(Su)(\theta)| \leq M_3$, so it follows that

$$\begin{aligned}|u(t)| \leq \|\widetilde{E}\|M_3 + c_1\|u_t\|_\infty + c_2 + M_2(c_1M_3 + c_2) \\ +M_2\int_0^t (c_1\|u_s\|_\infty + c_2)\|A(s)\|ds \\ +M_2\int_0^t p(s)\Psi(\|u_s\|_\infty + \textstyle\int_0^s \alpha(\tau)\Psi_0(\|u_\tau\|_\infty)d\tau)ds + M_2lD.\end{aligned}$$

Hence for all $t \in [-r, +\infty)$, we have:

$$\begin{aligned}|u(t)| \leq M_3 + \|\widetilde{E}\|M_3 + c_1\|u_t\|_\infty + c_2 + M_2(c_1M_3 + c_2) \\ +M_2\int_0^t (c_1\|u_s\|_\infty + c_2)\|A(s)\|ds \\ +M_2\int_0^t p(s)\Psi(\|u_s\|_\infty + \textstyle\int_0^s \alpha(\tau)\Psi_0(\|u_\tau\|_\infty)d\tau)ds + M_2lD. \qquad (1.4.14)\end{aligned}$$

Consider the function $\mu : [0, +\infty) \to [0, +\infty)$ defined by

$$\mu(t) := \sup\{|u(\xi)| : -r \leq \xi \leq t\}.$$

Observe that μ is not necessarily continuous only in the points t_k but $\mu(t_k^+)$ and $\mu(t_k^-) = \mu(t_k)$ exist, $k = 1, \dots, l$.
Then for $\xi \in [-r, t]$ from (1.4.14) we have:

$$|u(\xi)| \leq M_3 + \|\widetilde{E}\| M_3 + c_1 \|u_\xi\|_\infty + c_2 + M_2(c_1 M_3 + c_2) + M_2 \int_0^\xi (c_1 \|u_s\|_\infty + c_2)\|A(s)\| ds + M_2 \int_0^\xi p(s)\Psi(\|u_s\|_\infty + \textstyle\int_0^s \alpha(\tau)\Psi_0(\|u_\tau\|_\infty) d\tau) ds + M_2 l D. \tag{1.4.15}$$

Now, note that for all $\eta \in [0, t]$

$$\|u_\eta\|_\infty = \sup_{-r \leq \theta \leq 0} |u(\eta + \theta)| = \sup_{-r+\eta \leq \xi \leq \eta} |u(\xi)| \leq \sup_{-r \leq \xi \leq \eta} |u(\xi)| = \mu(\eta)$$

so, taking the $\sup_{-r \leq \xi \leq t}$ in the inequality (1.4.15), we obtain:

$$\mu(t) \leq M_3 + \|\widetilde{E}\| M_3 + c_1 \mu(t) + c_2 + M_2(c_1 M_3 + c_2) + c_1 M_2 \int_0^t \|A(s)\| \mu(s) ds + c_2 M_2 \int_0^t \|A(s)\| ds + M_2 \int_0^t p(s)\Psi(\mu(s) + \textstyle\int_0^s \alpha(\tau)\Psi_0(\mu(\tau)) d\tau) ds + M_2 l D.$$

that implies

$$\mu(t) \leq \frac{1}{1 - c_1} \Big[M_3 + \|\widetilde{E}\| M_3 + c_2 + M_2(c_1 M_3 + c_2) + c_1 M_2 \int_0^t \|A(s)\| \mu(s) ds + \ + c_2 M_2 \int_0^\infty \|A(s)\| ds + M_2 \int_0^t p(s)\Psi(\mu(s) + \textstyle\int_0^s \alpha(\tau)\Psi_0(\mu(\tau)) d\tau) ds + M_2 l D \Big].$$

Denoting by $v(t)$ the right-hand side of the last inequality, we have that $v(t)$ is a continuous function,

$$c := v(0) = \Big(\frac{1}{1 - c_1} \Big) \Big(M_3 + \|\widetilde{E}\| M_3 + c_2 + M_2(c_1 M_3 + c_2) + M_2 c_2 \int_0^\infty \|A(s)\| ds + M_2 l D \Big),$$

does not depend from u and $\mu(t) \leq v(t)$ for $t \geq 0$.
Moreover, for $t \neq t_1, \dots, t_l$,

$$\begin{aligned} v'(t) &= \frac{M_2 c_1}{1 - c_1} \mu(t) \|A(t)\| + \frac{M_2}{1 - c_1} p(t)\Psi(\mu(t) + \textstyle\int_0^t \alpha(s)\Psi_0(\mu(s)) ds) \\ &\leq \frac{M_2 c_1}{1 - c_1} v(t) \|A(t)\| + \frac{M_2}{1 - c_1} p(t)\Psi(v(t) + \textstyle\int_0^t \alpha(s)\Psi_0(v(s)) ds). \end{aligned}$$

Putting:

$$\omega(t) = v(t) + \int_0^t \alpha(s)\Psi_0(v(s))ds,$$

we get, for the continuous function ω,

$$\omega(0) = v(0) = c,$$
$$v(t) \le \omega(t)$$

and for $t \neq t_1, \dots, t_l$

$$\omega'(t) = v'(t) + \Psi_0(v(t))\alpha(t)$$
$$\le \frac{c_1 M_2}{1-c_1}\|A(t)\|\omega(t) + \frac{M_2}{1-c_1}p(t)\Psi(\omega(t)) + \alpha(t)\Psi_0(\omega(t)).$$

Let now

$$\xi(t) := \max\Big\{\frac{M_2 c_1}{1-c_1}\|A(t)\|, \frac{M_2}{1-c_1}p(t), \alpha(t)\Big\}.$$

Then from the previous inequality we have

$$\omega'(t) \le \xi(t)[\omega(t) + \Psi(\omega(t)) + \Psi_0(\omega(t))].$$

This implies that

$$\frac{\omega'(t)}{\omega(t) + \Psi(\omega(t)) + \Psi_0(\omega(t))} \le \xi(t), \qquad t \neq t_1, \dots, t_l,$$

and so, for any $b > 0$,

$$\int_0^b \frac{\omega'(t)}{\omega(t) + \Psi(\omega(t)) + \Psi_0(\omega(t))}dt \le \int_0^b \xi(t)dt \le \int_0^\infty \xi(t)dt := \Gamma < \infty.$$

Note that $\omega'(t)$ is a continuous function for all $t \neq t_1, \dots, t_l$. So,

$$\int_c^{\omega(b)} \frac{ds}{s + \Psi(s) + \Psi_0(s)} \le \Gamma.$$

This, together with hypothesis $(h_3)(ii)$, permits us to conclude that $\omega(t)$ is bounded by a constant Γ_0, say, depending on the functions $\Psi,\ \Psi_0,\ A,\ p$ and α only. Summarizing, $u \in \zeta(S)$ implies that $\|u\|_\infty \le \|\mu\|_\infty \le \|v\|_\infty \le \|\omega\|_\infty \le \Gamma_0$. □

Corollary 1.4.4 *Suppose that $(h_1) - (h_4)$ and (h_6) hold. Let $\phi \in C([-r,0];\mathbb{R}^n)$. Then the Cauchy problem:*

$$\begin{cases} \frac{d}{dt}[x(t) - g(t,x_t)] = A(t)x(t) + f(t, x_t, \int_0^t h(t,s,x_s)ds), \\ \qquad\qquad t \ge 0,\ t \neq t_k,\ k = 1, \dots, l \\ x(t_k{}^+) - x(t_k{}^-) = I_k(x(t_k{}^-)),\ \ k = 1, \dots, l \\ x|_{[-r,0]} = \phi \end{cases}$$

admits at least one solution in Ω.

Proof of Corollary (1.4.4). If we define:

$$Lx := x|_{[-r,0]}, \qquad x \in \Omega,$$
$$H(x) := \phi, \qquad x \in \Omega,$$

we see easily that (h_5) is satisfied and therefore, the thesis follows from theorem (1.4.2).

□

Remark 1.4.5 *The previous corollary extends the theorem (3.3) of Benchohra et al [7]: our operator* $A(t)$ *is time-dependent; our equation is integro-differential and the interval on which the equation is considered is infinite. Moreover in [7] the authors seem not realize that the functions* x_t *are not continuous, in general. So, their result seems to us correct, but the proof has probably some mistakes.*

1.5 Common fixed point theory and application to differential systems

By fixed point methods we investigate the existence of common solutions for first order differential Cauchy problems:

$$(a)\begin{cases} x' = f(t,x) \\ x(t_0) = x_0 \end{cases}, \quad (b)\begin{cases} x' = g(t,x) \\ x(t_0) = x_0 \end{cases}$$

where $t \in J = [t_0, t_0 + a]$, $x \in \mathcal{C}(J, E)$ and E is a partially ordered Banach space.

This problem requires a careful choice of the hypotheses on f, g.

In fact, too weak assumptions on the maps do not assure the existence of common solutions. On the other hand, too strong hypotheses can imply that f and g coincide on all of their domain.

An example of this possibility is in [58, 64] where the authors prove some existence and uniqueness results of common solutions for integro-differential system of the form:

$$(a)\begin{cases} u'(t) + Au(t) = f(t,u(t)) + \displaystyle\int_{t_0}^{t} g\left(t,s,u(s),\int_{t_0}^{s} K_1(s,\tau,u(\tau))d\tau\right)ds \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} h\left(t,s,u(s),\int_{t_0}^{\infty} K_2(s,\tau,u(\tau))d\tau\right)ds, \quad t > t_0 \geq 0 \\ u(t_0) = u_0 \end{cases}$$

$$(b)\begin{cases} u'(t) + Au(t) = \bar{f}(t, u(t)) + \displaystyle\int_{t_0}^{t} \bar{g}\left(t, s, u(s), \int_{t_0}^{s} \bar{K_1}(s, \tau, u(\tau))d\tau\right) ds \\ \qquad\qquad + \displaystyle\int_{t_0}^{t} \bar{h}\left(t, s, u(s), \int_{t_0}^{\infty} \bar{K_2}(s, \tau, u(\tau))d\tau\right) ds, \quad t > t_0 \geq 0 \\ u(t_0) = u_0 \end{cases}$$

with opportune hypotheses on the operator and the functions involved.

The authors use the following fixed point lemma for two contractive type operators:

Lemma 1.5.1 *Let T_1, T_2 be maps on a complete metric space (X, d). If there exists a positive integer m and a positive number $k < 1$ such that for any $x, y \in X$*

$$d(T_1^m x, T_2^m y) \leq kd(x, y),$$

then T_1, T_2 have a unique common fixed point.

Note that, if $x = y$ we have $T_1^m x = T_2^m x$, so previous lemma follows by Banach-Caccioppoli contraction's principle.

Moreover we remark that $T_1^m = T_2^m$ for some $m > 1$ do not implies $T_1 = T_2$.

To assure the existence of common solutions for (a) and (b) the authors consider hypotheses of metric type like:

$$\|f(t, x_1) - g(t, x_2)\| \leq L\|x_1 - x_2\|,$$

for any $x_1, x_2 \in E$ and $L \in \mathbb{R}^+$. The same previous considerations here produce $f = g$.

More interesting contributions to the above problem has been obtained recently by Dhage [25, 26, 27, 28] who relies on topological fixed point theory.

Dhage's proofs are based on Sadovskii's type theorems (see [61]). These theorems involve the measure of noncompactness, they generalize Darbo's fixed point theorem and include also Schauder theorem as special case.

In [28] Dhage introduced a new class of maps that we shall use later: the weakly isotone maps.

Let E an infinite dimensional real Banach space with norm $\|\cdot\|_E$. We recall the definition of *order cone* (see [42, 68]); a closed subset K of E is said an order cone if:

(i) $K + K \subset K$;

(ii) $\lambda K \subset K$ for $\lambda \geq 0$;

(iii) $K \cap -K = \{0\}$.

By order cones one can equip the space E with a partial order relation ($\leq$):

$$x \leq y \Leftrightarrow x - y \in K.$$

From Banas and Goebel [4] is well-known that the measure of non-compactness of a bounded set A in E is a nonnegative number for which:

(i) $\alpha(A) = 0$ iff A is precompact;

(ii) $\alpha(coA) = \alpha(\overline{co}A) = \alpha(A)$ where coA and $\overline{co}A$ are the convex and the closed convex hull of A respectively;

(iii) $A \subset B$ implies $\alpha(A) \leq \alpha(B)$;

(iv) $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$;

(v) $\alpha(\lambda A) = |\lambda|\alpha(A)$, with $\lambda \in \mathbb{R}$.

In the sequel α will be the Kuratowskii's measure of noncompactness ([50]).

Definition 1.5.1 *A mapping $T : E \to E$ is said to be $k-$set contraction if for any bounded set A in E, $T(A)$ is bounded and $\alpha(T(A)) \leq k\alpha(A)$ holds for some $k > 0$. In particular if $k < 1$ then T is called strict-set-contraction.*

Definition 1.5.2 *A mapping $T : E \to E$ is called condensing if for any bounded set A, $T(A)$ is bounded and $\alpha(T(A)) < \alpha(A)$ for $\alpha(A) > 0$.*

Remark 1.5.3 *It's clear that strict-set-contraction implies condensing mapping.*

Definition 1.5.4 *Let E be an ordered Banach space. Two mappings $S, T : E \to E$ are said to be weakly isotone increasing if $Sx \leq TSx$ and $Tx \leq STx$ hold for all $x \in E$. Similarly the mappings $S, T : E \to E$ are said to be weakly isotone decreasing if $Tx \geq STx$ and $Sx \geq TSx$ hold for all $x \in E$. We say two mappings S, T are weakly isotone if they are either weakly isotone increasing or weakly isotone decreasing on E.*

Dhage gives an example on the straight line:

Example 1.5.5 *([28]])*
Consider the real line $\mathbb{R}$ with the usual norm and the usual order relation. Let $X = [0, 1]$ and the mapping $f, g : [0, 1] \to [0, 1]$ defined by:

$$f(x) = \frac{x}{2}, \quad g(x) = \frac{x^2}{3}.$$

It's very simple to verify that f and g are weakly isotone increasing on X.

Example 1.5.6 *Let f, g be two real functions defined on $\mathbb{R}$ by:*

$$f(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases} \quad , \quad g(x) = 1.$$

We immediately verify that f and g are two increasing functions and weakly isotone increasing.

In the next results Dhage proves a common fixed point theorem for condensing mappings and an existence and uniqueness theorem for common solutions for differential equations. For the proofs one can see [28].

Theorem 1.5.7 *Let E be an ordered Banach space and let X denote a non-empty, closed, convex and bounded subset of E. Let $S, T : X \to X$ be two continuous and condensing mappings. Furthermore if S and T are weakly isotone, then they have a common fixed point, i.e. there is a point x^*such that $Tx^* = x^* = Sx^*$.*

Theorem 1.5.8 *Let E be an ordered Banach space. Consider the Cauchy problems,*

$$(a) \begin{cases} x' = f(t,x) \\ x(0) = x_0 \end{cases} , \qquad (b) \begin{cases} x' = g(t,x) \\ x(0) = x_0 \end{cases} , \qquad t \in J = [0,a], x \in \mathcal{C}(J,E)$$

under the following assumptions:

- *(h1) The functions $f, g : J \times E \to E$ are uniformly continuous and bounded on $J \times E$.*
- *(h2) The functions $f(t,x)$ and $g(t,x)$ are nondecreasing in $x \in E$ for all $t \in J$.*
- *(h3) $f(t,x) \le g(t, f(t,x))$ and $g(t,x) \le f(t, g(t,x))$ for all $(t,x) \in J \times E$.*
- *(h4) $f(t,x(t)) \le x_0 + \int_0^t f(\tau, x(\tau))d\tau$ and $g(t,x(t)) \le x_0 + \int_0^t g(\tau, x(\tau))d\tau$ for all $(t,x) \in J \times \mathcal{C}(J,E)$ and for the fixed element $x_0 \in E$ given in (a) and (b).*
- *(h5) For $t \in J$, $\alpha(f(t,B)) \le \Psi_f(\alpha(B))$ and $\alpha(g(t,B)) \le \Psi_g(\alpha(B))$ for any bounded set $B \subset E$, where α is the measure of noncompactness of Kuratowskii and Ψ_f,Ψ_g are non-negative continuous and nondecreasing real functions on $\mathbb{R}_+$, such that $a\Psi_f(r) < r$ and $a\Psi_g(r) < r$ for $r > 0$.*

Then the problems (a) and (b) have a common solution on J.

Our contribution to this topics is to show that in Banach lattices (in particular in $\mathbb{R}^n$) the only hypotheses (h1)-(h4) of theorem (1.5.8) are sufficient to characterize the common solutions of problems (a) and (b).

We start with the following:

Example 1.5.9 *Let us consider two autonomous Cauchy problems,*

$$\begin{cases} x' = f(x) \\ x(0) = \frac{1}{2} \end{cases}, \qquad \begin{cases} x' = g(x) \\ x(0) = \frac{1}{2} \end{cases} \qquad \text{on } J = [0,1]$$

where f, g are functions defined in example (1.5.6).
They have respectively the solutions:

$$x_f(t) = \begin{cases} \frac{1}{2}e^t & , 0 \le t \le \log 2 \\ t + 1 - \log 2 & , \log 2 \le t \le 1 \end{cases} \quad , \quad x_g(t) = \frac{1}{2} + t$$

and therefore do not have a common solution. (In this case all hypotheses of theorem 1.5.8 are satisfied except (h4)).

If we consider the two autonomous Cauchy problems:

$$\begin{cases} x' = f(x) \\ x(0) = 2 \end{cases}, \begin{cases} x' = g(x) \\ x(0) = 2 \end{cases}$$

they have the common solution $x(t) = t + 2$. We note that now all hypotheses of theorem (1.5.8) are satisfied.

Example (1.5.9) illustrates what happens in general; in fact the following result holds:

Proposition 1.5.10 *Let E be a Banach lattice. Let $S, T : E \to E$ be two bounded, increasing maps (i.e. $x \le y$ implies $Sx \le Sy$ and $Tx \le Ty$) and weakly isotone increasing maps. If $x^* = \sup_{x \in E} Tx$, then $Tx^* = x^* = Sx^*$.*

We note that proposition (1.5.10) holds without any compactness or continuity hypotheses on S and T (compare with theorem (1.5.7)).

Moreover, the proof of proposition (1.5.10) gives the idea to prove the following proposition:

Proposition 1.5.11 *Let E be a Banach lattice. Consider the Cauchy: problems*

$$(a) \begin{cases} x' = f(t,x) \\ x(0) = x_0 \end{cases}, \qquad (b) \begin{cases} x' = g(t,x) \\ x(0) = x_0 \end{cases}, \qquad t \in J = [0,a], x \in \mathcal{C}(J,E)$$

under the following assumptions:

(H1) The functions $f, g : J \times E \to E$ are continuous and bounded on $J \times E$.

(H2) The functions $f(t,x)$ and $g(t,x)$ are nondecreasing in $x \in E$ for all $t \in J$.

(H3) $f(t,x) \leq g(t, f(t,x))$ *and* $g(t,x) \leq f(t, g(t,x))$ *for all* $(t,x) \in J \times E$.

(H4) $f(t,x(t)) \leq x_0 + \int_0^t f(\tau, x(\tau))d\tau$ *and* $g(t,x(t)) \leq x_0 + \int_0^t g(\tau, x(\tau))d\tau$ *for all* $(t,x) \in J \times \mathcal{C}(J,E)$ *and for the fixed element* $x_0 \in E$ *given in (a) and (b).*

Then, if $M := sup\{f(t,x) : (t,x) \in J \times E\}$*, both the problems (a) and (b) reduce to the problem*

$$(c)\left\{\begin{array}{l} x' = f(t,M) \\ x(0) = x_0 \end{array}\right.$$

and thus have the common solution $x^*(t) = x_0 + \int_0^t f(\tau, M)d\tau$.

Remark 1.5.12

1. *The result of proposition (1.5.11) is stronger than theorem (1.5.8).*
2. *A key hypothesis is* $(H4)$ *(compare with example (1.5.9)), that is a condition on* x_0 *and is satisfied if* x_0 *is sufficiently large:*

$$x_0 \geq \max\Big\{ \sup\{f(t,M) : t \in J\} - \inf\Big\{\int_0^t f(\tau, x(\tau))d\tau : x \in C(J,E)\Big\}, \\ \sup\{g(t,M) : t \in J\} - \inf\Big\{\int_0^t g(\tau, x(\tau))d\tau : x \in C(J,E)\Big\}\Big\}.$$

3. *In the case of autonomus systems:*

$$(a')\left\{\begin{array}{l} x' = f(x) \\ x(0) = x_0 \end{array}\right. \quad (b')\left\{\begin{array}{l} x' = g(x) \\ x(0) = x_0 \end{array}\right. , \quad J = [0,a]$$

proposition (1.5.10) and proposition (1.5.11) assure that under the hypotheses $(H1)-(H2)-(H3)$ *the common solution of the systems* (a') *and* (b') *is the straight line*

$$x(t) = Mt + x_0$$

, where $M = \sup\{f(x) : x \in E\}$ *and:*

$$x_0 \geq \max\Big\{M - \inf_{0 \leq t \leq a}\{m_f t\},\ M - \inf_{0 \leq t \leq a}\{m_g t\}\Big\},$$

with $m_f = \inf\{f(x) : x \in E\}$, $m_g = \inf\{g(x) : x \in E\}$.

4. *The result and the proof of proposition (1.5.11) show that hypotheses like weak isotonie perhaps are too strong. To obtain interesting results for common solutions of differential systems in* $\mathbb{R}^n$.

1.5.1 Proofs

Proof.[Proof of Proposition 1.5.10]. Let

$$M_T := \sup\{Tx : x \in E\}$$

$$M_S := \sup\{Sx : x \in E\}.$$

Then for all $x \in E$,

$$Sx \leq (since\ S, T\ are\ weakly\ isotone\ increasing) \leq TSx \leq M_T$$

that implies $M_S \leq M_T$. Analogously one obtains $M_T \leq M_S$. So

$$x^* := M_T = M_S.$$

Moreover, from $Sx \leq x^*$, since T is increasing, one has:

$$TSx \leq Tx^*,$$

and from this, since S, T are weakly isotone increasing, it follows that:

$$Sx \leq Tx^*,$$

for all $x \in E$, i.e. Tx^* is a majorant of $\{Sx : x \in E\}$, so $x^* \leq Tx^*$.

On the other hand $Tx^* \leq x^*$, from definition of x^* and so $x^* = Tx^*$. Analogously one obtains $x^* = Sx^*$. □

Proof.[Proof of Proposition 1.5.11]. Let $M_f = \sup\{f(t,x) : (t,x) \in J \times E\}$, $M_g = \sup\{g(t,x) : (t,x) \in J \times E\}$.
Now,

$$f(t,x) \leq (from\ (H3)) \leq g(t, f(t,x)) \leq M_g,$$

for all $(t,x) \in J \times E$ implies $M_f \leq M_g$. Analogously $M_g \leq M_f$ and so:

$$M := M_f = M_g.$$

Put $\phi(t) := f(t,M)$, $\psi(t) := g(t,M)$.
Then:

$$\phi(t) = f(t,M) \leq (from\ (H3)) \leq g(t, f(t,M)) \leq (from\ (H2)) \leq g(t,M) = \psi(t).$$

Analogously one obtains $\psi(t) \leq \phi(t)$, so:

$$\phi(t) = \psi(t).$$

Moreover,

$$\phi(t) = g(t, M) \le (from\ (H3)) \le f(t, g(t, M)) = f(t, \phi(t)) \le$$

$$\le (from\ (H2)) \le f(t, M) = \phi(t),$$

i.e.,

$$\phi(t) = f(t, \phi(t)).$$

Analogously one obtains:

$$\phi(t) = g(t, \phi(t)),$$

and, yet from $(H2)$,

$$f(t, x) = g(t, x) = \phi(t), \tag{1.5.1}$$

for all $x \ge \phi(t)$.

Thus let $\bar{x}_f$ be a solution of (a). We show that $\bar{x}_f(t) \ge \phi(t)$ for all $t \in J$. First of all, if $t = 0$, $\bar{x}_f(0) = x_0$ and from $(H4)$ it follows:

$$\phi(0) = f(0, M) \le x_0 = \bar{x}_f(0).$$

Thus let $t > 0$. Let $\varepsilon > 0$. Take the function:

$$\tilde{x}(\tau) = \begin{cases} \bar{x}_f(\tau) & 0 \le \tau \le t - \varepsilon \\ \frac{t-\tau}{\varepsilon}\bar{x}_f(t-\varepsilon) + \frac{\tau - t + \varepsilon}{\varepsilon}\phi(t) & t - \varepsilon \le \tau \le t \\ \phi(t) & t \le \tau \le a \end{cases}.$$

Then $\tilde{x} \in C(J, E)$ and, from $(H4)$,

$$f(t, \tilde{x}(t)) \le x_0 + \int_0^t f(\tau, \tilde{x}(\tau))d\tau,$$

i.e.

$$\begin{aligned} \phi(t) &\le x_0 + \int_0^{t-\varepsilon} f(\tau, \bar{x}_f(\tau))d\tau + \int_{t-\varepsilon}^t f(\tau, \tilde{x}(\tau))d\tau \le \\ &\le x_0 + \int_0^{t-\varepsilon} f(\tau, \bar{x}_f(\tau))d\tau + \varepsilon\phi(t). \end{aligned}$$

So, for $\varepsilon \downarrow 0$ we obtain:

$$\phi(t) \le x_0 + \int_0^t f(\tau, \bar{x}_f(\tau))d\tau = \bar{x}_f(t),$$

since $\bar{x}_f$ is solution for (a).

Analogously we obtain $\phi(t) \leq \bar{x}_g(t)$ for any solution $\bar{x}_g$ of the Cauchy problem (b).

This, together with the relation (1.5.1), is sufficient to conclude that for all solutions x_f and x_g of the problems (a) and (b):

$$f(t, x_f(t)) = g(t, x_g(t)) = \phi(t).$$

So, both the Cauchy problems (a) and (b) reduce to the problem:

$$\begin{cases} x' = \phi(t) \\ x(0) = x_0 \end{cases},$$

and thus have the common solution $x^*(t) = x_0 + \int_0^t \phi(\tau)d\tau$. $\square$

Chapter 2

Some topics of MFPt that arises from TFPt

2.1 Preface

Brouwer theorem states that *for any convex bounded closed subset of a* finite *dimensional Banach space X have the TFPp*.
We focus our attention on two results that are equivalent to the Brouwer theorem. [1]
The first theorem involves the *retractions*.

Definition 2.1.1 *Let X be a Banach spaces and Y a subspace of X. The mapping $R : X \to Y$ is a retraction if R is continuous and $Rx = x$ for any $x \in Y$ (i.e. $R|_Y = Id$). The subspace Y is called retract of X.*

If B is the unit ball of the space X and S the unit sphere, we claim that the following are equivalent: *Brouwer's theorem $\Leftrightarrow$ the sphere S cannot be a retract of the ball B.*

($\Rightarrow$) If $R : B \to S$ is a retraction then $T : B \to S \subset B$ with $T := -R$ is a fixed point free mapping.

($\Leftarrow$)On the other hand, if we suppose that there exists $T : B \to B$ continuous and fixed point free map we can define the mapping:

$$T_1 x = \begin{cases} Tx & \|x\| \geq 1 \\ (2 - \|x\|)T\left(\dfrac{x}{\|x\|}\right) & \|x\| \in [1, 2]. \end{cases}$$

This map has no fixed points in $2B$ and $T_1(2S) = \{0\}$. Let us consider the mapping $T_2 : B \to B$ defined by:

$$T_2 x = \frac{1}{2} T_1(2x).$$

[1]For details one can see Goebel and Kirk's book [34] and Goebel's book [35].

T_2 holds the proprieties of T_1 and so $T_2(S) = \{0\}$. Thus, we define:

$$Rx = \frac{x - T_2 x}{\|x - T_2 x\|},$$

and Rx is a retraction of the ball onto the sphere.

Definition 2.1.2 *A topological space X is said to be contractible (to a point $z \in X$) if there exists a continuous function (an homotopy) $H : X \times [0,1] \to X$ such that $H(x,0) = x$ and $H(x,1) = z$ for all $x \in X$.*

For sake of completeness we prove that: *The sphere S is not retract of B $\Leftrightarrow$ S is not contractible.*

To show ($\Rightarrow$) is enough to observe that if S is contractible and H is the related homotopy then the mapping:

$$Rx = \begin{cases} z & \|x\| \leq \frac{1}{2} \\ H\left(\frac{x}{\|x\|}, 2(1 - \|x\|)\right) & \|x\| \geq \frac{1}{2} \end{cases},$$

is a retraction from B to S.
On the other hand if $R : B \to S$ is a retraction then

$$H(x,t) = R((1-t)x)$$

retracts S to the point $R(0)$.

We summarize the above in the following:

Theorem 2.1.3 *Let X be a finite dimensional Banach space. The following are equivalent and true:*

1. *(**Brouwer Theorem**) Any bounded closed and convex subset of X have the TFPp.*
2. *(**No retraction Theorem**) The unit sphere S is not a retract of the ball B.*
3. *(**Non contractibility of spheres**) The unit sphere S is not contractible.*

The situation changes if X is an *infinite* dimensional Banach space.

Example 2.1.4 *(Kakutani (1943))*
Let us consider the Hilbert space $l^2(\mathbb{N})$, its unit ball B_{l^2} and its unit sphere S_{l^2}. We define the mapping $T : B_{l^2} \to B_{l^2}$ by:

$$Tx = T(\eta_1, \eta_2, \ldots) = (\sqrt{(1 - \|x\|^2)}, \eta_1, \eta_2, \ldots).$$

T is continuous and fixed point free. In particular we observe that $T(B_{l^2}) = S_{l^2}$.

The previous example shows that it is not possible to extend the Brouwer theorem to non-compact setting. Moreover using the previous map it is possible to define a retraction from the ball to the sphere by:

$$\tilde{R}x = x + \lambda(x)u(x) \tag{2.1.1}$$

where:

$$u(x) = \frac{x - Tx}{\|x - Tx\|},$$
$$\lambda(x) = -\langle x, u(x)\rangle + \sqrt{1 - \|x\|^2 + \langle x, u(x)\rangle^2}.$$

More examples of self-mapping of the unit ball (of Hilbert spaces) without fixed points are given by Kakutani in [45]. In Hilbert spaces Bessaga and Pelczynski in [9], and Bessaga in [10] construct retractions from the ball onto the sphere. Other references are contained in bibliography of [34, 35].

Since a Banach space X is infinite dimensional iff Brouwer theorem does not hold, (2) of theorem (2.1.3) assures that, in infinite dimensional Banach spaces, there exist retractions from the ball to the sphere.

Nowak [56], Benyamini and Sternfeld [8] and Lin and Sternfeld [51] show much more:

Theorem 2.1.5 (Nowak (1979), Benyamini and Sternfeld (1983)) *For any infinite dimensional Banach space X there exists a lipschitzian retraction of the unit ball onto the unit sphere.*

Proof. See [8, 56] or Goebel and Kirk's book [34] □

Theorem 2.1.6 (Lin and Sternfeld (1985)) *For any noncompact, bounded, closed and convex subset K of a Banach space there is a lipchitzian mapping $T : K \to K$ for which*

$$\inf\{\|x - Tx\| : x \in K\} = d > 0. \tag{2.1.2}$$

Proof. See the authors's proof [51] or [34] □

These two theorems arise two principal questions:

(OR) (from (2.1.5)) **What is the Lipschitz constant of the retraction constructed above?**

(MD) (from (2.1.6)) **What is the *precise* minimum distance at which a Lipschitzian mapping can move all points of its domain?**

In next section we start with the (MD) problem, since the first is related to this. However it is important point out that: **for both problems no general answer is known**.

One final note.

An extensively examination of the proofs of theorem (2.1.5) and theorem (2.1.6) joint to the following assertion:

For any $\varepsilon > 0$, the ball $B(0, r) \subset X$ contains a sequence (x_n) of points such that
$$dist(x_{n+1}, span\{x_1, \dots, x_n\}) \geq r - \varepsilon,$$

permits to prove that:

Theorem 2.1.7 *There exists a universal constant k_0 such that for any Bancach space X there is a lipschitzian retraction R of the unit ball of X onto its boundary with $k(R) \leq k_0$.*

So this arises another question:

- **What is the exact value of k_0?**

Nowadays only one upper bound is known and it is due to Annoni [1]:

$$k_0 \leq 256 \cdot 10^9.$$

2.2 Minimal displacement problem

Let X be an infinite dimensional Banach space and let K be a closed, bounded, convex subset of X.

We restrict our interests to lipschitzian mappings and, reformulating (MD), we investigate on the problem: *to measure how near, certain mapping, are to have fixed points.*

We will find out that frequently this measure depends by the lipschitzian constant of the mapping. In general this does not hold as the next example shows:

Example 2.2.1 *([34])*
Let $X = C[0,1]$ and K defined by:

$$K = \{f \in X : 0 = f(0) \leq f(t) \leq f(1) = 1\}.$$

Let $i(t) = t$ and $\alpha \in K,\ \alpha \neq i$. Let us define $T_\alpha : K \to K$ by $T_\alpha(t) = \alpha(f(t))$.
This mapping inherits the behavior of the function α so if α is $m-$lipschitzian mapping the same holds for T_α. Moreover:

$$\|T_\alpha f - f\| = \max_{t\in[0,1]}\{|\alpha(f(t)) - f(t)|\} = \max_{t\in[0,1]}\{|\alpha(t) - t|\} = \|\alpha - i\| := d_1 > 0,$$

for which $\inf\{\|x - T_\alpha x\| : x \in K\} = d_1$.

For any $x \in K$ define:

$$r(x, K) = \sup\{\|x - y\| : y \in K\};$$
$$r(K) = \inf\{r(x, K) : x \in K\};$$
$$C(K) = \{x \in K : r(x, K) = r(K)\}.$$

The number $r(K)$ is said Chebyshev radius of K and the elements in $C(K)$ are said Chebyshev centers of K.

To completeness we remark that if X is reflexive, $C(K)$ is convex and closed while if X is uniformly convex, $C(K)$ consists of only one point (for reference see [57]).

For any $K \subset X$ we say:

$$\mathcal{L}_K(k) = \{\, T : K \to K, \quad k - lipschitzian\ map\},$$

and for any $T \in \mathcal{L}_K(k)$ define:

$$d(T) = \inf\{\|x - Tx\| : x \in K\}.$$

In [32] Goebel shows that:

$$d(T) \le r(K)\left(1 - \frac{1}{k}\right),$$

so, in the case $K = B$, $r(K) = 1$ and

$$d(T) \le \left(1 - \frac{1}{k}\right),$$

holds for any $T \in \mathcal{L}_B(k)$.

Our aim is to introduce some functions that characterize the problem. Put:

$$\varphi_K^*(k) = \sup_{T \in \mathcal{L}_K(k)} \{d(T)\},$$

and note that if K and J are such that $K = u + aJ$ with $u \in X$ and $a \in \mathbb{R}$ then

$$\varphi_K^*(k) = |a|\varphi_J^*(k).$$

For this reason we restrict our analysis to:

$$\varphi_K(k) = \frac{\varphi_K^*(k)}{r(K)}$$

for which also

$$\varphi_K(k) \le 1 - \frac{1}{k}$$

holds. For the entire space X we define:

$$\varphi(k) = \varphi_X(k) = \sup\{\varphi_K(k) : K \subset X \text{ is convex closed and bounded}\}.$$

Remark 2.2.2

- *If* $k \in (0,1)$ *the mapping* $T \in \mathcal{L}_K(k)$ *is a contraction than* $\varphi(k) = 0$ *(*T *have a unique fixed point);*
- $\varphi(k) \leq 1 - \dfrac{1}{k}$*, for any* $k > 1$*.*

When $K = B$, the unit ball of X, we rename:

$$\varphi_B(k) := \psi_X(k).$$

Let us start with some proprieties of the above functions .
In literature few qualitative proprieties are proved for the functions φ and ψ_X and many of these are used to recognize lipschitzian retraction. We collect the results of Goebel [32] in the following theorem:

Theorem 2.2.3 *[32] If* η *denotes any functions* $\varphi_K, \varphi, \psi_X$ *then:*

(i) $\eta(1 - \alpha + \alpha k) \geq \alpha\eta(k)$ *if* $\alpha \in [0,1]$*;*

(ii) $\dfrac{\eta(k)}{k-1}$ *is nonincreasing for* $k > 1$*;*

(iii) $\dfrac{k \cdot \eta(k)}{k-1}$ *is nondecreasing for* $k > 1$*;*

(iv) Always exists $\lim_{k\to 1} \dfrac{\eta(k)}{k-1} := \eta'(1)$*.*

Proof. See Goebel [32] or Goebel and Kirk [34]. □

Theorem (2.2.3) permits to formulate a upper/lower bound for $\varphi_K, \varphi, \psi_X$.

Theorem 2.2.4 *[32] For any bounded, convex, subset* K *of a Banach space* X *(with* $r(K) = 1$*):*

$$\varphi'_K(1)\left(1 - \tfrac{1}{k}\right) \leq \varphi_K(k) \leq \left(1 - \tfrac{1}{k}\right);$$
$$\varphi'(1)\left(1 - \tfrac{1}{k}\right) \leq \varphi(k) \leq \left(1 - \tfrac{1}{k}\right);$$
$$\psi'_X(1)\left(1 - \tfrac{1}{k}\right) \leq \psi_X(k) \leq \left(1 - \tfrac{1}{k}\right).$$

Goebel gives the following question: what is the minimal value for $\varphi'_K(1)$, $\varphi'(1)$ and $\psi'_X(1)$?
According to theorem (2.2.4) and the definition of $\varphi'(1)$ one has:

Corollary 2.2.5 *[32] For any Banach space X, $\varphi(k) = 1 - \frac{1}{k}$ iff $\varphi'(1) = 1$.*

If the left assertion of corollary does not hold one can find few answers to the problem that depend on the geometric propriety of the space. Here we cite only one of this results.

Theorem 2.2.6 *[32] If X is any Banach space for which $\varepsilon_0(X) < 1$ then $\varphi'(1) < 1$.*

Corollary 2.2.7 *[32] For any Banach space X with $\varepsilon_0(X) < 1$ we have*

$$\varphi(k) < 1 - \frac{1}{k}.$$

Definition 2.2.8 *An infinite dimensional Banach space X is extremal if $\psi_X(k) = 1 - \frac{1}{k}$.*

Goebel in [32] gives many examples of the computation of $\varphi(k)$ in certain Banach space, furthermore, he shows that the spaces $C[0,1], L^1[0,1]$ and c_0 are extremal.

Are these the only extremal spaces?

In his Ph.D. thesis [15] Bolibok, proves that every subspace of $C[0,1]$ with finite codimension is extremal. For example, $C_0[0,1]$ (the space of continuous functions on $[0,1]$ with $f(0) = 0$) is extremal. In [15] Bolibok shows that the space of all differentiable functions on $[0,1]$ with standard norms is also extremal.

Example 2.2.9 *Let $C^1[0,1]$ the space of differentiable function with norm:*

$$\|f\|_{C^1} = \max\{\|f\|_\infty, \|f'\|_\infty\}.$$

Let $\alpha : \mathbb{R} \to [0,1]$ the "cut" function:

$$\alpha(t) = \begin{cases} 1 & t \geq 1 \\ t & t \in [-1,1] \\ -1 & t \leq -1 \end{cases}, \tag{2.2.1}$$

that is a nonexpansive mapping. We consider the maps $T_k : C^1[0,1] \to C^1[0,1]$ defined by:

$$(T_k f)(t) = \int_0^t \alpha(k(f'(s) + g(s)))ds,$$

where $g(t)$ is a continuous strictly increasing function such that $g(0) = -2$ and $g(1) = 2$. It is important to observe that:

$$\|T_k f\|_\infty \leq \|\alpha(k(f' + g))\|_\infty = \|(T_k f)'\|_\infty \Rightarrow \|T_k f\|_{C^1} = \|(T_k f)'\|_\infty.$$

We note that:

(i) $T_k : B_{C^1} \to B_{C^1}$. *In fact* $\|(T_k f)'\|_\infty = \|\alpha(k(f' + g))\|_\infty \le 1$ *and therefore* $\|T_k f\|_{C^1} \le 1$.

(ii) For $f, h \in C^1$ *we have* $\|(T_k f)' - (T_k h)'\|_\infty \le k\|f' - h'\|_\infty = k\|f - h\|_{C^1}$ *hence*

$$\|T_k f - T_k h\|_{C^1} \le k\|f - h\|_{C^1}.$$

(iii) For any $f \in B_{C^1}$*, following [37] page 144 we have*

$$\|f - T_k f\|_{C^1} = \|f' - (T_k f)'\|_\infty = \sup_{t\in[0,1]} |f'(t) - \alpha(k(f' + g))| \ge 1 - \frac{1}{k}.$$

Therefore $(C^1[0,1], \|\cdot\|_{C^1})$ *is extremal space.*

In next examples we prove that $BC(\mathbb{R})$ and $BC_0(\mathbb{R})$ are extremal and the proofs use ideas of [35, 38].

Example 2.2.10 *Let* $g(t) \in BC(\mathbb{R})$ *be a fixed function strictly increasing with:*

$$\lim_{t\to\infty} g(t) = 2^-, \qquad \lim_{t\to-\infty} g(t) = -2^+$$

Let $\alpha : \mathbb{R} \to [0,1]$ *be the function defined in (2.2.1) and let us consider* $T_k : B \to B$ *the maps defined by:*

$$(T_k f)(t) = \alpha(k(f(t) + g(t))).$$

We observe that:

1. *Since the function* α *is a nonexpansive mapping and* $[k(f + g)]$ *is, obviously, a* $k-$*lipschitzian map, then* $T_k \in \mathcal{L}_B(k)$.

2. $\|T_k f - f\| \ge 1 - \frac{1}{k}$ *for all* $f \in B$.
   Proof. *Since* $g(t) \to \pm 2$ *if* $t \to \pm\infty$ *then there exists a unique* $c \in \mathbb{R}$ *such that* $g(c) = 0$. *We must consider two cases:*

   (a) If $f(c) \ge 0$ *we have* $(f(c) + g(c)) \ge 0$. *Moreover:*

   $$\lim_{t\to-\infty} g(t) = -2, \qquad \limsup_{t\to-\infty} f(t) \le 1,$$

   then there exists $s \in (-\infty, c]$ *with* $g(s) < 0$ *and* $f(s) + g(s) = -\dfrac{1}{k}$ *so that* $(T_k f)(s) = \alpha(k(f(s) + g(s)) = -1$ *and*

   $$\begin{aligned}\|f - T_k f\| &\ge |f(s) - (T_k f)(s)| \\ &= \left|-\frac{1}{k} - g(s) + 1\right| = 1 - \frac{1}{k} + |g(s)| > 1 - \frac{1}{k}.\end{aligned}$$

(b) *If* $f(c) \leq 0$ *following (a) there exists* $s \in [c, \infty)$ *such that* $g(s) > 0$ *and* $f(s) + g(s) = \dfrac{1}{k}$ *and*

$$\|f - T_k f\| \geq |(T_k f)(s) - f(s)| = 1 - \frac{1}{k} + g(s) > 1 - \frac{1}{k}.$$

□

Then $d(T_k) = 1 - \dfrac{1}{k}$ *that implies* $\varphi(k) = \psi_{BC(\mathbb{R})}(k) = 1 - \dfrac{1}{k}$ *and the space* $BC(\mathbb{R})$ *is extremal.*

Example 2.2.11 *Let us consider the sub-space of* $BC(\mathbb{R})$ *of all functions such that* $f(0) = 0$ *with the supremum norm. We indicate this space with* $BC_0(\mathbb{R})$.
Let us call $g : \mathbb{R} \to [0, 1)$ *the continuous function: such that*

$$\lim_{t \to \pm\infty} g(t) = 1 \text{ and } g(0) = 0.$$

Let T_k *be a mapping defined on* $BC_0(\mathbb{R})$ *by:*

$$(T_k f)(t) = \Delta_k(|f(t)| + g(t)),$$

where $k > 1$ *and* $\Delta_k : [0, +\infty) \to [0, 1]$ *is defined in [38] by:*

$$\Delta_k(t) = \begin{cases} kt & t \in \left[0, \frac{1}{k}\right] \\ 2 - kt & t \in \left[\frac{1}{k}, \frac{2}{k}\right] \\ 0 & t \geq \frac{2}{k} \end{cases}.$$

Remark 2.2.12

1. T_k *is* $k-$*lipschitzian because* Δ_k *is a* $k-$*lipschitzian map and* $(|f(t)| + g(t))$ *is a nonexpansive map.*

2. $(T_k f)(0) = \Delta_k(|f(0)| + g(0)) = 0$. *It follows that* $T_k : BC_0(\mathbb{R}) \to B$.

3. $\psi_{BC_0(\mathbb{R})}(k) = 1 - \dfrac{1}{k}$.
   Proof. *Let* $\varepsilon > 0$ *be sufficiently small such that:*

$$1 - \varepsilon > \frac{1}{k}.$$

Since $g(t) \to 1$ when $t \to \pm\infty$, there exists $\overline{t}$ such that $g(\overline{t}) > 1 - \varepsilon$ and, a fortiori, $|f(\overline{t})| + g(\overline{t}) > 1 - \varepsilon > \frac{1}{k}$.
Then there exists $t_1 \in [0, \overline{t}]$ with $|f(t_1)| + g(t_1) = \frac{1}{k}$ from which:

$$\|T_k f - f\| \geq |(T_k f)(t_1)| - |f(t_1)| = 1 - \frac{1}{k} + g(t_1) \geq 1 - \frac{1}{k}.$$

It follows that $d(T_k) \geq 1 - \frac{1}{k}$ and the thesis: $BC_0(\mathbb{R})$ is extremal. □

The extremality of the space in previous examples gives the equality:

$$\varphi(k) = \psi_X(k). \tag{2.2.2}$$

We remark that in general (2.2.2) does not hold.

Example 2.2.13 *([34])*
Let us $X = l^1$ and S^+ the subset of the unit ball B given by:

$$S^+ = \overline{conv}\{e_1, e_2, \ldots\} = \left\{ x = (\xi_i) : \xi_i \geq 0, \sum_{i=1}^{\infty} \xi_i = 1 \right\}.$$

We note that $r(S^+) = diam(S^+) = 2$. We suppose $k > 1$ and, from the proprieties of the series, let us $i_0 = i_0(x)$ a maximal index such that $\sum_{j=i_o}^{\infty} x_j > \frac{1}{k}$.

We call $\mu(x)$ the value of $[0,1]$ for which:

$$\mu(x)\xi_{i_0} + \sum_{j=i_o+1}^{\infty} x_j = \frac{1}{k}.$$

If we define $T_k : S^+ \to S^+$ by taking:

$$T_k x = T_k(\xi_1, \xi_2, \ldots) = k(\underbrace{0, \ldots, 0}_{i_0\text{-times}}, \mu(x)\xi_{i_0}, \xi_{i_0+1}, \ldots),$$

it is not difficult to prove (but it is long) that T_k is $k-$lipschitzian and $\|x - T_k x\| \geq 2\left(1 - \frac{1}{k}\right)$ for every $x \in S^+$. This implies that:

$$\varphi_{\frac{1}{2}S^+}(k) = \varphi_{l^1}(k) = 1 - \frac{1}{k}.$$

To give an upper bound to $\psi_{l^1}(k)$ we consider a mapping $T : B \to B$ with $T \in \mathcal{L}_B(k)$ and a number $m > k$.

Let us consider the implicit function $F : B \to B$ given by:

$$Fx = \left(1 - \frac{1}{m}\right)x + \frac{1}{m}TFx.$$

One can verify that:

1. *Since* $F \in \mathcal{L}_B\left(\frac{m-1}{m-k}\right)$, $TF \in \mathcal{L}_B\left(k \cdot \frac{m-1}{m-k}\right)$.
2. *Since* $\|x - Tx\| \geq d > 0$ *for all* $x \in B$ *thus* $\|x - Fx\| \geq \frac{d}{m-1}$ *and* $\|x - TFx\| \geq \frac{md}{m-1}$.

It is possible distinguish two possibility; firstly we suppose that $TF0$ *is finite dimensional (we consider* $dim TF0 = n$*). In this case, by Brouwer theorem and using the natural projection* P_n *of* l^1*, we prove that:*

$$d \leq \left(\frac{m-1}{m}\right)\left(\frac{m(k+1)-2k}{m(k+3)-4k}\right). \tag{2.2.3}$$

Secondly we suppose, obviously, that $TF0$ *is not finite dimensional. Taking* $\varepsilon > 0$ *and choosing* n *sufficiently big such that* $\|TF0 - P_nTF0\| < \varepsilon$ *we can repeat the previous prove with* $\varepsilon-$*proximity.*

In each case, if $T \in \mathcal{L}_B(k)$ *is such that* $d \geq \psi_{l^1}(k) - \varepsilon$*, minimizing with respect to* m *the right side of (2.2.3) one finds:*

$$\psi_{l^1}(k) \leq \begin{cases} \frac{2+\sqrt{3}}{4}\left(1 - \frac{1}{k}\right) & 1 \leq k \leq 3 + 2\sqrt{3} \\ \frac{k+1}{k+3} & k \geq 3 + 2\sqrt{3} \end{cases} < 1 - \frac{1}{k}. \tag{2.2.4}$$

Thus

$$\psi_{l^1}(k) < \varphi(k) = 1 - \frac{1}{k}.$$

Moreover l^1 *is not an extremal space.*

Remark 2.2.14 *We do not know if (2.2.4) is sharp but, at this time, this is the unique upper bound known in literature. The same can be said for lower bounds. It is known only that:*

$$\psi_{l^1}(k) \geq \begin{cases} (3 - 2\sqrt{2})(k-1) & k \in [1, 2 + \sqrt{2}] \\ 1 - \frac{2}{k} & k \in (2 + \sqrt{2}, \infty) \end{cases}$$

proved by Bolibok in [14].

In order to previous example one can ask: Does (2.2.2) imply that X is an extremal space?

Goebel in [32] gives the negative answer showing that:

Theorem 2.2.15 *In Hilbert space the equality* $\varphi(k) = \psi_H(k)$ *holds.*

Theorem 2.2.16 *In Hilbert space*

$$\varphi(k) \leq \left(1 - \frac{1}{k}\right)\sqrt{\frac{k}{k+1}}.$$

Theorems (2.2.15) and (2.2.16) imply that Hilbert spaces cannot be extremal.

Let us finish this section recalling some other proprieties for the function $\psi_H(k)$ in Hilbert spaces.

Casini [19] assure that the evaluation in theorem (2.2.16) is not sharp since:

Theorem 2.2.17 *There exists a function* $F : (1, +\infty) \to (0, 1)$ *such that:*

$$\psi_H(k) \leq \left(1 - \frac{1}{k}\right)\sqrt{\frac{k}{k+1}}F(k).$$

This represents the best upper bound estimate known for $\psi_H(k)$.
On the other side (lower bounds) we note that there are in literature few results and many of these are devoted to give a numerical estimate of the function $\psi_H(k)$.

The first approximation on $\psi_H(k)$ is due to Bolibok [11]. He proves that,

$$\psi_H(k) \geq \sup_{\varepsilon \in (0,2)} \left(1 - \frac{2+\varepsilon}{\sqrt{1+\varepsilon(\varepsilon+2)k^2} - 1} - \varepsilon(k+1)\right).$$

In a later paper again Bolibok improves his estimate showing that:

Theorem 2.2.18 *Let H an infinite-dimensional Hilbert space. Then*

$$\psi_H\left(\sqrt{2}(k+1)k^{\frac{3}{2}}\right) \geq \frac{k-1}{k+1} \tag{2.2.5}$$

for $k \geq 2$.

Proof. See [16] □

This formula seems to us very difficult to handle, but less than the first.

The last approximation is due to Casini [18]. This semms simpler than: (2.2.5)

Theorem 2.2.19 *If H is an infinite dimensional Hilbert space*

$$\psi_H(k) \geq 1 - \frac{2\sqrt{\sqrt{2}(k+1)}}{k}.$$

2.3 Optimal retraction problem

In theorem (2.1.5) and (2.1.6) Benyanimi, Lin, Nowak and Sternfeld assure that for any infinite dimensional Banach space there exists a lipschitzian retraction from the unit ball ball B onto the sphere S.

Let us define:

$$k_0(X) = \inf\{k > 1 : \exists \text{ a retraction } R : B \to S, R \in \mathcal{L}_B(k)\}.$$

In literature we found this value as *optimal retraction constant* for the space X.

To give an exact value for this number means to give an answer to the problem (OR) in § 2.1.

The aim of this section is to present the state of the research on this topic. Such researches turn on upper bounds and lower bounds for $k_0(X)$ for X a given space because: **exact values are unknown for any Banach spaces** included for extremal spaces.

Mainly we will put our attention on the methods by these estimates are obtained starting with lower bounds one.

The first lower bound for $k_0(X)$ is due to Goebel and Kirk [34].

Lemma 2.3.1 *For any infinite dimensional Banach space X, $k_0(X) \geq 3$.*

Proof. Let R be a $k-$lip retraction and let $T = -R$. We have $T : B \to S$, $T^2 = R$ and T is fixed point free.

For $\varepsilon > 0$ let $x_\varepsilon \in B$ such that $\|x_\varepsilon - Tx_\varepsilon\| \leq d(T) + \varepsilon$ with $d(T) \geq 0$.

Define the curve $\gamma : [0,1] \to S$ by $\gamma(t) = T((1-t)x_\varepsilon + tTx_\varepsilon)$ that lies on S. Note that γ inherits the lipschitzian constant from T, so is a rectifiable curve. Moreover it joins the antipodal points of S because $\gamma(0) = -Rx_\varepsilon$ and $\gamma(1) = Rx_\varepsilon$.

Let us call $g(X)$ the infimum of the lengths of $\gamma(t)$.

Observe that $g(X) \geq 2$ for any Banach space X and in the case of Hilbert spaces $g(X) = \pi$.

Using the definition of length of a curve ($l(\gamma)$) we remark that $g(X) \leq l(\gamma) \leq k(d(T) + \varepsilon)$. Moreover by definition of $\psi_X(k)$ and since $T \in \mathcal{L}_B(k)$ we have $d(T) \leq \psi_X(k)$.

These two relations imply that if we choose ε such that $(d(T) + \varepsilon)$ is sufficiently close to $\psi_X(k)$ we have:

$$k\psi_X(k) \geq 2 \Rightarrow k\left(1 - \frac{1}{k}\right) \geq k\psi_X(k) \geq 2 \Rightarrow k - 1 \geq 2,$$

that is $k \geq 3$. □

The same idea, joined with the lower estimate of $\psi_X(k)$ in section 2.2, permits to obtain the following results:

Corollary 2.3.1 *1.* $k_0(l^1) \geq 3.143$*;*

2. $k_0(H) \geq 4.5$*.*

The estimate of (2.3.1) (1) has been improved by Bolibok to the end of [14]. Bolibok proves that:

$$k_0(l^1) \geq 4.$$

This is the best lower approximation known so far for $k_0(l^1)$.

To review the research on upper bounds for $k_0(X)$ we discuss the three principal methods used.

It is important to remark that these technics are not the *only* methods that we find in literature. For example we cite Annoni and Casini [2] and Komorowski and Wosko [48] for different approach.

2.3.1 By lipschitzian homotopy

A first method to construct upper bounds estimate for $k_0(X)$ utilizes the lipschitzian homotopies.

Since, by theorem (2.1.3), the existence of a retraction is related to the contractibility of the sphere, it is not surprising that, by this method, one obtains bounds for lispchitzian constants for retractions.

Since the sphere is Lipschitz contractible (see theorem (11.3) in [37]), there exists a homotopy such that:

$$\|H(t,x) - H(s,y)\| \leq M|t-s| + N\|x-y\|. \tag{2.3.1}$$

It is clear that $M \geq 2$ since $M \geq g(X) \geq 2$; by the definition of homotopy and by (2.3.1) one obtains:

$$\|x-y\| = \|H(0,x) - H(0,y)\| \leq N\|x-y\| \text{ and so } N \geq 1.$$

The connection between M, N and $k_0(X)$ is due to the following:

Theorem 2.3.2 *Let* $H : [0,1] \times S \to S$ *be a homotopy such that for each* $x \in S$*,* $H(0,x) = x$ *and* $H(1,x) = x_0 \in S$*. Suppose that* H *is a lispchitzian homotopy for all* $x, y \in S$*,* $t, s \in [0,1]$ *and* M, N *are nonnegative constants. Then there is a retraction* $R : B \to S$ *satisfying for all* $x, y \in B$*:*

$$\|Rx - Ry\| \leq \frac{2N}{r}\|x-y\|, \tag{2.3.2}$$

where r *is the unique solution of*

$$\frac{2N}{r} = \frac{M - 2N\ln r}{1-r}. \tag{2.3.3}$$

Proof. See [37] □

In next example, we use theorem (2.3.2) to construct an estimate for $k_0(L^1[0,1])$. Surprisingly this estimate due to Goebel and Komorowski [33] has been the best known estimate for a Banach space until 2006. This has been improved in [39].

Example 2.3.3 *([34])*
Let $X = L^1[0,1]$, B the unit ball and S the unit sphere of X.
For any $f \in S$ and for any $c \in [0,1]$ let:

$$t_f(c) = \sup\left\{t : \int_0^t |f(t)|dt = c\right\}.$$

Set:

$$H(c,f) = \begin{cases} |f(t)| & t \leq t_f(c) \\ f(t) & t > t_f(c) \end{cases}.$$

We note that the homotopy does not join the identity to a given point of S but to the map $|\cdot| : S \to S^+$. To prove that H is a lipschitzian homotopy, without loss in generality, we consider $f, g \in S$ and $c \in [0,1]$ with $t_f(c) \leq t_g(c)$. It is not difficult verify that:

$$\|H(c,f) - H(c,g)\| \leq \|f-g\| + 2\int_{t_f(c)}^{t_f(c)} |g(t)|dt.$$

Previous integral can be increased in two way that produce:

$$\int_{t_f(c)}^{t_f(c)} |g(t)|dt \leq \int_{t_f(c)}^{1} |f-g|dt,$$
$$\int_{t_f(c)}^{t_f(c)} |g(t)|dt \leq \int_{0}^{t_f(c)} |f-g|dt,$$

for which:

$$\begin{aligned} \|H(c,f) - H(c,g)\| &\leq \|f-g\| + 2\min\left(\int_{t_f(c)}^{1} |f-g|dt, \int_0^{t_f(c)} |f-g|dt\right) \\ &\leq 2\|f-g\|. \end{aligned}$$

On the other hand, one verifies that, if $0 \leq c < d \leq 1$:

$$\|H(c,f) - H(d,f)\| \leq 2(d-c),$$

and so

$$\|H(c,f) - H(d,g)\| \leq 2|c-d| + 2\|f-g\|.$$

Exploiting this homotopy we define the retraction by:

$$Rf = \begin{cases} \dfrac{r - \|f\| + |f|}{r} & \|f\| \leq r \\ H\left(1 - a(\|f\|), \frac{f}{\|f\|}\right) & \|f\| > r \end{cases}$$

where the function $a(t)$ is an increasing convex and differentiable function defined on $[r, 1]$ for which $a(r) = 0$ and $a(1) = 1$. When $\|g\|, \|f\| \leq r$ we have:

$$\|Rf - Rg\| \leq \frac{2}{r}\, \|f - g\|,$$

while, the hypothesis $\|f\|, \|g\| \geq r$ lead to:

$$\|Rf - Rg\| \leq \frac{4}{r}\, \|f - g\|,$$

where r, for lemma (2.3.2), satisfies:

$$\frac{2 - r \ln r}{1 - r} = \frac{4}{r}.$$

Numerical method assure that $r \simeq 0.424$ and we conclude that:

$$k_0(L^1[0, 1]) \leq 9.43.$$

By a similar construction Komorowski [47] obtains a lipschitzian retraction in Hilbert space showing that $k_0(H) \leq 64.25$. This estimate has been the best approximation until 2001 when Bolibok [16] put the upper bound to ≃32.26.

Two brief notes.

Firstly we underline that almost all constructions known on Hilbert spaces present an important difficulty.

It is very usual that the maps constructed are not lipschitzian mappings on their domain. They benefits of:

$$\|Tx - Ty\|_H^2 \leq \beta(k)\|x - y\|_H + \gamma(k)\|x - y\|_H^2.$$

To avoid this problem the authors use the following extension theorem due to Kirzbraun and Valentine:

Theorem 2.3.4 *Let $A \subset H$ be an arbitrary set and let $T : A \to H$ be $k-$lispchitzian. Then there exists a $k-$lipschitzian extension $\tilde{T} : H \to \overline{conv}\, T(A) \subset H$ of T.*

Proof. See [57] □

Thus, to obtain a lipschitzian map on the entire Hilbert space we restrict our map on an opportune subset of H where it is lipschitzian, and we apply the previous extension theorem. An example of opportune subset of H we can look to proximinal sets (see [5] page 305).

The second observation arises from the technics used to define the homotopy; this new approach is found in [12] and [14] where Bolobok shows that $k_0(c_0) \leq 35.18$ and $k_0(l^1) \leq 31.64$.

We illustrate this approach in Hilbert spaces from [16].

Bolibok do not start defining directly the homotopy. Firstly he shows first that there exists a lipschitzian mapping $T : S^+ \to S^+$ with minimal displacement

$$d(T) = \frac{\sqrt{2}(\sqrt{k}-1)}{\sqrt{k+1}} \qquad k \geq 2.$$

By this mapping he defines the homotopy by:

$$H(c,x) = \frac{(1-c)x + cT(|x|)}{\|(1-c)x + cT(|x|)\|},$$

that is lipschitzian with constant:

$$M(k) = \frac{2\sqrt{2(k+1)}}{\sqrt{k}-1} \qquad N(k) = \sup_{c\in[\frac{1}{2},1]} \frac{1 + c(k^{\frac{3}{2}}-1)}{\sqrt{1 - 2c(1-c)\frac{(\sqrt{k}+1)^2}{k+1}}}.$$

Later Bolibok considers the 2-lipschitzian mapping $T_2 : B^+(r) \to B^+(r)$ by:

$$(T_2x)(t) = r - \|x\| + x(t),$$

where r is the solution of (2.3.3).

Finally, he defines the lipschitzian retraction depending by the homotopy and by T_2 as in example (2.3.3) and proves that:

$$k_0(H) \leq 32.26$$

holds.

By previous approach, Bolibok moves the difficult problem to define the homotopy onto to define a mapping $T : S \to S$ with minimal displacement known. This is simpler, as an example, if one has some bounds for $\psi_X(k)$ (see [11, 12]).

2.3.2 By perturbations of identity

The second method that we present, finds its roots in paper by Bolibok and Goebel [13] and its generalization by Baronti, Casini and Franchetti [5].

This approach, consists to consider a perturbation of the identity of the type:

$$Ax = x - (1 - \beta(\|x\|))Tx,$$

where $\beta(t) : [0,1] \to [0,1]$ is a continuous increasing function such that $\beta(1) = 1,\ \beta(0) = \gamma \in \left[0, 1 - \frac{1}{2k}\right)$ and $T : B \to B$ a $k-$lipschitzian mapping with minimal displacement $d(T) = d$. It gives an *upper bound for the retraction constant for every extremal space, for Hilbert space and for* l^1*-space*.

Note that $Ax = x$ if $x \in S$.

We observe that:

$$\|Ax\| = \|x - (1 - \beta(\|x\|))Tx\| \geq \|x\| + \beta(\|x\|) - 1,$$
$$\|Ax\| \geq \|x - Tx\| - \beta(\|x\|) \geq d - \beta(\|x\|),$$

and so

$$\|Ax\| \geq \min_{t \in [0,1]} \max\{\beta(t) + t - 1, d - \beta(t)\} := M(d, \beta). \tag{2.3.4}$$

It is easy to prove that $M(d, \beta)$ always exists and if $M(d, \beta) > 0$ and if A is $k(A)$-lipschitzian we can define a retraction by:

$$Rx = \frac{Ax}{\|Ax\|}.$$

We denote with

$$Px = \begin{cases} x & \|x\| \leq 1 \\ \dfrac{x}{\|x\|} & \|x\| > 1 \end{cases}$$

the radial projection of the space X onto B. Let us recall that P is a lipschitzian mapping and $k(P) \in [1, 2]$. In particular $k(P) = 1$ iff X is an Hilbert space and $k(P) < 2$ if X is uniformly non-square space (see [66, 31]). Moreover if $\|x\|, \|y\| \geq r$ one has:

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{k(P)}{r} \|x - y\|. \tag{2.3.5}$$

By this propriety we can assure that R is a lipshitzian map with:

$$k(R) \leq \frac{k(P)k(A)}{M(d, \beta)} \leq \frac{2k(A)}{M(d, \beta)}. \tag{2.3.6}$$

Bolibok and Goebel [13] choose $\beta(t) = t^n$ $(n \in \mathbb{N}, n \leq 20)$ and so

$$Ax = x - (1 - \|x\|^n)Tx.$$

Moreover, since the authors consider only the case X extremal space, we can choose T with minimal displacement d close to $1-\frac{1}{k}$. Thus in the sequel $d := 1-\frac{1}{k}$

Bolibok et al. exhibit a way to construct retractions in the case $n=1$ and $n=2$ and they state that computer elaborations with $n \leq 20$ give the best result for $n=4$. Here we do not point out neither the numerical experiments, nor the construction for $n=1$ or $n=2$ but we present the calculus parts for the best case $n=4$.

Note that A is a lipschitzian mapping since is a composition of lipschitzian mappings. We compute the constant:

$$\begin{aligned} \|Ax-Ay\| &\leq \|x-y\| + \left\| -(1-\|x\|^4)Tx + (1-\|y\|^4)Ty \pm \|x\|^4 ty \right\| \\ &\leq \|x-y\| + (1-\|x\|^4)k\|x-y\| + \left| \|x\|^4 - \|y\|^4 \right| \\ &\leq \left[1+k-k\|x\|^4 + (\|x\|^3 + \|y\|^3 + \|x\|\|y\|^2 + \|x\|^2\|y\|) \right] \|x-y\|. \end{aligned}$$

Similarly:

$$\|Ax-Ay\| \leq \left[1+k-k\|y\|^4 + (\|x\|^3 + \|y\|^3 + \|x\|\|y\|^2 + \|x\|^2\|y\|) \right] \|x-y\|,$$

thus

$$\begin{aligned} \|Ax-Ay\| &\leq \left[1+k-k\max\{\|y\|^4, \|x\|^4\} + 4\max\{\|y\|^3, \|x\|^3\} \right] \|x-y\| \\ &\leq \max_{t\in[0,1]} (1+k-kt^4+4t^3)\|x-y\| = \left(1+k+\frac{27}{k^3} \right) \|x-y\|. \end{aligned}$$

Moreover one observes, following (2.3.4), that for an opportune value of k, Ax is non vanishing for any $x \in B$ and so one can provide a lower bound for $\|Ax\|$ depending only by k:

$$\|Ax\| \geq \min_{t\in[0,1]} \max\{t^4+t-1, 1-\frac{1}{k}-t^4\} := M(k).$$

Thus, defining the map $R_0 : B \to S$ by

$$R_0x = \frac{Ax}{\|Ax\|},$$

so (2.3.6) assures that R_0 is a lipschitzian retraction with:

$$k(R_0) \leq \frac{k(P)(1+k+27/k^3)}{M(k)} \leq \frac{2(1+k+27/k^3)}{M(k)}.$$

Numerical experiments give that for $k \simeq 4.02$, $M(k)$ is strictly positive and the minimum value for $k(R_0)$ is obtained. We conclude that:

Theorem 2.3.5 *For every extremal space X:*

$$k_0(X) \leq k(R_0) \leq 37.74.$$

Remark 2.3.6 *Theorem (2.3.5) improves Franchetti [30] for which $k_0(X) < 933$, if X is an extremal space.*

In 2003, Baronti, Casini and Franchetti in [5] generalize the previous approach improving the estimates on the extremal spaces, on the Hilbert spaces and on the space l^1.

We prefer split the method of Caronti et al. in steps.

STEP 1. The author prove the following minimun principle.

Proposition 2.3.7 *Define:*

$$G = \{g \in C^1[0,1] : g(0) = \gamma,\ g(1) = 1\},$$
$$\Phi : G \to C[0,1]; \qquad \Phi(g) = (g-1)' - L(g-1),$$

where γ and $L > 0$ are fixed constants. Then

$$\inf\{\|\Phi(g)\| : g \in G\} = \frac{L|1-\gamma|}{1-e^{-L}} = \|\Phi(\overline{g})\|,$$

where the norm is the usual supremum norm and

$$\overline{g}(t) = 1 + e^{Lt}\left[\int_0^t e^{Ls}\frac{L(1-\gamma)}{1-e^{L}}ds + (\gamma - 1)\right].$$

STEP 2. They show that it is possible to construct a lipschitzian mapping, using perturbations of identity map.

Proposition 2.3.8 *Let $T : B \to B$ be a $L-$lipschitzian map. Suppose that there exists a function $d : [0,1] \to \mathbb{R}$ such that:*

$$\|x - Tx\| \geq d(\|x\|),$$

for any $x \in B$.

Define, for $x \in B$:

$$Ax = x - (1 - \overline{g}(\|x\|))Tx,$$

then (trivially) $Ax = x$ for $x \in S$ and moreover:

(i) For any $x \in B$, $\|Ax\| \geq M$ where $M = \min_{t\in[0,1]} \max\{t - 1 + \overline{g}(t), d(t) - \overline{g}(t)\}$.

(ii) A is $k(A)-$lipschitzian where:

$$k(A) = 1 + \frac{L(1-\gamma)}{1-e^{-L}}.$$

(iii) If $M > 0$ then $Rx = \frac{Ax}{\|Ax\|}$ defines a $k(R)-$lipschitzian retraction with:

$$k(R) = \frac{k(P)}{M} \left(1 + \frac{L(1-\gamma)}{1-e^{-L}}\right).$$

STEP 3. The authors apply the previous steps when X is an extremal space showing that:

Theorem 2.3.9

$$k_0(X) \leq 30.84.$$

When $X = H$ is an Hilbert space they prove that:

Theorem 2.3.10

$$k_0(H) \leq 28.99.$$

And in the end, if $X = l^1$, they obtain that:

Theorem 2.3.11

$$k_0(l^1) \leq 22.45.$$

Remark 2.3.12 *The main differences between the above lie in the choice of the function β. Taking $\overline{g}(t)$ as in Step 1 means to decrease the lipschitzian constant of A and in consequence to decrease the value of $k_0(X)$.*

2.3.3 By radial projection on maps that vanish on the sphere

Let us conclude this description of the optimal retraction problem with a glance to a last method to construct retractions in infinite dimensional Banach space.

The first approach to this method is given by Goebel [35] (2001), when he prove that it is possible improve the optimal retraction constant for $C[0, 1]$ with the value $k_0 \simeq$23.31.

Let us describe the main idea supposing that $T : B \to B$ is a mapping in $\mathcal{L}(k)$.

Let us suppose that the minimal displacement of T is strictly greater than zero and suppose that for any $x \in S$, $Tx = 0$ (why does this map exists? We have constructed one of these maps in the preface to show theorem (2.1.3)).

Consider the retraction:

$$Rx = \frac{x - Tx}{\|x - Tx\|}. \tag{2.3.7}$$

By the radial projection, we observe that:

$$Rx = \frac{(x - Tx)d(T)}{\|x - Tx\|d(T)} = \frac{x - Tx}{d(T)} \cdot \frac{d(T)}{\|x - Tx\|} = P\left(\frac{x - Tx}{d(T)}\right),$$

since
$$\frac{\|x - Tx\|}{d(T)} \geq 1.$$

In this way we can estimate the Lipschitz constant of the retraction in terms of k and $d(T)$:
$$k(R) \leq 2\frac{k+1}{d(T)}.$$

This permits to state that:
$$k_0(X) \leq \min_{k>1} \ 2\frac{k+1}{d(T)}. \tag{2.3.8}$$

What is the maximum value that the estimate (2.3.8) can achieve?

Observing that $d(T) \leq 1 - \dfrac{1}{k}$, we get:
$$\min_{k>1} 2\frac{k+1}{d(T)} \geq \min_{k>1} \ 2\frac{k(k+1)}{k-1} = 11.66.$$

Thus, the best retraction constant obtainable by this method cannot be lower than 11.66.

Therefore, this idea cannot be used to improve approximations of $k_0(L^1[0,1])$.

Goebel and Marino [38] went very close to this "limit" showing that:

Theorem 2.3.13 *Let us consider the space of continuous functions in* $[0,1]$ *vanishing in zero. Then:*
$$k_0(C_0[0,1]) \leq 12.$$

This improves the estimate $k_0(C_0[0,1]) \leq 17.38$ given in [34].

In next section we will prove that this result can be extended to a larger class of spaces: the space of bounded and continuous functions defined on a connected metric space K that vanish in a point $z \in K$.

In next example, to apply the method above, we prove that for the space of real sequences converging to zero (c_0) we have $k_0(c_0) \leq 23.31$.

This proof traces a original idea of Goebel [35].

Example 2.3.14 *Let* c_0 *be the space of real sequences convergent to 0 with the standard norm of maximum and* B *be the unit ball and* S *be the unit sphere in* c_0.
We recall that $\alpha : \mathbb{R} \to [-1,1]$ *is a nonexpansive function defined by:*
$$\alpha(t) = \begin{cases} 1 & t \geq 1 \\ t & t \in [-1,1] \\ -1 & t \leq -1 \end{cases}.$$

Let $T_{0,k} : B \to S \subset B$ be a mapping defined by:

$$T_{0,k}x = T_{0,k}(\eta_1, \eta_2, \ldots) = (1, \alpha(k|\eta_1|), \alpha(k|\eta_2|), \ldots).$$

Following [12, 34] we note that:

(1.) $T_{0,k}$ *is* $k-$*lipschitzian.*

(2.) $d(T_{0,k}) = 1 - \dfrac{1}{k}$.
Proof. *of (2).*
Let $x \in c_0$. *Then there exists an index* $i_k \neq 1$ *for which* $|\eta_{i_k}| \leq \frac{1}{k}$ *but* $|\eta_{i_k-1}| \geq \frac{1}{k}$. *We conclude that:*

$$\|x - T_{0,k}x\|_\infty \geq |\alpha(k|\eta_{i_k-1}|) - \eta_{i_k}| \geq |\alpha(k|\eta_{i_k-1}|)| - |\eta_{i_k}| \geq 1 - \frac{1}{k}.$$

□

(3.) There does not exist $x \in B$ *such that* $\|x - T_{0,k}x\|_\infty = 1 - \dfrac{1}{k}$.

Remark 2.3.15 *(2.) means that* c_0 *is an extremal space, hence the estimate of* $k_0(c_0)$ *by Baronti, Casini and Franchetti (*$k_0(c_0) \leq 30.84$*) holds. Now we improve it.*

Let $Q : c_0 \to B$ *be a nonexpansive mapping defined by* $Qx = (\alpha(|\eta_i|))_{n\in\mathbb{N}}$ *and for* $r \geq 0$ *let* $Q_r : c_0 \to rB$ *be:*

$$\begin{cases} Q_r x = \left(r\alpha\left(\frac{|\eta_i|}{r}\right)\right)_{i\in\mathbb{N}} & r > 0 \\ \\ Q_0 x = 0 & r = 0 \end{cases}.$$

We can show that if $r_1, r_2 > 0$ *and* $x, y \in c_0$ *we have:*

$$\|Q_{r_1}x - Q_{r_2}y\|_\infty \leq \max\{\|x - y\|_\infty, |r_1 - r_2|\}.$$

Proof. *Let* $r_2 > r_1$. *Let us consider:*

$$\left| r_1\alpha\left(\frac{|\eta_i|}{r_1}\right) - r_2\alpha\left(\frac{|\xi_i|}{r_2}\right)\right| = (*).$$

If we take $f(t) = |\eta_i|$ *and* $g(t) = |\xi_i|$ *for all* $t \in [0, 1]$*, following [35] we obtain:*

$$\begin{aligned} (*) &= \left\| r_1\alpha\left(\frac{f}{r_1}\right) - r_2\alpha\left(\frac{g}{r_2}\right)\right\|_\infty \\ &\leq \max\{\|f - g\|_\infty, |r_1 - r_2|\} = \max\{|\eta_i - \xi_i|, |r_1 - r_2|\}, \end{aligned}$$

and we conclude that:

$$\begin{aligned} \|Q_{r_1}x - Q_{r_2}y\|_\infty &= \max_{i\in\mathbb{N}} \left| r_1\alpha\left(\frac{|\eta_i|}{r_1}\right) - r_2\alpha\left(\frac{|\xi_i|}{r_2}\right)\right| \\ &= \max_{i\in\mathbb{N}}(\max\{|\eta_i - \xi_i|, |r_1 - r_2|\}) \\ &= \max\{\|x-y\|_\infty, |r_1 - r_2|\}. \end{aligned}$$

□

Now, we extend $T_{0,k}$ on the ball with radius 2 (B_2) as in [35], i.e. we define $T_{1,k} : B_2 \to B$ by:

$$T_{1,k}x = \begin{cases} T_{0,k}x, & \|x\|_\infty \le 1 \\ T_{0,k}Qx, & \|x\|_\infty \in \left[1, 2-\frac{1}{k}\right] \\ Q_{k(2-\|x\|_\infty)}T_{0,k}Qx, & \|x\|_\infty \in \left[2-\frac{1}{k}, 2\right] \end{cases} .$$

Remark 2.3.16

1. *If $x \in S_2$ then $T_{1,k}x = 0$;*
2. *$T_{1,k}$ is $k-$lipchitzian;*
3. $\inf\{\|x - T_{1,k}x\|_\infty : x \in B_2\} = 1 - \dfrac{1}{k}$.
   Proof. *of (3).*
   We distinguish 3 cases:
   (a) *If $\|x\|_\infty \le 1$ then $\|x - T_{1,k}x\|_\infty = \|x - T_{0,k}x\|_\infty > 1 - \dfrac{1}{k}$.*
   (b) *If $\|x\|_\infty \in \left[2 - \dfrac{1}{k}, 2\right]$, firstly we observe by the definition of $Q_r x$ that for all $T_{0,k}Qx \in B$:*

$$\|Q_{k(2-\|x\|_\infty)}T_{0,k}Qx\|_\infty \le k(2 - \|x\|_\infty) \le k\left(2 - 2 + \frac{1}{k}\right) = 1,$$

thus:

$$\begin{aligned} \|x - T_{1,k}x\|_\infty &= \|x - Q_{k(2-\|x\|_\infty)}T_{0,k}Qx\|_\infty \\ &\ge \|x\|_\infty - \|Q_{k(2-\|x\|_\infty)}T_{0,k}Qx\|_\infty \\ &\ge \left(2 - \frac{1}{k}\right) - 1 = 1 - \frac{1}{k}. \end{aligned}$$

(c) Let $\|x\|_\infty \in \left[1, 2 - \frac{1}{k}\right]$.
Since $x = (\eta_n) \in c_0$ *and* $\max_{i\in\mathbb{N}} |\eta_i| \geq 1$, *there exists an index* $i_0 \in \mathbb{N}$ *such that:*

$$\forall i \geq i_0,\ |\eta_i| \leq \frac{1}{k} \text{ and } |\eta_{i_o-1}| \geq \frac{1}{k}.$$

It follows that:

$$\begin{aligned}\|x - T_{1,k}x\|_\infty &= \max\{|1-\eta_1|, |\alpha(k\alpha(|\eta_i|)) - \eta_{i+1}|, i \in \mathbb{N}\} \\ &\geq |\alpha(k\alpha(\eta_{i_0-1})) - \eta_{i_0}| \\ &\geq |\alpha(k\alpha(\eta_{i_0-1})| - \frac{1}{k}.\end{aligned}$$

Now, if $|\eta_{i_0-1}| > 1$, $\alpha(|\eta_{i_0-1}|) = 1$ *and* $\alpha(k\alpha(|\eta_{i_0-1}|)) = \alpha(k) = 1$. *On the other side if* $|\eta_{i_0-1}| \in \left[\frac{1}{k}, 1\right]$ *we have* $\alpha(|\eta_{i_0-1}|) = |\eta_{i_0-1}|$ *and* $\alpha(k\alpha(|\eta_{i_0-1}|) = \alpha(k|\eta_{i_0-1}|) = 1$.
So in both cases:

$$\alpha(k\alpha(|\eta_{i_0-1}|)) - \frac{1}{k} = 1 - \frac{1}{k} \Rightarrow \|x - T_{1,k}x\|_\infty \geq 1 - \frac{1}{k}.$$

□

Let $T_{2,k} : B \to B$ *defined by:*

$$T_{2,k}x = \frac{1}{2}T_{1,k}(2x).$$

We note that:

1. $T_{2,k} \in \mathcal{L}_B(k)$ *and* $T_{2,k}S = \{0\}$;

2. $\|x - T_{2,k}x\|_\infty \geq \frac{1}{2}\left(1 - \frac{1}{k}\right)$;

   Proof.

$$\|x - T_{2,k}x\|_\infty = \left\|\frac{1}{2}2x - \frac{1}{2}T_{1,k}(2x)\right\|_\infty \geq \frac{1}{2}\left(1 - \frac{1}{k}\right)$$

   for all $x \in B$. □

We can conclude that:

$$\psi_{c_0}(k) \geq d(T_{2,k}) \geq \frac{1}{2}\left(1 - \frac{1}{k}\right),$$

and from this:

$$k_0(c_0) \leq \min_{k\geq 1}\left\{\frac{2(k+1)}{\frac{1}{2}\left(1 - \frac{1}{k}\right)}\right\} \simeq 23.31.$$

2.4 Contributions to the problem

The aim of this section is to explain our contributions to the problem of calculus of optimal retraction constant providing some improved upper estimates for a number of Banach spaces.

The results in the first subsection are suitable modifications of ideas contained in previous works of Goebel [34, 35, 37, 38] and Bolibok [11, 12, 13, 15] while the results in the second subsection are based on suitable modifications of the trick due to Annoni and Casini [2].

2.4.1 Spaces with uniform norm which are cut invariant

Let K be an arbitrary nonempty and infinite set and let $B(K)$ be the Banach space of all real bounded functions on K furnished with the uniform norm $\|f\| = \sup\{|f(t)| : t \in K\}$. Let us call a subspace $X \subset B(K)$ to be *cut invariant* if with any $f \in X$ also the function $Qf = \alpha \circ f \in X$ where α is the *cut function* defined in (2.2.1):

$$\alpha(t) = \begin{cases} -1 & \text{for } t < -1, \\ t & \text{for } -1 \leq t \leq 1, \\ 1 & \text{for } t > 1. \end{cases}$$

Obviously, the operation Q cutting each function on the level -1 and 1 is a nonexpansive retraction of X onto it's unit ball $B = B_X$ (i.e. $Q : X \to B$ and $Q|_B = Id$). Also Q generates the family of retractions $Q_r : X \to rB,\ r \geq 0$ defined by:

$$Q_r f = \begin{cases} 0 & \text{if } r = 0, \\ rQ(\frac{1}{r}f) & \text{if } r > 0 \end{cases}.$$

It is just a technicality to prove that

$$\|Q_{r_1}x - Q_{r_2}y\| \leq \max\{\|x - y\|, |r_1 - r_2|\}, \tag{2.4.1}$$

for all $r_1, r_2 \geq 0$ and $x, y \in X$.

For any set K the whole space $B(K)$ is cut invariant but there are other subspaces with the same property. For example if we fix any subset $A \subset K$ then the spaces $X_{0,A} = \{f \in X : f(t) = 0 \text{ for } t \in A\}$, $X_{const,A} = \{f \in X : f(t) = const \text{ on } A\}$ are invariant under Q. If the set K is a topological space, then the space of continuous functions $BC(K)$ and many of their subspaces are also cut invariant. In particular the classical sequence space c_0 (like the last example in previous section shows) and the space of continuous functions $C[0, 1]$ together with it is subspaces $C_0[0, 1] = \{f \in C[0, 1] : f(0) = 0\}$, $C_{const\{0,1\}} = \{f \in C[0, 1] : f(0) = f(1)\}$ are cut invariant. The same can be said for the space of bounded continuous functions on the whole line $BC(\mathbb{R})$ and it is analogously defined subspaces.

Some cut invariant spaces are also extremal with respect to the function ψ_X. For the convenience of the reader we recall three facts giving only hints for the proof.

Claim 2.4.1 $\psi_{c_0}(k) = 1 - \frac{1}{k}$.

Proof.[Hint] For any $k > 1$ consider the $k-$lipschitzian mapping $T_k : B \to B$

$$T_k x = T_k(\xi_1, \xi_2, \ldots) = (1, \alpha(k|\xi_1|), \alpha(k|\xi_2|), \ldots),$$

and observe that for any $x = (\xi_1, \xi_2, \ldots) \in c_0$, $\|x - T_k x\| > 1 - \frac{1}{k}$. □

Claim 2.4.2 $\psi_{C[0,1]}(k) = 1 - \frac{1}{k}$.

Proof.[Hint] Let $g \in C[0,1]$ be a strictly increasing function such that $g(0) \le -2$ and $g(1) \ge 2$. Then for any $k > 1$ the mapping

$$(T_k f)(t) = Q(k(f+g))(t) = \alpha(k(f(t) + g(t))),$$

is $k-$lipschitzian and satisfies $\|f - T_k f\| \ge 1 - \frac{1}{k}$. □

Claim 2.4.3 $\psi_{BC(\mathbb{R})}(k) = 1 - \frac{1}{k}$.

Proof. See example (2.2.10). □

Let us pass to the main observation.

Claim 2.4.4 *Let X be an extremal and cut invariant subspace of $B(K)$. Then:*

$$k_0(X) \le 4(1+\sqrt{2})^2 = 23.31..$$

Proof. Let $T : B \to B, T \in \mathcal{L}(k)$ be such that $d(T) = d > 0$.Extend T to the map $T_1 : 2B \to B$ by putting:

$$T_1 x = \begin{cases} Tx, & \|x\| \le 1 \\ TQx, & \|x\| \in [1, 2 - \frac{1}{k}] \\ Q_{k(2-\|x\|)} TQx, & \|x\| \in [2 - \frac{1}{k}, 2] \end{cases} .$$

We leave to the reader to check that T_1 is $k-$lipschitzian on $2B$ and that, since $d \le 1 - \frac{1}{k}$, $\|x - T_1 x\| \ge d$ for all $x \in 2B$. Observe also that T_1 maps the doubled unit sphere $2S$ into the origin, $T_1(2S) = \{0\}$.

Let $P : X \to B$ denotes the radial retraction

$$Px = \begin{cases} x & \text{if } \|x\| \le 1 \\ \frac{x}{\|x\|} & \text{if } \|x\| > 1 \end{cases}.$$

Now, denoting:

$$T_2 x = \frac{1}{2} T_1(2x),$$

we obtain a $k-$lipschitzian mapping $T_2 : B \to B$ with $d(T_2) = \frac{1}{2} d(T)$ and $T_2(S) = \{0\}$. By T_2, we can generate a retraction $R : B \to S$ upon setting:

$$Rx = P\left(\frac{x - T_2 x}{\frac{1}{2}d}\right).$$

Since P is $2-$lipschitzian, standard calculations show that the Lipschitz constant $k(R)$ of R satisfies:

$$k(R) \le 4\frac{k+1}{d(T)}.$$

Hence, since the space is extremal we can select mapping T with $d(T)$ arbitrarily close to $1 - \frac{1}{k}$. In view of this we can write:

$$k_0(X) \le 4\frac{k(k+1)}{k-1},$$

for each $k > 1$. Finally, choosing $k = 1 + \sqrt{2}$ which minimizes last estimate we get:

$$k_0(X) \le 4 \min_{k>1} \frac{k(k+1)}{k-1} = 4(1+\sqrt{2})^2 = 23.31..$$

□

Remark 2.4.5 *The spaces $BC(0,1)$, $BC(0,\infty)$, $BC(a,b)$, $BC(-\infty,a)$ are isometric to $BC(\mathbb{R})$ so we can obtain the same estimate for $k_0(X)$ (i.e. $k_0(X) \le 23.31$) when X is one of them.*

Remark 2.4.6 *Observe that the standard projection P from X on B does not satisfy (2.4.1) in previous proposition when $P_r x = rP\left(\frac{x}{r}\right)$ but satisfy the weaker inequalities:*

$$\|P_{r_1}x - P_{r_2}y\| \le \max\{4\|x-y\|, 2|r_1 - r_2|\}.$$

2.4.2 Spaces $BC_z(K)$ on connected metric spaces K

In this section we modify the idea contained in [38] to discuss the special case of spaces $BC_z(K)$ of bounded continuous functions $f : K \to \mathbb{R}$ defined on a connected metric space K and vanishing at a given point z, $f(z) = 0$. We give a direct construction leading to:

Claim 2.4.7 *For each connected metric space K and each $z \in K$:*

$$k_0(BC_z(K)) \leq 12.$$

Proof. Let ρ be the metric on K and let $r \in [0, \infty]$ be defined by $r = \{\rho(x, z) : x \in K\}$. Fix a number $a > 0$ such that $ar > 1$. Now consider the function $\Lambda : [0, +\infty) \to \left[0, \frac{3}{2}\right]$ defined by:

$$\Lambda(t) = \begin{cases} 3t & \text{for } 0 \leq t \leq \frac{1}{2} \\ 3(1-t) & \text{for } \frac{1}{2} \leq t \leq 1 \\ 0 & \text{for } t \geq 1 \end{cases}$$

Using the above, let us define a mapping $T_0 : BC_z(K) \to \frac{3}{2}B$ by:

$$(T_0 f)(x) = \Lambda(|f(x)| + a\rho(z, x)).$$

Observe that T_0 satisfies Lipschitz condition with constant $k = 3$. Also observe that, since K is connected, for any function $f \in BC_z(K)$ there exists a point $x_1 \in K - \{z\}$ such that $|f(x_1)| + a\rho(z, x_1) = \frac{1}{2}$. Hence, for all $f \in BC_z(K)$ we have:

$$\|T_0 f - f\| \geq |(T_0 f)(x_1) - f(x_1)| \geq |(T_0 f)(x_1)| - |f(t_1)| = \frac{3}{2} - \frac{1}{2} + a\rho(z, x_1) > 1.$$

For functions satisfying $\|f\| > 1$, there is a point x_2 such that $|f(x_2)| > 1$, and we have $(T_0 f)(x_2) = 0$.
Now, define the mapping $T_1 : \frac{3}{2}B \to \frac{3}{2}B$,

$$(T_1 f)(x) = \begin{cases} (T_0 f)(x) & \text{if } \|f\| \leq 1 \\ \min\{(T_0 f)(x), 3(\frac{3}{2} - \|f\|)\} & \text{if } 1 \leq \|f\| \leq \frac{3}{2} \end{cases}.$$

Observe the following facts:

1. The mapping T_1 is lipschitzian with Lipschitz constant $k = 3$.
2. For all $f \in \frac{3}{2}B$, $\|T_1 f - f\| > 1$.
3. T_1 sends the boundary of the ball $\frac{3}{2}B$, the sphere $\frac{3}{2}S$ into the origin. In other words $T_1\left(\frac{3}{2}S\right) = \{0\}$.

Items 1 and 3 are obvious, item 2 follows by the general evaluation for functions $f \in B$ and by the existence of the point x_2 for functions with norm satisfying $1 < \|f\| \leq \frac{3}{2}$. Consequently the mapping $T : B \to B$ defined by:

$$T(f) = \frac{2}{3} T_1 \left(\frac{3}{2} f \right),$$

has the same Lipschitz constant $k = 3$, and it satisfies $\|T_1 f - f\| \geq \frac{2}{3}$, for all $f \in B$. Moreover, it sends the unit sphere S to the origin, $T(S) = \{0\}$. Now we can define the retraction $R : B \to S$. Put:

$$Rf = \frac{f - Tf}{\|f - Tf\|} = P \left(\frac{3}{2}(f - Tf) \right).$$

Here, as before, $P : BC_z(K) \to B$ denotes the radial projection.
Now we have:

$$\begin{aligned} \|Rf - Rg\| &= \left\| P(\frac{3}{2}(f - Tf)) - P(\frac{3}{2}(g - Tg)) \right\| \leq 3\|(f - Tf) - (g - Tg)\| \\ &\leq 3\|f - g\| + 3\|Tf - Tg\| \leq 3\|f - g\| + 9\|f - g\| = 12\|f - g\|. \end{aligned}$$

Summing up. For the space $BC_z(K)$ there exists the retraction $R : B \to S$ satisfying:

$$\|Rf - Rg\| \leq 12\|f - g\|,$$

which implies our claim. □

2.4.3 The space $L^1[0, 1]$

The content of this section presents a modification of the trick of M. Annoni and E. Casini given in [2]. They have proved that $k_0(l^1) \leq 8$. Similar estimate holds for the function space $L^1[0, 1]$.

Theorem 2.4.8 *For the space* $L^1[0, 1]$

$$k_0(L^1[0, 1]) \leq 8.$$

Proof. Let us define:

$$B_1 = \left\{ f \in L_1 : \|f\|_1 \leq \frac{1}{2} \right\},$$

$$B_2 = \left\{ f \in L_1 : \frac{1}{2} \leq \|f\|_1 \leq 1 \right\}.$$

For each $f \in B_2$ define $t_f \in [0, 1]$:

$$t_f = \sup \left\{ t \in [0, 1] : \int_t^1 |f(s)| ds = 1 - \|f\|_1 \right\}.$$

Let us be T a mapping defined on B_2 as

$$(Tf)(t) = \begin{cases} 0 & t \in (0, t_f) \\ f(t) & t \in [t_f, 1) \end{cases}$$

Remark 2.4.9

1. *$T : B_2 \to B_1$. In fact $\|Tf\|_1 = 1 - \|f\|_1 \leq \frac{1}{2}$.*
2. *If $\|f\|_1 = 1$ then $t_f = 1$ and $(Tf)(t) = 0$ for all $t \in [0, 1]$.*
3. *By (1.) and by the definition of L^1-norm we have $t_f = t_{|f|}$ and moreover, $(Tf)(t) = (T|f|)(t) = 0$ for all $t \in [0, 1]$ if $\|f\| = 1$.*

Proposition 2.4.10 *The mapping T is 3-Lipschitz and $(I - T)$ (where I is the identity map) is 2-Lipschitz.*

Proof. If $\|f\|_1 = \|g\|_1 = 1$ there is nothing to show.
We only consider the case $t_f \neq t_g$ and $\|f\|_1 \neq \|g\|_1 \neq 1$ because the other cases are not difficult to verify. Without lost in generality let's suppose $t_f < t_g$. So:

$$\begin{aligned} \|Tf - Tg\|_1 &= \int_{t_f}^{t_g} |f(s)|ds + \int_{t_g}^{1} |f(s) - g(s)|ds \\ &= \int_{t_f}^{1} |f(s)|ds - \int_{t_g}^{1} |f(s)|ds + \int_{t_g}^{1} |f(s) - g(s)|ds \\ &= 1 - \|f\|_1 - \int_{t_g}^{1} |f(s)|ds + \int_{t_g}^{1} |f(s) - g(s)|ds \\ &= \int_{t_g}^{1} |g(s)|ds + \|g\|_1 - \|f\|_1 - \int_{t_g}^{1} |f(s)|ds + \int_{t_g}^{1} |f(s) - g(s)|ds \\ &\leq 3\|f - g\|_1. \end{aligned}$$

$$\begin{aligned}
\|(I-T)f-(I-T)g\|_1 = & \int_0^{t_f} |f(s)-g(s)|ds + \int_{t_f}^{t_g} |g(s)|ds \\
= & \int_0^{t_f} |f(s)-g(s)|ds + \int_{t_f}^{1} |g(s)|ds - \int_{t_g}^{1} |g(s)|ds \\
= & \int_0^{t_f} |f(s)-g(s)|ds + \int_{t_f}^{1} |g(s)|ds - (1-\|g\|_1) \\
= & \int_0^{t_f} |f(s)-g(s)|ds + \int_{t_f}^{1} |g(s)|ds \\
& - \left(\int_{t_f}^{1} |f(s)|ds + \|f\|_1 - \|g\|_1\right) \\
\leq & \int_0^{t_f} |f(s)-g(s)|ds + \int_{t_f}^{1} |g(s)-f(s)|ds + \|f-g\|_1 \\
\leq & 2\|f-g\|_1.
\end{aligned}$$

□

Now we observe that:

$$\|(I-T)f\|_1 = \int_0^{t_f} |f(s)|ds = \|f\|_1 - \int_{t_f}^{1} |f(s)|ds = 2\|f\|_1 - 1.$$

Let R_1 be a mapping defined on B_1:

$$(R_1 f)(t) = 2|f(t)| + 1 - 2\|f\|_1,$$

and we observe that:

(a) $\|(R_1)f\|_1 = 1$ for all $f \in B_1$;

(b) $\|R_1 f - R_1 g\|_1 \leq 4\|f-g\|_1$ for all $f, g \in B_1$.

Let us define $R_2 : B_2 \to S$ in this way:

$$(R_2 f)(t) = [(I-T)f](t) + 2(T|f|)(t) = \begin{cases} f(t) & t \in (0, t_f) \\ 2|f(t)| & t \in [t_f, 1) \end{cases}.$$

R_2 is well defined by (3.) in remark (2.4.9).
We note that:

(i) $\|R_2 f - R_2 g\|_1 \leq 2\|f-g\|_1 + 2 \cdot 3\|f-g\|_1 = 8\|f-g\|_1$.

(ii) If $f \in S$, $(R_2 f)(t) = f$. In fact, by (2.) and (3.) in remark 2.4.9 we have $(T|f|)(t) = (Tf)(t) = 0$ for all $t \in [0,1]$ and so $(R_2 f) = (I-T)f$.

(iii) If $\|f\|_1 = \dfrac{1}{2}$ then $R_2 f = 2|f| = R_1 f$ a.e. in [0,1].

In the end we consider $R : B \to S$ defined by:

$$Rf = \begin{cases} R_1 f & f \in B_1 \\ R_2 f & f \in B_2 \end{cases} .$$

By (ii) we conclude that R is a retraction for $L_1[0,1]$ and by (iii) we say that R is a 8-Lipschitzian retraction.
So:

$$k_0(L_1[0,1]) \leq k(R) \leq 8.$$

□

2.4.4 The spaces $AC_0[0,1]$ and $BV[0,1] \cap C_0[0,1]$

Another space which has as good as $l^1$and $L^1[0,1]$ evaluation of the retraction constant is $AC_0[0,1]$, the space of absolutely continuous functions on $[0,1]$ vanishing at 0 furnished with the $BV-$norm:

$$\|h\|_{BV} = Var_{[0,1]}(h) = \sup \sum_{j=1}^{n} |h(t_j) - h(t_{j-1})| = \int_0^1 |h'(s)| ds,$$

where the supremum is taken over all partition $0 = t_0 < t_1 < \ldots < t_n = 1$.

Claim 2.4.11

$$k_0(AC_0[0,1]) \leq 8.$$

Proof. The above comes from the fact that the retraction constant remains the same for isometric spaces. $AC_0[0,1]$ can be isometrically mapped onto $L^1[0,1]$ by isometry sending each function h into h'. □

Let us end with showing that almost the same technical tricks work for larger space $X = (BV[0,1] \cap C_0[0,1], \| \cdot \|_{BV})$.

Claim 2.4.12

$$k_0(BV[0,1] \cap C_0[0,1]) \leq 8.$$

Proof. For all $h \in X$ and for all interval $[a,b] \subset [0,1]$ denote

$$\|h\|_{[a,b]} := \|\chi_{[a,b]} h\|_{BV} = Var_{[a,b]}(h).$$

Then for any fixed $h \in X$, the function of t, $\|h\|_{[0,t]}$ is a continuous non decreasing functions with range $[0, \|h\|_{BV}]$.
Let us consider the number:

$$t_h = \sup\{t \in [0,1] : \|h\|_{[0,t]} \leq 1 - \|h\|_{BV}\}.$$

We observe that:

(1.) $\|h\|_{BV} \leq \frac{1}{2}$ implies $t_h = 1$.

(2.) $\|h\|_{BV} \geq \frac{1}{2}$ implies $\|h\|_{[0,t_h]} = 1 - \|h\|_{BV} \leq \frac{1}{2}$.

Let us define the mapping $W : B \to \frac{1}{2}B$ by

$$(Wh)(t) = \begin{cases} h(t) & \text{if } t \in [0, t_h] \\ h(t_h) & \text{if } t \in [t_h, 1] \end{cases} .$$

Then $\|Wh\|_{BV} = \|h\|_{[0,t_h]}$. Moreover:

(3.) If $\|h\|_{BV} \leq \frac{1}{2}$ then $Wh = h$.

(4.) If $\|h\|_{BV} = 1$ then $\|h\|_{[0,t_h]} = 0 \;\Rightarrow\; h(t) = 0$ on $[0, t_h] \;\Rightarrow\; Wh = 0$, i.e. $W : S \to \{0\}$.

(5.) W is 3-lipschitzian.
Proof.[Proof of (5.)] It is sufficient take $\|f\|_{BV}, \|g\|_{BV} \geq \dfrac{1}{2}$, and $t_f \leq t_g$. Then

$$(Wf - Wg)(t) = \begin{cases} f(t) - g(t) & \text{if } t \in [0, t_f] \\ f(t_f) - g(t) & \text{if } t \in [t_f, t_g] \\ f(t_f) - g(t_g) & \text{if } t \in [t_g, 1] \end{cases} .$$

$$\begin{aligned} \|Wf - Wg\|_{BV} &= \|f-g\|_{[0,t_f]} + \|g\|_{[t_f,t_g]} \\ &= \|f-g\|_{[0,t_f]} + \|g\|_{[0,t_g]} - \|g\|_{[0,t_f]} \\ &= \|f-g\|_{[0,t_f]} + 1 - \|g\|_{BV} - \|g\|_{[0,t_f]} \\ &= \|f-g\|_{[0,t_f]} + \|f\|_{BV} + \|f\|_{[0,t_f]} - \|g\|_{[BV]} - \|g\|_{[0,t_f]} \\ &\leq \|f-g\|_{[0,t_f]} + \|f\|_{BV} - \|g\|_{[BV]} + \|f-g\|_{[0,t_f]} \\ &\leq 3\|f-g\|_{BV}. \end{aligned}$$

□

(6.) $(I - W) = M$ is 2-lipschitzian.

Proof.[Proof of (6.)] Since $\|f\|_{BV} \leq \frac{1}{2} \Rightarrow (I-W)f = 0$, it is enough to consider $\|f\|_{BV}$, $\|g\|_{BV} \geq \frac{1}{2}$, $t_f \leq t_g$. Then

$$[Mf - Mg](t) = \begin{cases} 0 & \text{if } t \in [0, t_f] \\ f(t) - f(t_f) & \text{if } t \in [t_f, t_g] \\ f(t) - f(t_f) - g(t) + g(t_g) & \text{if } t \in [t_g, 1] \end{cases} .$$

Hence:

$$\begin{aligned} \|(I-W)f - (I-W)g\|_{BV} &= \|f-g\|_{[t_g,1]} + \|f\|_{[t_f,t_g]} \\ &= \|f-g\|_{[t_g,1]} + \|f\|_{[0,t_g]} - \|f\|_{[0,t_f]} \\ &= \|f-g\|_{[t_g,1]} - 1 + \|f\|_{BV} + \|f\|_{[0,t_g]} \\ &= \|f-g\|_{[t_g,1]} + \|f\|_{BV} + \|f\|_{[0,t_g]} \\ &\quad -\|g\|_{[BV]} - \|g\|_{[0,t_g]} \\ &\leq 2\|f-g\|_{BV}. \end{aligned}$$

□

Now let us consider the isometry $A : X \to X$ defined by:

$$Ah(t) = \begin{cases} h(2t) & t \in [0, \frac{1}{2}] \\ h(1) & t \in [\frac{1}{2}, 1] \end{cases},$$

and take the operator $R : B \to X$ defined by

$$(Rh)(t) = \begin{cases} 2\left(\chi_{[\frac{1}{2},1]}(t)\right)(1 - 2\|h\|_{BV})\left(t - \dfrac{1}{2}\right) + 2(Ah)(t) & \text{for } \|h\|_{BV} \leq \frac{1}{2}, \\ (I-W)h](t) + 2(AWh)(t) & \text{for } \|h\|_{BV} \geq \frac{1}{2} \end{cases} .$$

The definition is well posed since if $\|h\|_{BV} = \frac{1}{2}$ then

$$\begin{aligned} & (2Ah)(t) + 2\left(\chi_{[\frac{1}{2},1]}(t)\right)(1 - 2\|h\|_{BV})\left(t - \frac{1}{2}\right) \\ = \; & [(I-W)h](t) + 2(AWh)(t) = \text{(from(3))} = 2(Ah)(t). \end{aligned}$$

Now, if $\|h\|_{BV} \leq \frac{1}{2}$ then

$$(Rh)(t) = \begin{cases} 2h(2t) & t \in \left[0, \dfrac{1}{2}\right] \\ \\ 2(1 - 2\|h\|_{BV})\left(t - \dfrac{1}{2}\right) + 2h(1) & t \in \left[\dfrac{1}{2}, 1\right] \end{cases}$$

and

$$\begin{aligned}\|Rh\|_{BV} &= \|2h(2t)\|_{[0,\frac{1}{2}]} + \left\|2(1-2\|h\|_{BV})\left(t-\frac{1}{2}\right)+2h(1)\right\|_{[\frac{1}{2},1]} \\ &= 2\|h\|_{BV}+1-2\|h\|_{BV}=1,\end{aligned}$$

while $\|h\|_{BV} \geq \frac{1}{2}$ then

$$(Rh)(t) = \begin{cases} 2h(2t) & t \in \left[0, \frac{t_h}{2}\right] \\ 2h(t_f) & t \in \left[\frac{t_h}{2}, t_h\right] \\ h(t)+h(t_f) & t \in [t_h, 1] \end{cases},$$

and

$$\|Rh\|_{BV} = \|2h(2t)\|_{\left[0,\frac{t_h}{2}\right]} + \|h\|_{[t_h,1]} = 2\|h\|_{[0,t_h]} + \|h\|_{[t_h,1]} = 1.$$

Thus $R : B \to S$. Moreover from (4) it follow $R|_S = Id$.
Hence R is a retraction. At this point to evaluate the Lipschitz constant for R we can proceed as the retraction R in $L^1[0,1]$ concluding that R is 8-lipschitzian. So we proved the claim. □

Claim 2.4.13 *For* $X = BV[0,1] \cap C[0,1]$, $BV[0,1] \cap C_0[0,1]$, $AC[0,1]$, $AC_0[0,1]$*:*

$$\psi_X(k) \geq 1 - \frac{2}{k}.$$

Proof. For all $f \in X$ we define $\phi(t) = Var_{[0,t]}f + t$.
We observe that $\phi(0) = 0$, $\phi(1) = Var_{[0,1]}f + 1 \geq 1$ and ϕ is continuous and strictly increasing. Let's define $t_f \in [0,1]$ the unique value such that:

$$Var_{[0,t_f]}f + t_f = \frac{1}{k},$$

and

$$Tf(t) = \begin{cases} k(Var_{[0,t]}f + t) & t \geq [0, t_f] \\ 1 & t \in [t_f, 1] \end{cases}.$$

We note that $Tf(0) = 0$ and it's not difficult to verify that if f is in the ball of B of X then Tf belongs to the sphere S of X.

We can prove that T is $2k-$lipschitzian.
Indeed:

$$(Tf)(t)-(Tg)(t)=\begin{cases} k[(Var_{[0,t]}f+t)-(Var_{[0,t]}g+t)] & t\geq[0,t_f] \\ 1-k(Var_{[0,t]}g+t) & t\in[t_f,t_g] \\ 0 & t\in[t_g,1] \\ \end{cases}$$

hence

$$\begin{aligned}\|T_f-Tg\| &= Var_{[0,t_f]}(k(Var_{[0,t]}f-Var_{[0,t]}g))+kVar_{[t_f,t_g]}(Var_{[0,t_f]}g+t) \\ &\leq kVar_{[0,t_f]}(f-g) \\ &\quad +k(Var_{[0,t_g]}(Var_{[0,t]}g+t)-Var_{[0,t_f]}(Var_{[0,t]}g+t)) \\ &= kVar_{[0,t_f]}(f-g)+k\left[\frac{1}{k}-Var_{[0,t_f]}g-t_f\right] \\ &= kVar_{[0,t_f]}(f-g)+k[Var_{[0,t_f]}f-Var_{[0,t_f]}g] \\ &\leq 2kVar_{[0,t_f]}(f-g) \\ &\leq 2k\|f-g\|.\end{aligned}$$

Moreover:

$$\begin{aligned}\|f-Tf\| &= Var_{[0,t_f]}(f-Tf)+Var_{[t_f,1]}f \\ &\geq |f(t_f)-k(Var_{[0,t_f]}f+t_f)-f(0)| \\ &= |-1+f(t_f)-f(0)|\geq 1-|f(t_f)-f(0)|\geq 1-Var_{[0,t_f]}f \\ &> 1-\frac{1}{k},\end{aligned}$$

that proves the claim. □

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Index