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Qualitative and regularity properties of solutions to quasilinear elliptic problems

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I wish to dedicate this thesis to my parents
If I didn't give up, it is thanks to them.

Abstract

The thesis focuses on the study of the regularity and qualitative properties of solutions to quasilinear elliptic problems. The thesis is divided into two main parts. The first part (chapters 1–5) focuses on problems involving anisotropic functionals, where the main operator is governed by a norm that is generally non-Euclidean, leading to the p-Finsler Laplace operator. The second part (chapters 6–7) examines the regularity properties of solutions to equations involving both the scalar and vectorial Euclidean p-Laplacian operator.

- In Chapter 1, we introduce a suitable anisotropic Kelvin-type transformation applied to the Finsler Laplace operator for specific types of anisotropy, particularly the Riemannian case. This transformation leads to a solution that satisfies an anisotropic equation involving the dual norm.

- In Chapter 2, we prove Liouville-type theorems for stable solutions (and solutions stable outside a compact set) of quasilinear anisotropic elliptic equations. Specifically, for some general p-Finsler equations, we show that the only stable solutions to suitable nonlinear problems in $\mathbb{R}^N$ under certain conditions on the dimension of the space and the nonlinearity of the zero-order term of the equation, are identically zero.

- In Chapter 3, we establish rigidity results for Finsler p-Laplacian-type equations, including Liouville-type theorems and one-dimensional symmetry results. The proofs are based on a Poincaré-type inequality and an extension of a formula by Sternberg and Zumbrun, which relates the second derivatives of a weak solution to the principal curvatures of the associated level set, within the anisotropic framework.

- In Chapter 4, an asymptotic analysis of solutions to anisotropic doubly critical equations in $\mathbb{R}^N$ is provided, with particular focus on the behavior of the solutions and their gradients. Specifically, we examine these solutions both near the origin and at infinity, establishing that they exhibit properties analogous to their Euclidean counterparts.

- In Chapter 5, we analyze solutions to degenerate anisotropic elliptic equations with a focus on their regularity. Specifically, we derive second-order estimates and establish regularity results for the associated stress field.

- Chapter 6 concerns regularity and symmetry properties for the p-Laplacian system. More specifically, we focus on second-order estimates for the stress field. As a consequence of our regularity results, we deduce a weighted Sobolev inequality, which leads to weak comparison principles. Additionally, we apply the moving

plane technique to derive symmetry and monotonicity results for the solutions, under suitable assumptions.

- Chapter 7 is devoted to the study of solutions to p-Laplace equations as p approaches the semilinear limiting case $p = 2$. Our focus is on the integrability of the third derivatives of the solutions. As a corollary, the obtained estimates enable us to deduce regularity results for the stress field of the solution.

The results presented in this thesis were obtained during my three-year doctoral program. They stem from various scientific collaborations and are included in the following research papers: [15, 16, 77, 78, 91, 92, 128].

Riassunto

La tesi si concentra sullo studio della regolarità e delle proprietà qualitative delle soluzioni di problemi ellittici quasilineari. Essa è suddivisa in due parti principali: la prima parte (capitoli 1-5) è dedicata a problemi che coinvolgono funzionali anisotropi, in cui l'operatore principale è regolato da una norma che, in generale, non è euclidea, portando alla definizione dell'operatore di Laplace p-Finsler. La seconda parte (capitoli 6-7) esamina le proprietà di regolarità delle soluzioni di equazioni che coinvolgono gli operatori p-Laplaciano scalare e vettoriale euclidei.

- Nel Capitolo 1, introduciamo una trasformazione di tipo Kelvin anisotropo applicata all'operatore di Laplace Finsler per particolari tipi di anisotropia, in particolare nel caso riemanniano. La trasformazione conduce a una soluzione che soddisfa un'equazione anisotropa contenente la norma duale.
- Nel Capitolo 2, proviamo teoremi di tipo Liouville per soluzioni stabili (e soluzioni stabili fuori da un insieme compatto) di equazioni ellittiche quasilineari anisotrope. In particolare, per alcune equazioni p-Finsler generali, dimostriamo che l'unica soluzione stabile di problemi non lineari in $\mathbb{R}^N$ sotto determinate condizioni sulla dimensione dello spazio e sulla non linearità del termine di ordine zero dell'equazione è la soluzione identicamente nulla.
- Nel Capitolo 3, otteniamo risultati di rigidità per equazioni di tipo Finsler p-Laplaciano, inclusi teoremi di tipo Liouville e risultati di simmetria unidimensionale. Le dimostrazioni si basano su una disuguaglianza di tipo Poincaré e su un'estensione di una formula di Sternberg e Zumbrun, che collega le seconde derivate di una soluzione debole alle curvature principali dell'insieme di livello corrispondente, nel contesto anisotropo.
- Nel Capitolo 4, forniamo un'analisi asintotica delle soluzioni di equazioni anisotrope doppiamente critiche in $\mathbb{R}^N$, concentrandoci sul comportamento delle soluzioni e dei loro gradienti. In particolare, esaminiamo tali soluzioni sia vicino all'origine che all'infinito, mostrando che esse presentano proprietà analoghe a quelle dei corrispondenti casi euclidei.
- Nel Capitolo 5, analizziamo soluzioni di equazioni ellittiche anisotrope degeneri, con particolare attenzione alla loro regolarità. In particolare, otteniamo stime del secondo ordine e risultati di regolarità per il campo vettoriale della soluzione.
- Nel Capitolo 6, studiamo proprietà di regolarità e simmetria di soluzioni per il p-Laplaciano vettoriale. In particolare, ci concentriamo su stime del secondo ordine per il campo vettoriale associato alla soluzione vettoriale. Come conseguenza dei risultati di regolarità ottenuti, deduciamo una disuguaglianza di Sobolev pesata, che porta a principi di confronto deboli. Inoltre, applichiamo la tecnica del piano

mobile per ottenere risultati di simmetria e monotonicità per le soluzioni, sotto opportune ipotesi.

- Nel Capitolo 7, studiamo soluzioni di equazioni che coinvolgono l'operatore p-Laplaciano quando p tende al caso semilineare limite $p = 2$. Ci concentriamo sull'integrabilità delle derivate terze delle soluzioni. Come corollario, le stime ottenute ci permettono di dedurre risultati di regolarità per il campo vettoriale associato alla soluzione.

I risultati presentati in questa tesi sono stati ottenuti nel corso del mio programma di dottorato triennale. Essi derivano da diverse collaborazioni scientifiche e sono inclusi nei seguenti articoli di ricerca: [15, 16, 77, 78, 91, 92, 128].

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Notations

- For $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ open, and a function $w : \Omega \to \mathbb{R}$, we shall denote by ∇w its n-dimensional gradient, $D^2 w$ its hessian matrix. For $i, j, k = 1, ..., n$, we write the partial derivatives as
$$w_i = \partial_i w = \partial_{x_i} w = \frac{\partial w}{\partial x_i}$$
$$w_{ij} = \partial^2_{ij} w = \partial^2_{x_{ij}} w = \frac{\partial^2 w}{\partial x_{ij}}$$
$$w_{ijk} = \partial^3_{ijk} w = \partial^3_{x_{ijk}} w = \frac{\partial^3 w}{\partial x_{ijk}}.$$
Moreover we denote with $D^3 w = (D^3 w)_{ijk} := w_{ijk}$ the "tensor" third derivative of w.

- Generic fixed and numerical constants will be denoted by $\mathcal{C}$ (with subscript in some cases) and they will vary from line to line.

- By $|A|$ we denote the Lebesgue measure of a measurable set A.

- For $a, b \in \mathbb{R}^N$ we denote by $a \otimes b$ the matrix whose entries are $(a \otimes b)_{ij} = a_i b_j$. We remark that for any $v, w \in \mathbb{R}^N$ it holds that:
$$\langle a \otimes b\; v, w\rangle = \langle b, v\rangle \langle a, w\rangle.$$

- We denote with I the identity matrix of order N.

- The symbol f^+ stands for the positive part of a function, i.e. $f^+ = \max\{f, 0\}$.

- We use the bold style to stress the vectorial nature of different quantities. A N-vectorial function $\boldsymbol{w}$ defined in Ω will be written as
$$\boldsymbol{w}(x) = \left(w^1(x), \dots, w^N(x)\right),$$
where w^ℓ is a scalar function defined in Ω, for $\ell = 1, \dots, N$.
We say $\boldsymbol{w} := (w_1, \dots, w_n) \geq 0$ if
$$w_i \geq 0 \qquad \forall\, i = 1, \dots, n\,.$$
Moreover we denote with $D\boldsymbol{w}$ the Jacobian of $\boldsymbol{w}$ given by
$$D\boldsymbol{w} = \begin{pmatrix} \nabla w^1 \\ \vdots \\ \nabla w^N \end{pmatrix}.$$

We also use the notation $w^\ell_j = \dfrac{\partial w^\ell}{\partial x_j}$.

Moreover we define

$$\|D^2\boldsymbol{w}\| = \sqrt{\sum_{i=1}^{N}\sum_{j,k}(w^i_{jk})^2}.$$

- We point out that along this paper the symbol $:$ stands for the scalar product of the matrices rows, namely

$$\mathcal{M} : \mathcal{N} = \sum_{i=1}^{q} \mathcal{M}^i \cdot \mathcal{N}^i = \sum_{i=1}^{q}\sum_{j=1}^{r} \mathcal{M}^i_j \mathcal{N}^i_j,$$

  where $\mathcal{M}, \mathcal{N}$ are $(q \times r)$ real matrices, whereas $\langle, \rangle$ denotes the scalar product of two real vectors.

- For a matrix $\mathcal{M} = \mathcal{M}_{ij} \in \mathbb{R}^{qr}$, we denote with

$$|\mathcal{M}| = \sqrt{\sum_{i=1}^{q}\sum_{j=1}^{r} \mathcal{M}^2_{ij}},$$

  the Froebenius norm of a matrix.

Introduction

This work is devoted to the study of the regularity and qualitative proprieties of solutions to quasilinear elliptic problems.
In the first part we study qualitative and regularity proprieties of solutions to quasilinear problems involving anisotropic operators. Anisotropy introduces a richer geometric structure compared to the usual Euclidean geometry. However, the primary interest lies in practical applications, as anisotropic media occur naturally in various real-world phenomena. Anisotropic energies are extensively utilized in computer vision (see, for example, [11, 165, 168]) and in continuum mechanics, especially in materials that exhibit different behaviors in different directions, often due to the crystalline microstructure of the medium (see, for example, [19, 20, 26, 154] and references therein). In particular, anisotropic operators arise in Finsler geometry, see [21, 47, 110, 147, 148], as modelization e.g. of material science [34, 100], relativity [9], biology [6], image processing [74, 136].
In this initial part of the thesis, we will focus primarily on problems involving the Finsler norm H , i.e. a function satisfying following assumptions:

(h_H) (i) $H(\xi) > 0 \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}$;

(ii) $H(s\xi) = |s|H(\xi) \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \forall s \in \mathbb{R}$;

(iii) H is *uniformly elliptic*, that means the set $\mathcal{B}_1^H := \{\xi \in \mathbb{R}^N : H(\xi) < 1\}$ is *uniformly convex*

$$\exists \Lambda > 0 : \quad \langle D^2 H(\xi)v, v\rangle \geq \Lambda |v|^2 \quad \forall \xi \in \partial \mathcal{B}_1^H, \ \forall v \in \nabla H(\xi)^{\perp}. \tag{0.0.1}$$

Moreover we define the *dual norm* $H^\circ : \mathbb{R}^N \to [0, +\infty)$ as:

$$H^\circ(x) = \sup\{\langle \xi, x\rangle : H(\xi) \leq 1\}. \tag{0.0.2}$$

It is easy to prove that H° is also a Finsler norm and it has the same regularity properties of H. Moreover it follows that $(H^\circ)^\circ = H$. For $r > 0$ and $\overline{x} \in \mathbb{R}^N$ we define:

$$\mathcal{B}_r^H(\overline{x}) = \{x \in \mathbb{R}^N : H(x - \overline{x}) \leq r\}$$

and

$$\mathcal{B}_r^{H^\circ}(\overline{x}) = \{x \in \mathbb{R}^N : H^\circ(x - \overline{x}) \leq r\}.$$

For simplicity, when $\overline{x} = 0$, we set: $\mathcal{B}_r^H = \mathcal{B}_r^H(0)$, $\mathcal{B}_r^{H^\circ} = \mathcal{B}_r^{H^\circ}(0)$. In literature $\mathcal{B}_r^H$ and $\mathcal{B}_r^{H^\circ}$ are also called “Wulff shape” and “Frank diagram” respectively.

In the following, we will always consider anisotropies H with a uniformly convex unit ball, meaning that the principal curvatures of its boundary are bounded away from zero. We recall that Riemannian geometry is a particular case of the Finsler one and, in fact, also in this more general framework it is possible to define length of curves, geodesics, curvatures, etc. For our purposes, we focus the attention on Finsler norms not depending on the position, that means invariant by translations.

For more details on Finsler geometry see for instance [14, 21, 105].

We refer to [51, 21, 52, 94] for a discussion on the geometry of the Riemannian case and for interesting related results.

For additional properties or further details on the norm H, see Section 1.2, Chapter 1.

Having concluded this general overview of Finsler norms, we now proceed to analyze in detail the problems addressed in the first five chapters, where we delve into various theoretical and applicative aspects related to this topic.

- **In Chapter 1** we consider elliptic partial differential equations involving the Finsler Laplace operator:
$$-\Delta^H u := -\operatorname{div}(H(\nabla u)\nabla H(\nabla u))\,,$$
where H is a Finsler norm (see assumption (h_H) in Section 1.2).

  In the first part of the chapther, we shall consider solutions to the equation
$$-\Delta^H u = f(x) \qquad \text{in } \mathbb{R}^N, \quad N \geq 2\,, \tag{0.0.3}$$
with $f \in L^2_{loc}(\mathbb{R}^N)$.

  Here we provide a suitable Kelvin-type transformation in the Riemannian case,
$$H(\xi) = \sqrt{\langle M\xi, \xi\rangle} \tag{h_M}$$
for some symmetric and positive definite matrix M. In this setting, we define the anisotropic Kelvin-type transformation by
$$T_H : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}, \quad T_H(\xi) := \frac{\nabla H(\xi)}{H(\xi)},$$
and consequently
$$\hat{u} := \frac{u \circ T_H}{H^{N-2}} \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$
Our first result is the following

  Theorem 0.0.1 ([78]). *Let H be given by (h_M). If $u \in H^1_{loc}(\mathbb{R}^N)$ is a locally bounded solution of (0.0.3), with $f \in L^2_{loc}(\mathbb{R}^N)$, then $\hat{u}$ belongs to $D^{1,2}$ far from the origin, and weakly solves the dual equation*
$$-\Delta^{H^\circ} \hat{u} = \frac{f \circ T_H}{H^{N+2}} \quad \textit{in } \mathbb{R}^N \setminus \{0\}, \tag{0.0.4}$$
where H° is the dual norm.

  As noted by Ciraolo, Figalli, and Roncoroni in [49], a Kelvin-type approach is generally ineffective in the anisotropic context. This is also true in the Euclidean framework when $p \neq 2$.

  We remark that, for general Finsler norms, a pure invariance of the equation is not expected, even in the semilinear case $p = 2$ (for more details see Remark 1.3.4 in Section 1.3).

  The last part of the Chapter 1 is devoted to the study of the following quasilinear elliptic equation:
$$-\Delta^H_N u = g(x) \qquad \text{in } \mathbb{R}^N, \quad N \geq 3\,, \tag{0.0.5}$$
where $\Delta^H_N u := \operatorname{div}(H^{N-1}(\nabla u)\nabla H(\nabla u))$ is the Finsler N-Laplace operator.

  As in the previous case, we have the following result

Theorem 0.0.2 ([78]). *Let H be given be (h_M). If $u \in W^{1,N}_{loc}(\mathbb{R}^N)$ is a locally bounded solution of (0.0.5), with $g \in L^{\frac{N}{N-1}}_{loc}(\mathbb{R}^N)$, then $u^* := u \circ T_H$ weakly solves the dual equation*

$$-\Delta_N^{H^\circ} u^* = \frac{g \circ T_H}{H^{2N}} \quad in\ \mathbb{R}^N \setminus \{0\}, \tag{0.0.6}$$

where H° is the dual norm.

We emphasize that our results rely on the assumption that the norm H has a specific form, as mentioned in (h_M) above. This is, of course, a special case to consider, in contrast to the more general Finsler norms.

- **In Chapter 2** we consider weak solutions, possibly unbounded and sign-changing, to the quasilinear elliptic equation

$$-\Delta_B^H u = |u|^{q-1} u \quad \text{in } \Omega\,, \tag{0.0.7}$$

where Ω is a domain of $\mathbb{R}^N$, $N \geq 2$, $q > 1$ and Δ_B^H is the anisotropic operator

$$\Delta_B^H u = \operatorname{div}(B'(H(\nabla u))\nabla H(\nabla u)).$$

Here $H \in C^2(\mathbb{R}^N \setminus \{0\})$ is a Finsler norm and B satisfies the following assumptions

(h_B) (i) $B \in C^2((0, +\infty))$;

(ii) $B(0) = B'(0) = 0, \quad B(t), B'(t), B''(t) > 0 \quad \forall t \in (0, +\infty)$;

(iii) there exist $p > 1$, $\kappa \in [0, 1]$, $\gamma > 0$, $\Gamma > 0$ such that

$$\begin{aligned} \gamma(\kappa + t)^{p-2} t \leq & B'(t) \leq \Gamma(\kappa + t)^{p-2} t \quad \forall\, t > 0 \\ \gamma(p-1)(\kappa + t)^{p-2} \leq & B''(t) \leq \Gamma(p-1)(\kappa + t)^{p-2} \quad \forall\, t > 0\,. \end{aligned} \tag{0.0.8}$$

Throughout the chapter, we prove some Liouville-type theorems for stable solutions of quasilinear anisotropic elliptic equation (0.0.7). Liouville-type results for non-negative solutions and subcritical values of the exponent q have been previously obtained in renowned papers such as [97, 98] for $p = 2$ and [145] for $p \neq 2$. The investigation into Liouville-type theorems for stable solutions, or those stable outside a compact set, which may be unbounded and sign-changing, was first explored in [84, 85] for the semilinear case $p = 2$ (also see [71]). The quasilinear case for $p \neq 2$ has been developed in [54] (also see [86]).

Here, by utilizing the techniques developed in [54, 84, 85, 86], we will provide conditions on the dimension of the space N and q under which stable solutions of equation (0.0.7) are identically zero.

In particular we have the following

Theorem 0.0.3 ([91]). *Let us assume that (h_H) and (h_B) hold and let $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ be a weak stable (see Definition 2.1.5) solution of (0.0.7) with $p \geq 2$. If one of the following holds*

(i) $\Gamma\kappa^{p-2}(p-1) \geq 1$, $q > \Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2$ *and*

$$N < g(q) := 2\frac{\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2(q-1) + 2q + 2\sqrt{q(q - \Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2)}}{\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2(q-1)}$$

(ii) $0 < \Gamma\kappa^{p-2}(p-1) < 1$, $q > \kappa^{2-p}\gamma^{-1}$ *and*

$$N < f(q) := 2\frac{\kappa^{2-p}\gamma^{-1}(q-1) + 2q + 2\sqrt{q(q-\kappa^{2-p}\gamma^{-1})}}{\kappa^{2-p}\gamma^{-1}(q-1)}$$

(iii) $\kappa = 0$, $q > \Gamma\gamma^{-1}(p-1)$ *and*

$$N < s(q) := p\frac{\Gamma\gamma^{-1}(p-1)(q-1) + 2q + 2\sqrt{q(q-\Gamma\gamma^{-1}(p-1))}}{\Gamma\gamma^{-1}(p-1)(q-(p-1))}$$

Then $u \equiv 0$.

Moreover, we recover the case of pure Finsler p-Laplacian, that is $B(t) = t^p/p$.

Theorem 0.0.4 ([91]). *Let us assume that* (h_H) *holds and let* u *be a weak stable solution of*

$$-\operatorname{div}(H(\nabla u)^{p-1}\nabla H(\nabla u)) = |u|^{q-1}u \qquad \text{in } \mathbb{R}^N. \tag{0.0.9}$$

Assume that $p \geq 2$ *and*

$$\begin{cases} p-1 < q < \infty & \text{if } N \leq \frac{p(p+3)}{p-1} \\ \\ p-1 < q < q_c(N,p) & \text{if } N > \frac{p(p+3)}{p-1} \end{cases} \tag{0.0.10}$$

with

$$q_c(N,p) = \frac{((p-1)N-p)^2 + p^2(p-2) - p^2(p-1)N + 2p^2\sqrt{(p-1)(N-1)}}{(N-p)\left((p-1)N - p(p+3)\right)}. \tag{0.0.11}$$

Then $u \equiv 0$.

In the final part of this chapter we also establish a Liouville-type result for solutions that remain stable outside a compact set (see Definition 2.1.5). In this regard, using the Pohozaev identity in the anisotropic setting we get the following result

Theorem 0.0.5 ([91]). *Let us assume that* (h_H) *and* (h_B) *hold, with* $\kappa = 0$ *in* (h_B). *Let* $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ *be a weak solution of* (0.0.7) *which is stable outside a compact set (see Definition* 2.1.5*). Assume* $p \geq 2$ *and*

$$\begin{cases} \Gamma\gamma^{-1}(p-1) < q < \infty & \text{if } N \leq p\gamma\Gamma^{-1} \\ \\ \Gamma\gamma^{-1}(p-1) < q < \frac{N\left(p-\frac{\Gamma}{\gamma}\right)+p}{N\frac{\Gamma}{\gamma}-p}, & \text{if } N > p\gamma\Gamma^{-1}. \end{cases} \tag{0.0.12}$$

Then $u \equiv 0$.

We remark that Theorem 0.0.3 and Theorem 0.0.5 are new even in the Euclidean case, i.e. if $H(\xi) = \xi$.

- **In Chapter 3** we consider $u \in C^1(\mathbb{R}^N)$ weak solutions to the problem

$$-\operatorname{div}\left(B'(H(\nabla u))\nabla H(\nabla u)\right) = f(u) \quad \text{on } \mathbb{R}^n, \tag{0.0.13}$$

Here, $H \in C^{3,\beta}_{loc}(\mathbb{R}^N \setminus \{0\})$ is a Finsler norm, $f \in C^{1,\beta}_{loc}(0,+\infty)$ and B satisfies

(h$_B$) (i) $B \in C^{3,\beta}_{loc}(0,+\infty) \cap C^1([0,\infty))$;
(ii) $B(0) = B'(0) = 0, \quad B(t), B'(t), B''(t) > 0 \quad \forall t \in (0,+\infty)$;
(iii) there exist $p > 1$, $\kappa \in [0,1)$, $\gamma > 0$, $\Gamma > 0$ such that

$$\begin{aligned} \gamma(\kappa+t)^{p-2} &\leq B'(t) \leq \Gamma(\kappa+t)^{p-2}t \quad \forall\, t > 0 \\ \gamma(p-1)(\kappa+t)^{p-2} &\leq B''(t) \leq \Gamma(p-1)(\kappa+t)^{p-2} \quad \forall\, t > 0\,. \end{aligned} \tag{0.0.14}$$

In this chapter, we focus on rigidity properties, including Liouville-type theorems and one-dimensionality results.

Our first result is the following:

Theorem 0.0.6 ([92])**.** *Let $N = 2$. Let $u \in C^1(\mathbb{R}^2)$ be a weak stable (see Definition 3.1.1) solution of (0.0.13), with $\nabla u \in L^\infty(\mathbb{R}^2)$. Then there exist $u_0 : \mathbb{R} \to \mathbb{R}$ and $w \in S^1$ such that $u(x) = u_0(w \cdot x)$ for any $x \in \mathbb{R}^2$.*

The proof that we present is based on a Poincaré type inequality and it is a variation given in [89, 150, 151]. For this purpose, we will extend a formula by Sternberg and Zumbrun which relates the second derivatives of a weak solution of the equation (0.0.13) with the principal curvatures of the corresponding level set, in the anisotropic setting.

The second symmetry result is the following

Theorem 0.0.7 ([92])**.** *Let $N = 3$. Let $u \in C^1(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ be a weak solution of (0.0.13). Suppose that*

$$\partial_{x_3} u > 0. \tag{0.0.15}$$

Then u has one dimensional symmetry in the full space $\mathbb{R}^3$, namely there exist $u_0 : \mathbb{R} \to \mathbb{R}$ and $w \in S^2$ such that $u(x) = u_0(w \cdot x)$ for any $x \in \mathbb{R}^3$.

In the Euclidean framework and in the case of $B(t) = 1$ and $f(t) = t - t^3$, Theorem 0.0.6 and Theorem 0.0.7 reduce to a problem proposed by De Giorgi, [64]. The first result was obtained for $N = 2$ in [22, 99], and, subsequently, the case $N = 3$ was proved in [3, 5]. Moreover, assuming an additional limit requirement on the solutions, the De Giorgi's conjecture is true for $4 \leq N \leq 8$, (see [138]). The conjecture fails for $N \geq 9$, [65]. In the case of $p-$Laplace operator and non-uniformly elliptic operators, one dimensional symmetry for solutions has also been dealt with in [33, 83, 93, 89, 90, 160].

- **In Chapter 4** we consider weak solutions to the following anisotropic doubly critical problem

$$\begin{cases} -\Delta_p^H u - \dfrac{\gamma}{H^\circ(x)^p} u^{p-1} = u^{p^*-1} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}_H$}$$

where $1 < p < N$, $p^* := Np/(N-p)$ is the Sobolev critical exponent, $0 \leq \gamma < C_H := ((N-p)/p)^p$ is the Hardy constant and

$$\Delta_p^H u := \operatorname{div}(H^{p-1}(\nabla u)\nabla H(\nabla u)), \tag{0.0.16}$$

where Δ_p^H is the so-called anisotropic p-Laplacian or Finsler p-Laplacian. We point out that H is a Finsler type norm and H° is its the dual norm.

The aim of this chapter is to provide a complete asymptotic analysis of the weak solution $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and its gradient ∇u near the origin and at infinity, showing that this solution has the same features of its euclidean counterpart.

In the first part of this Chapter we deduce the precise asymptotic estimates for weak solutions u to the problem ($\mathcal{P}_H$).

In particular we have the following result

Theorem 0.0.8 ([77]). *Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak solution of ($\mathcal{P}_H$) with $1 < p < N$, $0 \leq \gamma < C_H$. Then there exist positive constants $0 < R_1 < 1 < R_2$ depending on N, p, γ and u, such that*

$$\frac{c_1}{[H^\circ(x)]^{\mu_1}} \leq u(x) \leq \frac{C_1}{[H^\circ(x)]^{\mu_1}} \qquad x \in \mathcal{B}^{H^\circ}_{R_1}, \tag{0.0.17}$$

and

$$\frac{c_2}{[H^\circ(x)]^{\mu_2}} \leq u(x) \leq \frac{C_2}{[H^\circ(x)]^{\mu_2}} \qquad x \in (\mathcal{B}^{H^\circ}_{R_2})^c, \tag{0.0.18}$$

where μ_1, μ_2 are the solutions of

$$\mu^{p-2}[(p-1)\mu^2 - (N-p)\mu] + \gamma = 0, \tag{0.0.19}$$

C_1, C_2 are positive constants depending on N, p, γ, H and u, c_1 is a positive constant depending on $N, p, \gamma, H, R_1, \mu_1$ and u, c_2 is a positive constant depending on N, p, γ, H, R_2, μ_2 and u.

In the proof of the Theorem 0.0.8, we will also leverage some clever ideas from [166] to address the challenges of the anisotropic problem.

A different approach is necessary for studying the asymptotic behavior of the gradient of the solution u. Notably, since the moving plane technique cannot be applied, a crucial aspect is provided by the following classification result

Theorem 0.0.9 ([77]). *Let $v \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{0\})$ be a positive weak solution of the equation*

$$-\Delta^H_p v - \frac{\gamma}{[H^\circ(x)]^p} v^{p-1} = 0 \text{ in } \mathbb{R}^N \setminus \{0\},$$

where $0 \leq \gamma < C_H$. Assume that there exist two positive constants C and c such that

$$\frac{c}{[H^\circ(x)]^{\mu_i}} \leq v(x) \leq \frac{C}{[H^\circ(x)]^{\mu_i}} \ \forall x \in \mathbb{R}^N \setminus \{0\},$$

where μ_i ($i = 1, 2$) are the roots of the equation (0.0.19), and suppose that there exists a positive constant $\hat{C}$ such that

$$|\nabla v(x)| \leq \frac{\hat{C}}{[H^\circ(x)]^{\mu_i + 1}} \ \ \forall x \in \mathbb{R}^N \setminus \{0\},$$

then

$$v(x) = \frac{\bar{c}}{[H^\circ(x)]^{\mu_i}},$$

for some $\bar{c} > 0$.

We remark that the proof of the Theorem 0.0.9 is very much different than the ones available in the euclidean case $H(\xi) = |\xi|$.

Finally, in the last part of the chapter, using the previous Theorem 0.0.9, we derive precise asymptotic estimates for the gradient, as stated in the following

Theorem 0.0.10 ([77]). *Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak solution of ($\mathcal{P}_H$) with $1 < p < N$, and $0 \leq \gamma < C_H$. Then there exist positive constants $\tilde{c}$, $\tilde{C}$ depending on N, p, γ, H and u such that*

$$\frac{\tilde{c}}{[H^\circ(x)]^{\mu_1+1}} \leq |\nabla u(x)| \leq \frac{\tilde{C}}{[H^\circ(x)]^{\mu_1+1}} \qquad x \in \mathcal{B}_{R_1}^{H^\circ}, \tag{0.0.20}$$

and

$$\frac{\tilde{c}}{[H^\circ(x)]^{\mu_2+1}} \leq |\nabla u(x)| \leq \frac{\tilde{C}}{[H^\circ(x)]^{\mu_2+1}} \qquad x \in (\mathcal{B}_{R_2}^{H^\circ})^c, \tag{0.0.21}$$

where μ_1, μ_2 are roots of (0.0.19), and $0 < R_1 < 1 < R_2$ are constants depending on N, p, γ and u.

- **In Chapter 5** we focus on the second-order regularity of the solutions to

$$-\operatorname{div}(A(H(\nabla u))H(\nabla u)\nabla H(\nabla u)) = f(x), \qquad \text{in } \Omega. \tag{0.0.22}$$

Here Ω is a domain of $\mathbb{R}^N$, and $H \in C^{2,\alpha}_{loc}(\mathbb{R}^n \backslash \{0\})$ is a Finsler norm. The function $A : (0, +\infty) \to (0, +\infty)$ is of class $C^{1,\alpha}_{loc}(0, +\infty)$ and satisfies the following Uhlenbeck structure

$$-1 < \inf_{t>0} \frac{tA'(t)}{A(t)} := m_A \leq M_A := \sup_{t>0} \frac{tA'(t)}{A(t)} < +\infty.$$

For the right-hand side of equation (0.0.22) we assume the following hypothesis

(H_f) the source term $f \in W^{1,\frac{N}{N-\gamma-s}}_{loc}(\Omega) \cap C^{0,\alpha}_{loc}(\Omega)$ where $s \in (0, N-\gamma)$ and $\gamma < N-2$ if $N > 2$ and $\gamma = 0$ if $N = 2$. If $\gamma = 0$, we consider $f \in W^{1,1}_{loc}(\Omega) \cap C^{0,\alpha}_{loc}(\Omega)$.

If $A(t) = t^{p-2}$ and in the isotropic case, i.e. when H is the classical Euclidean norm $H(\xi) = |\xi|$, our equation reduces to the well known p-Laplace equation.

In this chapther we analyze solutions to degenerate anisotropic elliptic equations to investigate their regularity. Specifically, we derive second-order estimates and establish regularity results for the stress field. To the best of our knowledge, second-order regularity for equations with Uhlenbeck structure is not fully understood, even in the isotropic case.

We present the following result regarding the regularity of families of stress fields of the form $A(H(\nabla u))^\beta H(\nabla u)\nabla H(\nabla u)$ which involves only the quantities m_A and M_A.

Theorem 0.0.11 ([16]). *Let $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$ be a weak solution to (0.0.22), where f satisfies (H_f).*
Then

(i) *If $0 < m_A < M_A$, then*

$$A(H(\nabla u))^{\frac{k-1}{m_A}} H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$

for any $k > 1 + \dfrac{m_A}{2} - \dfrac{m_A}{2M_A}$.

(ii) *If* $m_A < M_A < 0$, *then*

$$A(H(\nabla u))^{\frac{k-1}{m_A}} H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$

for any $k > \dfrac{1}{2} + \dfrac{m_A}{2}$.

(iii) *If* $m_A < 0 < M_A$, *then*

$$A(H(\nabla u))^{\frac{k-1}{M_A}} H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$

for any $k \in \left(\dfrac{1}{2} + \dfrac{M_A}{2}, 1 + \dfrac{M_A}{2} - \dfrac{M_A}{2m_A}\right)$.

(iv) *If* $m_A = 0$ *and* $M_A > 0$, *then*

$$A(H(\nabla u))^{k-1} H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$

for any $k > \dfrac{3}{2} - \dfrac{1}{2M_A}$.

(v) *If* $m_A < 0$ *and* $M_A = 0$, *then*

$$A(H(\nabla u))^{k-1} H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$

for any $k < \dfrac{3}{2} - \dfrac{1}{2m_A}$.

Naturally, appropriate choices of k yield the optimal results obtained in [8], i.e.

$$A(H(\nabla u))H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega).$$

Furthermore, in [8], global second order estimates for anisotropic problems, in the Uhlenbeck structure have been obtained under minimal assumptions on the regularity of $\partial\Omega$ and on H. In particular, the authors, in [8], proved that

$$f \in L^2 \iff A(H(\nabla u))H(\nabla u)\nabla H(\nabla u) \in W^{1,2}(\Omega).$$

Our second result concerns estimates on the second derivative of the solution to equation (0.0.22) and generalizes a result obtained in [35]. Specifically, we have the following

Theorem 0.0.12 ([16]). *Let* $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$ *be a weak solution to* (0.0.22) *where* f *satisfies* (H_f). *Suppose* $\inf_{t\in[0,M]} A(t) = 0$. *Then*

(i) *if* $M_A < 1$, *it holds*

$$u \in W^{2,2}_{loc}(\Omega). \tag{0.0.23}$$

(ii) *If* $M_A \geq 1$ *and if we assume that*

$$f \geq \tau > 0 \quad \text{a.e. in } \Omega. \tag{0.0.24}$$

it holds

$$u \in W^{2,q}_{loc}(\Omega), \tag{0.0.25}$$

with $q < \dfrac{M_A + 1}{M_A}$.

If $\inf_{t\in[0,M]} A(t) > 0$, *then* (0.0.23) *holds without further assumptions on* M_A.

In the second part of the thesis, we will examine the regularity properties of solutions involving both the scalar and vectorial p-Laplacian. In particular we will deal with weak solutions of

$$-\Delta_p u = f \quad \text{in } \Omega. \tag{0.0.26}$$

The p-Laplace operator is defined as follows

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\langle D^2 u \nabla u, \nabla u\rangle$$

where $\langle D^2 u\nabla u, \nabla u\rangle := \sum_{ij} \partial^2_{ij} u \partial_i u \partial_j u$ and Ω is a domain of $\mathbb{R}^n$ with $n \geq 2$. We remark that the p-Laplace operator becomes the classical Laplacian when $p = 2$, i.e.

$$\Delta_2 u = \operatorname{div}(\nabla u) = \Delta u := \sum_i \partial^2_{ii} u.$$

When $p \neq 2$ the situation is completely different and it is well known that, since the p-Laplace operator is singular or degenerate elliptic (respectively if $1 < p < 2$ or $p > 2$), solutions of (0.0.26) are of class $C^{1,\beta}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$, where Z_u denotes the set where the gradient vanishes, for every $p > 1$ and under suitable assumptions on the source term (see [67, 113, 157]).

The regularity of weak solutions to equation (0.0.26) has a long history. As a quasilinear, degenerate elliptic PDE, the linear theory developed by De Giorgi [63], Nash [131], and Moser [130] does not directly apply. However, Serrin established an Harnack inequality for positive weak solutions of (0.0.26) in 1964 [143], building on ideas introduced earlier by Moser, which led to the proof of Hölder regularity.

The Hölder regularity of the gradient is more complex. The range $1 < p \leq 2$ was studied by Uraltseva [158] and Uhlenbeck [159], while the general case for $1 < p < +\infty$ was addressed by Di Benedetto [67], Lewis [111], and Tolksdorff [157].

For further extensive discussion on regularity theory in the quasilinear context we refer to [10, 13, 41, 43, 37, 50, 59, 66, 72, 73, 108, 102, 107, 119, 121, 122, 149]. Recent significant results are presented in [60, 61, 62].

For $p = 2$, (0.0.26) becomes the linear Poisson equation

$$-\Delta u = f \quad \text{in } \Omega. \tag{0.0.27}$$

Early contributions on second order regularity for solution to (0.0.27) can be found in [24, 146]. For further generalization to linear equations in divergence form we refer to [95, 132, 133]. The classical result is the following: *if* u *is a weak solution of* (0.0.27) *then*

$$f \in L^2_{loc}(\Omega) \iff u \in W^{2,2}_{loc}(\Omega). \tag{0.0.28}$$

For a proof of the previous result, we refer to [30, 81, 101]. If we consider a bounded smooth domain Ω, in particular if $\partial\Omega \in C^2$, the result can be extended on the boundary, under Dirichlet boundary condition.

The extension of (0.0.28) for the p-Laplace equation (0.0.26), due to Cianchi and Maz'ya [45], is as follows

$$f \in L^2_{loc}(\Omega) \iff |\nabla u|^{p-2}\nabla u \in W^{1,2}_{loc}(\Omega). \tag{0.0.29}$$

Moreover, if we consider a bounded domain Ω, they proved a global result under minimal assumptions on the boundary of the domain, when the Dirichlet or Neumann boundary condition are imposed. For a similar result involving the stress field of the solution we refer to Lou [119]. Further generalizations of (0.0.29) to anisotropic operators, involving the Finsler norm, can be found in [7, 8]. Moreover, if $f \in W^{1,1}_{loc}(\Omega) \cap L^q_{loc}(\Omega)$, with $q > n$, similar results as in (0.0.29) have also been obtained (see [126]) for vector fields of the form $V_\alpha := |\nabla u|^{\alpha-1}\nabla u$, that is

$$V_\alpha \in W^{1,2}_{loc}(\Omega) \quad \text{for } \alpha > \frac{p-1}{2}. \tag{0.0.30}$$

Hidden in the result (0.0.29) and under suitable assumption on f, the solution u belongs to the Sobolev space $W^{2,2}_{loc}(\Omega)$. If the right-hand side of the $p-$Laplace equation (0.0.26) has more regularity, then we can deduce improved second-order estimates for the solution u. In particular, see [57, 141, 142], if $f \in W^{1,1}_{loc}(\Omega) \cap L^q_{loc}(\Omega)$, with $q > n$, then we have

$$u \in W^{2,2}_{loc}(\Omega) \quad \text{for } 1 < p < 3. \tag{0.0.31}$$

Moreover, if f has a sign, then

$$u \in W^{2,q}_{loc}(\Omega) \quad q < \frac{p-1}{p-2}. \tag{0.0.32}$$

We remark that the result expressed by (0.0.32) is sharp. This can be deduced by the example involving $u := x_1^{p'}/p'$, where $p' = p/(p-1)$ and $x_1 \geq 0$, which solves $\Delta_p u = 1$. In addition, the result can be extended on the boundary, when Dirichlet or Neumann boundary condition is imposed (see [126]). We also mention the regularity results in [37], achieved through a different approach.

The last part of the thesis is devoted to the study of the solutions to the following p-Laplace system

$$-\boldsymbol{\Delta}_p \boldsymbol{u} = -\,\mathbf{div}(|D\boldsymbol{u}|^{p-2} D\boldsymbol{u}) = \boldsymbol{f} \quad \text{in } \Omega, \tag{0.0.33}$$

where Ω is a domain of $\mathbb{R}^n$, $p > 1$ and $D\boldsymbol{u}$ is the Jacobian of the vector field $\boldsymbol{u} : \Omega \to \mathbb{R}^N$, $N \geq 2$.

Results from [2, 159] imply that any local weak solution $\boldsymbol{u}$ to (0.0.33) satisfies $D\boldsymbol{u} \in L^\infty_{loc}(\Omega; \mathbb{R}^{Nn})$, and additionally, $D\boldsymbol{u} \in C^{0,\alpha}_{loc}(\Omega; \mathbb{R}^{Nn})$ for some $\alpha \in (0,1)$. Recall that, in striking contrast to the scalar case solutions to nonlinear elliptic systems with a more general structure than that of (0.0.33) can be irregular. Examples in this context can be found in [153], where the authors prove the existence of nonlinear elliptic systems with smooth coefficients depending only on the gradient, but with solutions that are not even bounded.

The study of global regularity, meaning up to the boundary, in boundary value problems for nonlinear elliptic systems has a more recent history. Global gradient boundedness and gradient Hölder regularity for the p-Laplacian elliptic system (0.0.33) , under homogeneous Dirichlet boundary conditions, were obtained in [38] as a consequence of similar results for the associated parabolic problem. That paper considers right-hand sides that are bounded in x and domains whose boundary is of class $C^{1,\alpha}$. Global gradient Hölder regularity, both for Dirichlet and Neumann problems, is established in [103] for systems with Uhlenbeck structure in domains of class $C^{1,1}$.

Second-order estimates were established in [12, 46] for solutions to the p-Laplace system with a right-hand side in L^2. In particular, in [12] the authors proved that

$$|D\boldsymbol{u}|^{p-2} D\boldsymbol{u} \in W^{1,2}_{loc}(\Omega) \quad \text{for } p > 2(2-\sqrt{2}).$$

If the right hand-side of the p-Laplace system (0.0.33) has more regularity, a similar result was obtained in [123]. In particular, the author proves that for $p \geq 3$ and $f \in W^{1,p'}(\Omega)$ then Du locally lies in a suitable fractional Nikol'skii space. This result is obtained proving that $|D\boldsymbol{u}|^{s-1}D\boldsymbol{u} \in W^{1,2}_{loc}(\Omega)$ for every $s \in ((p-1)/2, p/2]$ when $p \geq 3$. As a rule, if $p \in [2,3)$, $|D\boldsymbol{u}|^{s-1}D\boldsymbol{u} \in W^{1,2}_{loc}(\Omega)$ for $s \in [1, p/2]$ follows too.

Other results concerning regularity of p-Laplace type systems can be found in [25, 27, 28, 29, 42, 44, 68, 69].

We now focus on the specific topics covered in the last two chapters.

- **In Chapter 6** we consider weak solutions $\boldsymbol{u} \in C^1(\overline{\Omega})$ to the p-Laplace system

$$\begin{cases} -\,\mathbf{div}(|D\boldsymbol{u}|^{p-2}D\boldsymbol{u}) = \boldsymbol{f}(x,\boldsymbol{u}) & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{p-S}$$

where Ω is a bounded smooth domain in $\mathbb{R}^n$, $\boldsymbol{u} = (u^1, ..., u^N) : \Omega \to \mathbb{R}^N$, $N \geq 2$, and $D\boldsymbol{u}$ is the Jacobian of $\boldsymbol{u}$.

Problems involving the p-Laplace system (p-S), present greater difficulties. The techniques generally used for the scalar case fail for the system, as it is coupled in the gradient term $|D\boldsymbol{u}|^{p-2}$.

The first step of our study addresses the regularity of solutions, a topic still explored in the literature, despite its importance. In addition to the works cited above, we refer to the reader to [107] and to the recent deep developments in [60, 61, 62].

Here, in the first part of the chapter, we extend the regularity estimates for the second derivatives of solutions to (p-S).

In particular, we have the following weighted estimate involving the Hessian $D^2\boldsymbol{u}$ of the solution to the system (p-S).

Theorem 0.0.13 ([128])**.** *Let Ω be a bounded smooth domain and $p > 1$. Let $\boldsymbol{u} \in C^1(\Omega)$ be a weak solution of (p-S), with*

$$\boldsymbol{f}(x,\boldsymbol{u}) := \boldsymbol{f}(x) \in W^{1,\frac{n}{n-\gamma-s}}_{loc}(\Omega) \cap C^{0,\beta}_{loc}(\Omega),$$

where $0 < s < n-\gamma$ and $\gamma < n-2$ ($\gamma = 0$ if $n = 2$).

If $1 < p \leq 2$ let us assume $0 \leq \alpha < p-1$ and if $p > 2$ let $\alpha \in [0,1)$. Then, for any $\tilde{\Omega} \subset\subset \Omega$, it follows that

$$\int_{\tilde{\Omega}\setminus Z_{\boldsymbol{u}}} \frac{|D\boldsymbol{u}|^{p-2-\alpha}\|D^2\boldsymbol{u}\|^2}{|x-y|^\gamma}\,dx \leq C, \tag{0.0.34}$$

uniformly for any $y \in \mathbb{R}^n$, with $C = C(\boldsymbol{f}, n, p, \alpha, \gamma, \tilde{\Omega})$.

Consequently, we also deduce the integrability properties of $|D\boldsymbol{u}|^{-1}$.

Theorem 0.0.14 ([128])**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and let $\boldsymbol{u} \in C^1(\Omega)$ be a weak solution of (p-S) with $\boldsymbol{f}(x) := \boldsymbol{f}(x,\boldsymbol{u}) \in W^{1,\frac{n}{n-\gamma-s}}_{loc}(\Omega) \cap C^{0,\beta}_{loc}(\Omega)$, where $0 < s < n-\gamma$ and $\gamma < n-2$ if $n \geq 3$ ($\gamma = 0$ if $n = 2$). Suppose that for some i:*

$$f^i \geq \tau > 0 \quad in \quad \Omega\,.$$

Then, for any $\tilde{\Omega} \subset\subset \Omega$

$$\int_{\tilde{\Omega}} \frac{1}{|D\boldsymbol{u}|^{\sigma}} \frac{1}{|x-y|^{\gamma}} \leq C, \tag{0.0.35}$$

for any

$$\sigma < \begin{cases} 2p-3 & \text{if } 1 < p \leq 2 \\ p-1 & \text{if } p > 2, \end{cases} \tag{0.0.36}$$

and $C = C(\boldsymbol{f}, n, p, \gamma, \sigma, \tau, \tilde{\Omega})$ *is a positive constant. In particular the Lebesgue measure of* $Z_{\boldsymbol{u}}$ *is zero.*

As an important consequence, exploiting our Theorem 0.0.13 and Theorem 0.0.14, we recover that

$$\boldsymbol{u} \in W^{2,2}_{loc}(\Omega) \quad \text{for } p \in (1,3)$$

and, under the assumption that $\boldsymbol{f}$ has a sign, we prove that for $p \geq 3$,

$$\boldsymbol{u} \in W^{2,q}_{loc}(\Omega) \quad \text{with } q < \frac{p-1}{p-2}.$$

The second part of this chapter is devoted to obtaining monotonicity and symmetry results, using the moving plane technique.

We recall that the scalar case $N = 1$ has been extensively treated in [56] for $1 < p < 2$, and subsequently extended in [57], for every $p > 1$. For the vectorial case $N > 1$ no general symmetry results are available in the literature and some achievement was obtained in [120], under stronger assumptions on the vectorial function $\boldsymbol{f}$ (namely assuming that $\boldsymbol{f}$ is strictly increasing respect to x) and in the singular case $1 < p < 2$. We emphasize that the strict monotonicity assumption on the nonlinearity in [120] prevents to consider the natural case of autonomous equations.

Using regularity proprieties we obtain a weighted Poincare inequality that allow us to establish a comparison principle in small domains.

Theorem 0.0.15 ([128]). *Let* $p > 1$ *and* $\boldsymbol{u}, \boldsymbol{v} \in C^1(\overline{\Omega})$ *such that either* $\boldsymbol{u}$ *or* $\boldsymbol{v}$ *is solution to* (p-S) *and*

$$-\boldsymbol{\Delta}_p \boldsymbol{u} - \boldsymbol{f}(x, \boldsymbol{u}) \leq -\boldsymbol{\Delta}_p \boldsymbol{v} - \boldsymbol{f}(x, \boldsymbol{v}) \quad \text{in } \Omega,$$

where $\boldsymbol{f}$ *satisfies* (h_f) *(see Section 6.1 in Chapter 6). Let* $\tilde{\Omega} \subset\subset \Omega$ *be open and assume that*

$$(u^i - v^i)(u^j - v^j) \geq 0 \quad \text{in } \tilde{\Omega}, \quad \text{for } i \neq j,$$

and $\boldsymbol{u} \leq \boldsymbol{v}$ *on* $\partial\tilde{\Omega}$.
Then there exist $\lambda = \lambda(\boldsymbol{f}, p, N, \|D\boldsymbol{u}\|_\infty) > 0$ *such that, if* $|\tilde{\Omega}| < \lambda$, *it follows that*

$$\boldsymbol{u} \leq \boldsymbol{v} \quad \text{in } \tilde{\Omega}.$$

If $\tilde{\Omega} \subseteq \Omega$, *the same result holds assuming, for* $p > 2$, *that* $Z_{\boldsymbol{u}} \subset\subset \Omega$.

Finally, we conclude the chapter deducing a symmetry and regularity result, under general assumptions on the source term $\boldsymbol{f}$ (see assumptions (h_f)).

We state the following

Theorem 0.0.16 ([128]). *Let Ω be a bounded smooth domain of $\mathbb{R}^n$, convex with respect to the x_1-direction and symmetric with respect to T_0, where*

$$T_0 = \{x \in \Omega : x_1 = 0\}.$$

Let $\boldsymbol{u}$ be a positive $C^1(\overline{\Omega})$ weak solution to (p-S) such that

$$|D\boldsymbol{u}|^{p-2}D\boldsymbol{u} \in W^{1,2}_{loc}(\Omega), \qquad Z_{\boldsymbol{u}} \subset\subset \Omega,$$

with $\boldsymbol{f}$ satisfying (h_f) (see Section 6.1 in Chapter 6) and

$$\boldsymbol{f}(x_1, x_2, \ldots, x_n, \boldsymbol{u}) < \boldsymbol{f}(y_1, x_2, \ldots, x_n, \boldsymbol{u}) \quad \text{where } x_1 < y_1 < 0,$$

$$\boldsymbol{f}(-x_1, \ldots, x_{n-1}, x_n, \boldsymbol{u}) = \boldsymbol{f}(x_1, \ldots, x_{n-1}, x_n, \boldsymbol{u}).$$

We set $a := \inf_{x\in\Omega} x_1$ and $b := \sup_{x\in\Omega} x_1$. Assume that

$$(u^i(x) - u^i(2\lambda - x_1, ..., x_n))(u^j(x) - u^j(2\lambda - x_1, ..., x_n)) \geq 0 \quad i, j = 1, ..., N,$$

either for any $a < \lambda < 0$ with $x \in \Omega$ such that $x_1 < \lambda$ or for any $0 < \lambda < b$ with $x \in \Omega$ such that $x_1 > \lambda$.
Then $\boldsymbol{u}$ is symmetric with respect to T_0 and non-decreasing with respect to the x_1-direction in $\Omega_0 = \{x \in \Omega \,:\, x_1 < 0\}$.

- **In Chapter 7** we deal with the regularity of the third derivatives of weak solutions, possibly sign-changing, to:

$$-\Delta_p u = f(x) \quad \text{in } \Omega \tag{0.0.37}$$

where $\Delta_p u$ is the p-Laplace operator and Ω is a domain of $\mathbb{R}^n$ with $n \geq 2$.

In this chapter, we examine the integrability of the third derivatives of the solutions when p approaches the semilinear limiting case $p = 2$, which can be understood as distributional derivatives in an appropriate sense. To our knowledge, no other results in the literature address this topic.

Our results are influenced by the classical theory of elliptic regularity, particularly by the constant $C(n, q)$ found in the Calderón-Zygmund estimate

$$\|D^2 w\|_{L^q(\Omega)} \leq C(n, q)\|\Delta w\|_{L^q(\Omega)}. \tag{0.0.38}$$

To achieve our main results, we first extend a result in [120], which does not consider the effects of the boundary of Ω. In particular, we have the following

Theorem 0.0.17 ([15]). *Let Ω be a domain of $\mathbb{R}^n$ and $u \in C^{1,\beta}_{loc}(\Omega)$ be a weak solution to (0.0.37), with $f(x) \in W^{1,1}_{loc}(\Omega) \cap C^{0,\beta'}_{loc}(\Omega)$. Let $q \geq 2$ and p be such that*

$$|p - 2| < \frac{1}{C(n, q)},$$

with $C(n, q)$ given by (0.0.38). Then, if $1 < p \leq 2$, we have

$$u \in W^{2,q}_{loc}(\Omega).$$

If $p > 2$ then the same conclusion holds if p satisfies, in addition, $q < \frac{p-1}{p-2}$.

In [45] the authors proved that $|\nabla u|^{p-2}\nabla u \in W^{1,2}_{loc}(\Omega)$, if $f \in L^2_{loc}(\Omega)$. The natural generalization could be

$$|\nabla u|^{p-2}\nabla u \in W^{1,\tilde{\alpha}}_{loc}(\Omega) \text{ iff } f \in L^{\tilde{\alpha}}_{loc}(\Omega).$$

Partial results have been obtained in [102], when $\tilde{\alpha} - 2$ is sufficiently small, with $f \in L^{\tilde{\alpha}}(\Omega) \cap W^{-1,p'}(\Omega)$. Using a third order estimate (see Theorem 7.1.4) we deduce improved regularity properties of the stress field. Here, we obtain a improved regularity result for the stress field $|\nabla u|^{p-2}\nabla u$

Theorem 0.0.18 ([15]). *Let $\Omega \subset \mathbb{R}^n$ be a domain and $u \in C^{1,\beta}_{loc}(\Omega)$ be a weak solution of* (0.0.37). *Assume that $\tilde{\alpha} \geq 3$ and we set $q = 2(\tilde{\alpha} - 1)$. Suppose that*

$$\max\left\{2 - \frac{1}{2\tilde{\alpha} - 1}, 2 - \frac{1}{C(n,q)}\right\} < p \leq 2. \tag{0.0.39}$$

where $C(n,q)$ is given by (0.0.38). *Let $f(x) \in W^{2,2(\tilde{\alpha}-1)/3}_{loc}(\Omega) \cap C^{1,\beta'}_{loc}(\Omega)$. Then*

$$|\nabla u|^{p-2}\nabla u \in W^{1,\tilde{\alpha}}_{loc}(\Omega). \tag{0.0.40}$$

In the last part of this chapter we obtain second order regularity results for the stress field $|\nabla u|^{k-1}\nabla u$, for some $k \in \mathbb{R}$. In particular, we get

Theorem 0.0.19 ([15]). *Let $\Omega \subset \mathbb{R}^N$ be a domain and $u \in C^{1,\beta}_{loc}(\Omega)$ be a weak solution of* (0.0.37). *We set*

$$\begin{cases} q_N = 2(q_{N-1} - 1) \\ q_0 = 4 \end{cases} \tag{0.0.41}$$

Let $q > q_0$, with $q_N < q \leq q_{N+1}$ for some $N \in \mathbb{N}_0$.
Assume p such that

$$2 - \frac{1}{C(n,q)} < p < \min\left\{2 + \frac{1}{q-1}, 2 + \frac{1}{C(n,q)}\right\}. \tag{0.0.42}$$

Let $f(x) \in W^{2,q/(q-1)}_{loc}(\Omega) \cap C^{1,\beta'}_{loc}(\Omega)$ and

$$f \geq \tau > 0 \quad in \quad \Omega\,, \quad (or\ f \leq \tau < 0 \quad in \quad \Omega\,).$$

Then we have

$$|\nabla u|^{k-1}\nabla u \in W^{2,1}_{loc}(\Omega, \mathbb{R}^N) \tag{0.0.43}$$

for any $k > (\alpha + 1)/2$, where

$$\alpha > \frac{(q - q_N)}{2^N q}(1 - p) + \frac{3 - p}{2^N} + 1. \tag{0.0.44}$$

In case f has no sign, let $q \geq q_0$, with $q_N \leq q < q_{N+1}$ for some $N \in \mathbb{N}_0$, and q_N given in (0.0.41). *Assume p satisfies* (0.0.42). *Then* (0.0.43) *holds for any $k \geq (p + \alpha)/2$, where $\alpha > (3 - p)/2^N + 1$ (if $N = 0$, $\alpha \geq 4 - p$).*

We remark that Theorems 0.0.18 and Theorem 0.0.19 follow from more general estimates concerning the third derivatives (see Theorem 7.1.4.)

The results presented here were obtained during the three years of my doctoral program. They are the product of various scientific collaborations and are included in the following research articles in [15, 16, 77, 78, 91, 92, 128]

Chapter 1

Anisotropic Kelvin-type transform for a special class of anisotropic elliptic operators

1.1 Main results

The aim of this chapter is to investigate elliptic partial differential equations involving the Finsler Laplace operator:

$$-\Delta^H u := -\operatorname{div}(H(\nabla u)\nabla H(\nabla u)),$$

where H is a Finsler norm (see assumption (h_H) in Section 1.2), and for $H(\xi) = |\xi|$, the operator Δ^H simplifies to the classical Laplacian. Elliptic equations involving anisotropic operators appear as the Euler-Lagrange equations of Wulff-type energy functionals.

Initially, we examine solutions to the equation

$$-\Delta^H u = f(x) \qquad \text{in } \mathbb{R}^N, \quad N \geq 2. \tag{1.1.1}$$

which is the Euler-Lagrange equation of the *Wulff-type energy functional*

$$\mathcal{W}(u) = \int_{\mathbb{R}^N} \left[\frac{H(\nabla u)^2}{2} - f(x)u\right] \mathrm{d}x. \tag{1.1.2}$$

As usual, a solution has to be understood in the weak distributional meaning, i.e. $u \in H^1_{loc}(\mathbb{R}^N)$ is a weak solution of (1.1.1) if

$$\int_{\mathbb{R}^N} H(\nabla u)\langle \nabla H(\nabla u), \nabla\varphi\rangle \, \mathrm{d}x = \int_{\mathbb{R}^N} f(x)\varphi \, \mathrm{d}x, \quad \forall \varphi \in C^1_c(\mathbb{R}^N). \tag{1.1.3}$$

As mentioned in the introduction, we tackle this issue by providing a suitable Kelvin-type transformation for the Riemannian case,

$$H(\xi) = \sqrt{\langle M\xi, \xi\rangle} \tag{h_M}$$

for some symmetric and positive definite matrix M. In this setting, we define the anisotropic Kelvin-type transformation by

$$T_H : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}, \quad T_H(\xi) := \frac{\nabla H(\xi)}{H(\xi)},$$

and consequently

$$\hat{u} := \frac{u \circ T_H}{H^{N-2}} \quad \text{in } \mathbb{R}^N \setminus \{0\},$$

see Section 1.3 for more details.

Remark 1.1.1. Although our main results are proven in the Riemannian case (h_M), parts of our construction apply to general Finsler norms. Thus, many preliminary results will be presented in a general framework, using Finsler geometry terminology where possible. This approach underscores the geometric concepts underlying our construction.

With this framework, we establish the following theorem.

Theorem 1.1.2. *Let H be given by (h_M). If $u \in H^1_{loc}(\mathbb{R}^N)$ is a locally bounded solution of (1.1.1), with $f \in L^2_{loc}(\mathbb{R}^N)$, then $\hat{u}$ belongs to $D^{1,2}$ far from the origin, and weakly solves the dual equation*

$$-\Delta^{H^\circ} \hat{u} = \frac{f \circ T_H}{H^{N+2}} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{1.1.4}$$

where H° is the dual norm, given by $H^\circ(x) = \sup\{\langle \xi, x\rangle : H(\xi) \leq 1\}$.

Remark 1.1.3. *Let us stress the fact that the result in Theorem 1.1.2 could be also obtained by using a suitable change of variable together with the techniques known for the standard Laplace operator. Such a proof would not be so easier or shorter. This led us to provide a proof that emphasizes the geometrical properties of the anisotropic Kelvin transformation.*

We remark that in the Euclidean framework, i.e. if $H(\xi) = \xi$, equation (1.1.1), with $f = u^{2^*-1}$, reduces to the classical Sobolev critical problem , for which the classification of positive solutions $u \in D^{1,2}(\mathbb{R}^N)$ was started in the seminal paper of [96]. Later on, Caffarelli, Gidas and Spruck [33], using the Kelvin transformation, classified all the solutions of the Sobolev critical equation without any a priori assumption on the energy. We refer to [76, 117, 127, 155] for an appropriate use of the Kelvin transformation. As we shall show adding an example in Remark 1.3.4, a pure invariance of the equation is not expected for general Finsler norms, even in the semilinear case $p = 2$.

To our knowledge, this is the first result in this direction within the anisotropic framework. We also acknowledge the work of Monti and Morbidelli [124], which presents a non-trivial application of the Kelvin transformation for equations involving Grushin operators.

The last part of our chapter is devoted to studying the following quasilinear elliptic equation:

$$-\Delta_N^H u = g(x) \qquad \text{in } \mathbb{R}^N, \quad N \geq 3\,, \tag{1.1.5}$$

where $\Delta_N^H u := \operatorname{div}(H^{N-1}(\nabla u)\nabla H(\nabla u))$ is the Finsler N-Laplace operator, and $g \in L^{\frac{N}{N-1}}_{loc}(\mathbb{R}^N)$.

As pointed out above, we say that $u \in W^{1,N}_{loc}(\mathbb{R}^N)$ is a weak solution of (1.1.1) if

$$\int_{\mathbb{R}^N} H^{N-1}(\nabla u)\langle \nabla H(\nabla u), \nabla\varphi\rangle \, \mathrm{d}x = \int_{\mathbb{R}^N} g(x)\varphi \, \mathrm{d}x, \quad \forall \varphi \in C^1_c(\mathbb{R}^N)\,. \tag{1.1.6}$$

In the Euclidean framework, i.e. if $H(\xi) = |\xi|$, we refer to the seminal paper by W. Chen and C. Li [39], where the, making use of the Kelvin transformation, classified solutions to the Liouville equation ($g = e^u$) via an adaptation of the moving plane method. As remarked above, in quasilinear case the situation is much more involved and we refer to the work of P. Esposito [80] for a classification result of the Liouville equation for $p = N > 2$, with the use of the Kelvin transform, but with a different approach to the one carried out in [39].

Theorem 1.1.4. *Let H be given be (h_M). If $u \in W^{1,N}_{loc}(\mathbb{R}^N)$ is a locally bounded solution of (1.1.5), with $g \in L^{\frac{N}{N-1}}_{loc}(\mathbb{R}^N)$, then $u^* := u \circ T_H$ weakly solves the dual equation*

$$-\Delta_N^{H^\circ} u^* = \frac{g \circ T_H}{H^{2N}} \quad \text{in } \mathbb{R}^N \setminus \{0\}, \tag{1.1.7}$$

where H° is the dual norm.

We point out that under suitable assumptions on f and g, due to the results in [7, 67, 113, 157], it is possible to assume that solutions of (1.1.1) and (1.1.5) are of class $\mathcal{C}^{1,\alpha}$.

In the following we devote Section 1.2 to the introduction of the main geometrical notion used in the thesis. In Section 1.3 we study the Kelvin-type transformation, defined above, and we prove our main results.

1.2 The norm H

The aim of this section is to introduce the Finsler norm H and to recall technical results about anisotropic elliptic operator that will be involved in the proof of the main results of the thesis.

Let H be a function belonging to $C^2(\mathbb{R}^N \setminus \{0\})$. H is said a "*Finsler norm*" if it satisfies the following assumptions:

(h_H) (i) $H(\xi) > 0 \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}$;

(ii) $H(s\xi) = |s|H(\xi) \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \forall s \in \mathbb{R}$;

(iii) H is *uniformly elliptic*, that means the set $\mathcal{B}_1^H := \{\xi \in \mathbb{R}^N : H(\xi) < 1\}$ is *uniformly convex*

$$\exists \Lambda > 0 : \quad \langle \mathrm{D}^2 H(\xi) v, v\rangle \geq \Lambda |v|^2 \quad \forall \xi \in \partial \mathcal{B}_1^H, \ \forall v \in \nabla H(\xi)^\perp. \tag{1.2.1}$$

Now we provide some remarks on Finsler norms.

Remark 1.2.1. A set is said uniformly convex if the principal curvatures of its boundary are all strictly positive. We point out that these kind of sets are also called "*strongly convex*", which is stronger a condition than the usual "*strict convexity*". Moreover, assumption (iii) is equivalent to assume that $\mathrm{D}^2(H^2)$ is definite positive, i.e. there exist $\lambda > 0$ such that, for all $\xi \neq 0$ and $\eta \in \mathbb{R}^N$,

$$\langle D^2 H^2(\xi)\eta, \eta\rangle \geq \lambda |\eta|^2. \tag{1.2.2}$$

Remark 1.2.2. Actually, in literature the definition of "*Finsler norm*" is not unique. In differential geometry, it is usual to define this class of norms satisfying directly the assumption that $\mathrm{D}^2(H^2)$ is definite positive. The last one allows to define the fundamental tensor, that is a very useful tool in order to show suitable properties of geometrical objects like curvatures, distances and so on. However, in many framework, it is enough to require that $\mathcal{B}_1^H$ is only strictly convex, and not strongly convex. Therefore, it is usual to give a definition of Finsler norm without the requirement (iii) (see for instance [18, 21, 94]), but with the only assumption that H^2 has to be strictly convex. For example, for $p \neq 2$, the p-norm in $\mathbb{R}^N$ is not uniformly elliptic, because its unit ball $\mathcal{B}_1^H$ is not strongly convex but only strictly convex. For our computations we need the stronger condition (iii) and we choose a particular class of Finsler norms satisfying it.

Since H a norm in $\mathbb{R}^N$, we immediately get that there exists $\alpha_1, \alpha_2 > 0$ such that:

$$\alpha_1|\xi| \le H(\xi) \le \alpha_2|\xi|, \quad \forall\, \xi \in \mathbb{R}^N. \tag{1.2.3}$$

The *dual norm* $H^\circ : \mathbb{R}^N \to [0, +\infty)$ is defined as:

$$H^\circ(x) = \sup\{\langle \xi, x\rangle : H(\xi) \le 1\}.$$

It is easy to prove that H° is also a Finsler norm and it has the same regularity properties of H. Moreover it follows that $(H^\circ)^\circ = H$. For $r > 0$ and $\overline{x} \in \mathbb{R}^N$ we define:

$$\mathcal{B}_r^H(\overline{x}) = \{x \in \mathbb{R}^N : H(x - \overline{x}) \le r\}$$

and

$$\mathcal{B}_r^{H^\circ}(\overline{x}) = \{x \in \mathbb{R}^N : H^\circ(x - \overline{x}) \le r\}.$$

For simplicity, when $\overline{x} = 0$, we set: $\mathcal{B}_r^H = \mathcal{B}_r^H(0)$, $\mathcal{B}_r^{H^\circ} = \mathcal{B}_r^{H^\circ}(0)$. In literature $\mathcal{B}_r^H$ and $\mathcal{B}_r^{H^\circ}$ are also called "Wulff shape" and "Frank diagram" respectively.

We recall that, since H is a differentiable and 1-homogeneous function, it holds the Euler's characterization result, i.e.

$$\langle \nabla H(\xi), \xi\rangle = H(\xi) \qquad \forall\, x \in \mathbb{R}^N, \tag{1.2.4}$$

and

$$\nabla H(t\xi) = \text{sign}(t) H(\xi) \qquad \forall\, \xi \in \mathbb{R}^N,\ \forall t \in \mathbb{R}, \tag{1.2.5}$$

and the same is true for H°.

Since H is 1-homogeneous, we have that ∇H is 0-homogeneous and it satisfies

$$\nabla H(\xi) = \nabla H\left(|\xi|\frac{\xi}{|\xi|}\right) = \nabla H\left(\frac{\xi}{|\xi|}\right) \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\}.$$

Hence, by the previous equality, we infer that there exists a constant $M > 0$ such that

$$|\nabla H(\xi)| \le M \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{1.2.6}$$

Since H is 1-homogeneous function we get

$$\mathrm{D}^2 H(t\xi) = \frac{1}{|t|}\mathrm{D}^2 H(\xi) \qquad \forall\, \xi \in \mathbb{R}^N \setminus \{0\},\ \forall t \neq 0, \tag{1.2.7}$$

For the same reasons there exists a constant $\overline{M} > 0$ such that:

$$|D^2 H(\xi)| \le \frac{\overline{M}}{|\xi|} \qquad \forall\, \xi \in \mathbb{R}^N \setminus \{0\}, \tag{1.2.8}$$

where $|\cdot|$ denotes the usual Euclidean norm of a matrix, and

$$D^2 H(\xi)\xi = 0 \qquad \forall\, \xi \in \mathbb{R}^N \setminus \{0\}. \tag{1.2.9}$$

Moreover, there holds following identities:

$$H(\nabla H^\circ(x)) = 1 = H^\circ(\nabla H(x)) \tag{1.2.10}$$

and

$$H(x)\nabla H^\circ(\nabla H(x)) = x = H^\circ(x)\nabla H(\nabla H^\circ(x)), \tag{1.2.11}$$

and we refer the reader to [21] for a complete proof.

1.2.1 Useful inequalities for the norm H

We start with some elliptic estimates that can be proved in the same spirit of the Euclidean framework.

Proposition 1.2.3. *For any $p > 1$ and $\eta, \eta' \in \mathbb{R}^N$ such that $|\eta| + |\eta'| > 0$, it holds*

$$|H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \le \tilde{C}_p(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|. \tag{1.2.12}$$

Moreover, any $p \ge 2$ it holds the following inequality

$$H^p(\eta) \ge H^p(\eta') + pH^{p-1}(\eta')\langle \nabla H(\eta'), \eta - \eta'\rangle + \hat{C}(p)H^p(\eta - \eta'), \tag{1.2.13}$$

for any $\eta, \eta' \in \mathbb{R}^N$. Furthermore, if $1 < p < 2$ we have that

$$H^p(\eta) \ge H^p(\eta') + pH^{p-1}(\eta')\langle \nabla H(\eta'), \eta - \eta'\rangle + C_p[H(\eta) + H(\eta')]^{p-2}H^2(\eta - \eta'), \tag{1.2.14}$$

for any $\eta, \eta' \in \mathbb{R}^N$ such that $|\eta| + |\eta'| > 0$.

Proof. We start the proof showing (1.2.12). First of all we note that (1.2.12) is symmetric in η, η'. Hence, without loss of generality, we can assume that $|\eta'| \ge |\eta| > 0$. We note that for $j = 1, \ldots, N$:

$$\begin{aligned} &H^{p-1}(\eta)\frac{\partial H}{\partial \eta_j}(\eta) - H^{p-1}(\eta')\frac{\partial H}{\partial \eta_j}(\eta') = \\ &= \int_0^1 \sum_{i=1}^N \Bigg[(p-1)H^{p-2}(\eta' + t(\eta - \eta'))\left(\frac{\partial H}{\partial \eta_i}\cdot\frac{\partial H}{\partial \eta_j}\right)(\eta' + t(\eta - \eta')) \\ &\qquad + H^{p-1}(\eta' + t(\eta - \eta'))\frac{\partial^2 H}{\partial \eta_i \partial \eta_j}(\eta' + t(\eta - \eta'))\Bigg]\cdot(\eta_i - \eta_i')\,dt. \end{aligned} \tag{1.2.15}$$

By (1.2.15), using (1.2.3), (1.2.6) and (1.2.8) we have

$$\begin{aligned} &|H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \le \\ &\le \int_0^1 |(p-1)H^{p-2}(\eta' + t(\eta - \eta'))\nabla H(\eta' + t(\eta - \eta')) \otimes \nabla H(\eta' + t(\eta - \eta')) \\ &\qquad + H^{p-1}(\eta' + t(\eta - \eta')))D^2H(\eta' + t(\eta - \eta'))|\cdot|\eta - \eta'|\,dt \\ &\le ((p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'|\int_0^1 (|\eta' + t(\eta - \eta')|)^{p-2}\,dt, \end{aligned} \tag{1.2.16}$$

where $|\cdot|$ denotes the standard matrix Euclidean norm.

Now, we observe that

$$|\eta' + t(\eta - \eta')| \le |\eta| + |\eta'|, \tag{1.2.17}$$

and, since $|\eta'| \ge |\eta|$, we have either

$$|\eta - \eta'| \le \frac{|\eta'|}{2} \implies |\eta' + t(\eta - \eta')| \ge |\eta'| - |\eta - \eta'| \ge \frac{|\eta'|}{2} \ge \frac{|\eta'| + |\eta|}{4} \tag{1.2.18}$$

or, putting $t_0 := \dfrac{|\eta'|}{|\eta - \eta'|} \in (0, 2)$,

$$\begin{aligned} |\eta - \eta'| > \frac{|\eta'|}{2} \implies |\eta' + t(\eta - \eta')| &\ge \big||\eta'| - t|\eta - \eta'|\big| = |t_0 - t|\cdot|\eta - \eta'| \\ &\ge |t_0 - t|\cdot\frac{|\eta'|}{2} = |t_0 - t|\frac{|\eta'|}{2} \\ &\ge |t_0 - t|\frac{|\eta| + |\eta'|}{4}. \end{aligned} \tag{1.2.19}$$

If $p > 2$, using (1.2.17) in (1.2.16) we have

$$|H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \le \tilde{C}_p(|\eta'| + |\eta|)^{p-2}|\eta - \eta'| \tag{1.2.20}$$

where $\tilde{C}_p = (p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M}$. Hence (1.2.12) holds.

If $p \le 2$ and (1.2.18) holds, by (1.2.16) we obtain

$$\begin{aligned}
&|H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \le \\
&\le ((p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'| \int_0^1 (|\eta' + t(\eta - \eta')|)^{p-2}\, dt. \\
&\le ((p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'| \int_0^1 \left(\frac{|\eta'| + |\eta|}{4}\right)^{p-2} dt \\
&\le \tilde{C}_p(|\eta'| + |\eta|)^{p-2}|\eta - \eta'|,
\end{aligned} \tag{1.2.21}$$

where $\tilde{C}_p = ((p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})/4^{p-2}$. Hence (1.2.12) holds.

If $p \le 2$ and (1.2.19) holds, by (1.2.16) we obtain

$$\begin{aligned}
&|H^{p-1}(\eta)\nabla H(\eta) - H^{p-1}(\eta')\nabla H(\eta')| \le \\
&\le ((p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'| \int_0^1 (|\eta' + t(\eta - \eta')|)^{p-2}\, dt. \\
&\le ((p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M})|\eta - \eta'| \int_0^1 |t_0 - t|^{p-2} \left(\frac{|\eta'| + |\eta|}{4}\right)^{p-2} dt \\
&= \frac{(p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M}}{4^{p-2}}(|\eta'| + |\eta|)^{p-2}|\eta - \eta'| \int_0^1 |t_0 - t|^{p-2}\, dt \\
&\le \frac{(p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M}}{4^{p-2}}(|\eta'| + |\eta|)^{p-2}|\eta - \eta'|\; 2\int_0^1 z^{p-2}\, dz \\
&= \tilde{C}_p(|\eta'| + |\eta|)^{p-2}|\eta - \eta'|,
\end{aligned} \tag{1.2.22}$$

where $\tilde{C}_p = (2(p-1)\alpha_2^{p-2}M^2 + 2\alpha_2^{p-1}\overline{M})/4^{p-2}$.

Collecting the estimates above, we deduce that inequality (1.2.12) holds for every $p > 1$ and for $\tilde{C}_p = ((p-1)\alpha_2^{p-2}M^2 + \alpha_2^{p-1}\overline{M}) \cdot \max\{1, 4^{2-p}, 2 \cdot 4^{2-p}\}$.

Now we will show (1.2.13) and (1.2.14). For $\eta, \eta' \in \mathbb{R}^N$, we define

$$f(t) = H^p(\eta' + t(\eta - \eta'))$$

and Taylor's formula yields

$$H^p(\eta) = H^p(\eta') + pH^{p-1}(\eta')\langle \nabla H(\eta'), \eta - \eta'\rangle + \int_0^1 (1-t)f''(t)\, dt, \tag{1.2.23}$$

provided $|\eta' + t(\eta - \eta')| \ne 0$, for $0 \le t \le 1$. But the case when $\eta' + t(\eta - \eta') = 0$ can be easily verified. By [52, Theorem 1.5] we obtain that

$$\begin{aligned}
f''(t) &= \langle D^2(H^p(\eta' + t(\eta - \eta')))(\eta - \eta'), \eta - \eta'\rangle \\
&\ge C(p)|\eta' + t(\eta - \eta')|^{p-2}|\eta - \eta'|^2,
\end{aligned} \tag{1.2.24}$$

where $C(p)$ is a constant depending on p. We remark that

$$\int_0^1 (1-t)f''(t)\, dt \ge \frac{3}{4}\int_0^{\frac{1}{4}} f''(t)\, dt. \tag{1.2.25}$$

If $1 < p < 2$, by (1.2.17) we have

$$(H(\eta) + H(\eta'))^{p-2} \le \alpha_2^{p-2} |\eta' + t(\eta - \eta')|^{p-2}$$

and using (1.2.25) we arrive at (1.2.14).

If $p \ge 2$, using a similar argument as in the proof of the inequality (1.2.12), we obtain

$$\int_0^{\frac{1}{4}} f''(t) \ge \overline{C}(p)(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2, \tag{1.2.26}$$

with $\overline{C}(p)$ constant depending on p. Since $|\eta - \eta'| \le |\eta| + |\eta'|$ and using (1.2.3), we get (1.2.13). $\square$

1.3 Anisotropic Kelvin transformation

The aim of this section is to prove Theorem 1.1.2.

As pointed out in Remark 1.1.1, a part of our construction hold for generic Finsler norms, but the main results are proved in the Riemannian case (h_M). In this case

$$H^\circ(\xi) = \sqrt{\langle M^{-1}\xi, \xi\rangle}.$$

In order to achieve our claim, we introduce the formal definition of Kelvin transformation in the context of Finsler geometry.

Definition 1.3.1. *Let H be a Finsler norm. We define the map $T_H : \mathbb{R}^N \setminus \{0\} \to \mathbb{R}^N \setminus \{0\}$ as*

$$T_H(\xi) = \frac{\nabla H(\xi)}{H(\xi)}. \tag{1.3.1}$$

Note that, when $H(\xi) = |\xi|$, T_H coincides with the usual Euclidean Kelvin transform. Hence, we observe that it holds the following result.

Proposition 1.3.2. *The map T_H is a global diffeomorphism with inverse given by T_{H°.*

Proof. We show that, if $y = T_H(x)$, then $x = T_{H^\circ}(y)$. By (1.2.5),(1.2.10) and (1.2.11) we have:

$$\begin{aligned} T_H(T_{H^\circ}(y)) &= \nabla H\left(\frac{\nabla H^\circ(y)}{H^\circ(y)}\right) H^{-1}\left(\frac{\nabla H^\circ(y)}{H^\circ(y)}\right) \\ &= \nabla H(\nabla H^\circ(y)) H^\circ(y) H^{-1}(\nabla H^\circ(y)) \\ &= H^\circ(y)\nabla H(\nabla H^\circ(y))) = y\,. \end{aligned} \tag{1.3.2}$$

$\square$

Now, we denote by $\mathrm{D}T_H$ the Jacobian matrix of T_H, namely:

$$\mathrm{D}T_H = \frac{1}{H^2}\left[H\mathrm{D}^2 H - \nabla H \otimes \nabla H\right] = \frac{1}{H^2}\left[\mathrm{D}\left(\frac{1}{2}\nabla(H^2)\right) - 2\nabla H \otimes \nabla H\right].$$

In particular, if H satisfies (h_M), we have

$$\mathrm{D}T_H(x) = \frac{1}{H(x)^2}\left[M - \frac{2}{H(x)^2} Mx \otimes Mx\right]. \tag{1.3.3}$$

Moreover, we set $J(x) = |\det \mathrm{D}T_H(x)|$. Using these notations, we are ready to prove the following key lemma.

Lemma 1.3.3. *If H is a Finsler norm that satisfy (h_M), then*

$$J(x) = \frac{\det M}{H(x)^{2N}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}. \tag{1.3.4}$$

Proof. Let L be a diagonal matrix with entries $d_1, \dots, d_N$ and denote by $\sqrt{L}$ the diagonal matrix whose entries are $\sqrt{d_1}, \dots, \sqrt{d_N}$. Suppose $H(x) = \sqrt{\langle Lx, x\rangle}$. In view of (1.3.3) we have:

$$\begin{aligned} \mathrm{D}T_H(x) &= \frac{1}{H(x)^2}\left(L - \frac{2}{\langle Lx, x\rangle} Lx \otimes Lx\right) = \\ &= \frac{1}{H(x)^2}\sqrt{L}\left(I - \frac{2}{\langle \sqrt{L}x, \sqrt{L}x\rangle}\sqrt{L}x \otimes \sqrt{L}x\right)\sqrt{L} \end{aligned} \tag{1.3.5}$$

Since

$$\det\left(I - \frac{2}{|y|^2} y \otimes y\right) = 1, \quad \forall y \in \mathbb{R}^N \setminus \{0\}, \tag{1.3.6}$$

we get (1.3.4) in the case L diagonal. Let H be of the form (h_M). The spectral theorem ensures that it holds the decomposition $C^tMC = L$, where C and D are respectively an orthogonal and diagonal matrices. Having in mind this fact, we have

$$C^t\mathrm{D}T_H(x)C = \frac{1}{H(x)^2}\sqrt{L}\left(I - \frac{2}{\langle \sqrt{L}C^tx, \sqrt{L}C^tx\rangle}\sqrt{L}C^tx \otimes \sqrt{L}C^tx\right)\sqrt{L}. \tag{1.3.7}$$

By (1.3.6) we get the thesis. □

Assumption (h_M) is crucial in the previous lemma, and this result is essential also in the proof of Theorem 1.1.2 and Theorem 1.1.4. Indeed, our approach doesn't work without Lemma 1.3.3, as pointed out in the next remark.

Remark 1.3.4. In the proof of Theorem 1.1.2 it will be crucial the fact that $H(x)^{2N}J(x)$ is constant, see (1.3.4). For a generical Finsler norm this property is not true. For example, if we consider the norm

$$H(x) = \sqrt[4]{x_1^4 + 3x_1^2x_2^2 + x_2^4}$$

for $x = (x_1, x_2) \in \mathbb{R}^2$, then it follows that $H(x)^{2N}J(x)$ is not constant, even if H satisfies all the conditions (i)-(ii)-(iii) stated above in the definition of a Finsler norm.

Now, in the same spirit of the seminal paper of Caffarelli, Gidas and Spruck [33], given a function $u(x)$, we define the action of the Kelvin transform on u as follows

$$\hat{u}(y) := \frac{1}{H(y)^{N-2}} u(T_H(y)), \tag{1.3.8}$$

for every $y \in \mathbb{R}^N \setminus \{0\}$.

Remark 1.3.5. It is easy to check that

$$u(x) := \frac{1}{H^\circ(x)^{N-2}} \hat{u}(T_{H^\circ}(x)), \tag{1.3.9}$$

for every $x \in \mathbb{R}^N \setminus \{0\}$. In fact, using (1.2.10), by (1.3.8) we have:

$$\begin{aligned} u(x) &= \hat{u}\left(T_{H^\circ}(x)\right) H\left(T_{H^\circ}(x)\right)^{N-2} = \hat{u}\left(T_{H^\circ}(x)\right) H\left(\frac{\nabla H^\circ(x)}{H^\circ(x)}\right)^{N-2} \\ &= \hat{u}\left(T_{H^\circ}(x)\right) \frac{1}{H^\circ(x)^{N-2}} H(\nabla H^\circ(x))^{N-2} = \frac{\hat{u}\left(T_{H^\circ}(x)\right)}{H^\circ(x)^{N-2}}. \end{aligned} \tag{1.3.10}$$

Now we are ready to prove the first result of our chapter.

Proof of Theorem 1.1.2. We use $\psi \in C_c^1(\mathbb{R}^N \setminus \{0\})$ as test function in (1.1.3). Then $\varphi(x) := \psi(T_{H^\circ}(x))$ can also be used as test function in (1.1.3), obtaining:

$$\begin{aligned}\int_{\mathbb{R}^N} H(\nabla u(x))\langle \nabla H(\nabla u(x)), \mathrm{D}T_{H^\circ}(x)\nabla\psi(T_{H^\circ}(x))\rangle \,\mathrm{d}x \\ = \int_{\mathbb{R}^N} f(x)\psi(T_{H^\circ}(x))\,\mathrm{d}x.\end{aligned} \tag{1.3.11}$$

Using the change of variable $y = T_{H^\circ}(x)$, (1.3.11) becomes

$$\begin{aligned}\int_{\mathbb{R}^N} H(\nabla u(T_H(y)))\langle \mathrm{D}T_{H^\circ}(T_H(y))\nabla H(\nabla u(T_H(y))), \nabla\psi(y)\rangle J(y)\,\mathrm{d}y \\ = \int_{\mathbb{R}^N} f(T_H(y))\psi(y)J(y)\,\mathrm{d}y.\end{aligned} \tag{1.3.12}$$

Under our assumption (h_M) we deduce that

$$\begin{aligned}&H(\nabla u(T_H(y)))\mathrm{D}T_{H^\circ}(T_H(y))\nabla H(\nabla u(T_H(y))) \\ &\quad = H(y)^4 H^\circ(\mathrm{D}T_H(y)\nabla u(T_H(y)))\nabla H^\circ(\mathrm{D}T_H(y)\nabla u(T_H(y))).\end{aligned} \tag{1.3.13}$$

If we set $u^*(y) := u(T_H(y))$ and $f^*(y) := f(T_H(y))$, by using (1.3.13) and Lemma 1.3.3, (1.3.12) becomes:

$$\begin{aligned}\int_{\mathbb{R}^N} \frac{1}{H(y)^{2N-4}} H^\circ(\nabla u^*(y))\langle \nabla H^\circ(\nabla u^*(y)), \nabla\psi(y)\rangle\,\mathrm{d}y \\ = \int_{\mathbb{R}^N} \frac{1}{H(y)^{2N}} f^*(y)\psi(y)\,\mathrm{d}y.\end{aligned} \tag{1.3.14}$$

Let $\phi \in C_c^1(\mathbb{R}^N \setminus \{0\})$ be such that $\psi(y) = H(y)^{N-2}\phi(y)$. Then ψ can be used as test function in (1.3.14), getting:

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{1}{H(y)^{N-2}} H^\circ(\nabla u^*(y))\langle \nabla H^\circ(\nabla u^*(y)), \nabla\phi(y)\rangle \\ &\quad - \int_{\mathbb{R}^N} H^\circ(\nabla u^*(y))\langle \nabla H^\circ(\nabla u^*(y)), \nabla\left(\frac{1}{H(y)^{N-2}}\right)\rangle\phi(y)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} \frac{1}{H(y)^{N+2}} f^*(y)\phi(y)\,\mathrm{d}y\end{aligned} \tag{1.3.15}$$

Recalling (1.2.5), the fact that H is a 1-homogeneous function and that

$$\nabla\hat{u} = \nabla\left(\frac{1}{H^{N-2}}\right)u^* + \frac{1}{H^{N-2}}\nabla u^*,$$

we get

$$\begin{aligned}&\int_{\mathbb{R}^N} H^\circ\left(\nabla\hat{u}(y) - \nabla\left(\frac{1}{H(y)^{N-2}}\right)u^*\right)\times \\ &\qquad \times \langle \nabla H^\circ\left(\nabla\hat{u}(y) - \nabla\left(\frac{1}{H(y)^{N-2}}\right)u^*\right), \nabla\phi(y)\rangle\,\mathrm{d}y \\ &\quad - \int_{\mathbb{R}^N} H^\circ(\nabla u^*(y))\langle \nabla H^\circ(\nabla u^*(y)), \nabla\left(\frac{1}{H(y)^{N-2}}\right)\rangle\phi(y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^N} \frac{1}{H(y)^{N+2}} f^*(y)\phi(y)\,\mathrm{d}y\end{aligned} \tag{1.3.16}$$

By (h_M), we deduce that $H^\circ \nabla H^\circ$ is linear and symmetric with respect to the Euclidean inner product of $\mathbb{R}^N$, i.e.

$$\langle H^\circ(x)\nabla H^\circ(x), y\rangle = \langle x, H^\circ(y)\nabla H^\circ(y)\rangle \quad \forall x, y \in \mathbb{R}^N.$$

Hence, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} H^\circ(\nabla \hat{u}(y))\langle \nabla H^\circ(\nabla \hat{u}(y)), \nabla\phi(y)\rangle \, \mathrm{d}y \\
&- \int_{R^N} H^\circ\left(\nabla\left(\frac{1}{H(y)^{N-2}}\right)\right)\left\langle \nabla H^\circ\left(\nabla\left(\frac{1}{H(y)^{N-2}}\right)\right), \nabla(u^*(y)\phi(y))\right\rangle \, \mathrm{d}y \\
&= \int_{R^N} \frac{1}{H(y)^{N+2}} f^*(y)\phi(y) \, \mathrm{d}y.
\end{aligned} \tag{1.3.17}$$

Integrating by parts (1.3.17), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N} H^\circ(\nabla \hat{u}(y))\langle \nabla H^\circ(\nabla \hat{u}(y)), \nabla\phi(y)\rangle \, \mathrm{d}y \\
&\quad + \int_{\mathbb{R}^N} \operatorname{div}\left(H^\circ\left(\nabla\left(\frac{1}{H(y)^{N-2}}\right)\right)\nabla H^\circ\left(\nabla\frac{1}{H(y)^{N-2}}\right)\right) u^*(y)\phi(y) \, \mathrm{d}y \\
&= \int_{\mathbb{R}^N} \frac{1}{H(y)^{N+2}} f^*(y)\phi(y) \, \mathrm{d}y.
\end{aligned} \tag{1.3.18}$$

Since $w(x) := H(x)^{2-N}$ is an H°-harmonic function, namely $\Delta^{H^\circ} w = 0$, we get the thesis. □

To conclude, with the same spirit of the previous proof, we prove our last result.

Proof of Theorem 1.1.4. Let us consider $\psi \in C_c^1(\mathbb{R}^N \setminus \{0\})$ and we set $\varphi := \psi(T_{H^\circ})$. We note that φ is a good test function and substituting in (1.1.5), we get

$$\begin{aligned}
\int_{\mathbb{R}^N} H(\nabla u(x))^{N-1}\langle \nabla H(\nabla u(x)), \mathrm{D}T_{H^\circ}(x)\nabla\psi(T_{H^\circ}(x))\rangle \, \mathrm{d}x \\
= \int_{\mathbb{R}^N} g(x)\psi(T_{H^\circ}(x)) \, \mathrm{d}x.
\end{aligned} \tag{1.3.19}$$

We set $y = T_{H^\circ}(x)$, and using this variable change (1.3.19) becomes

$$\begin{aligned}
\int_{\mathbb{R}^N} H(\nabla u(T_H(y)))^{N-1}\langle \mathrm{D}T_{H^\circ}(T_H(y))\nabla H(\nabla u(T_H(y))), \nabla\psi(y)\rangle J(y) \, \mathrm{d}y \\
= \int_{\mathbb{R}^N} g(T_H(y))\psi(y)J(y) \, \mathrm{d}y.
\end{aligned} \tag{1.3.20}$$

If we set $g^*(y) := g(T_H(y))$, and recalling (1.3.13) and Lemma 1.3.3, (1.3.20) becomes:

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{H^{N-2}(\nabla u(T_H(y)))}{H(y)^{2N-4}} H^\circ(\nabla u^*(y))\langle \nabla H^\circ(\nabla u^*(y)), \nabla\psi(y)\rangle \, \mathrm{d}y \\
= \int_{\mathbb{R}^N} \frac{g^*(y)\psi(y)}{H(y)^{2N}} \, \mathrm{d}y.
\end{aligned} \tag{1.3.21}$$

Under assumption (h_M), it is easy to check

$$H^\circ(DT_H(y)\xi) = \frac{H(\xi)}{H^2(y)} \quad \forall \xi \in \mathbb{R}^N. \tag{1.3.22}$$

By (1.3.22) we get the thesis. □

Chapter 2

Liouville-type results for some quasilinear anisotropic elliptic equations

2.1 Main results

As mentioned in the introduction, in this chapter we will study solutions, possibly unbounded and sign-changing, to some quasilinear anisotropic elliptic equations in general Euclidean domains and we obtain new Liouville-type theorems. Specifically, for any domain Ω of $\mathbb{R}^N$, $N \geq 2$, and $q > 1$ we investigate the quasilinear elliptic equation:

$$-\Delta_B^H u = |u|^{q-1}u \quad \text{in}\, \Omega\,, \tag{2.1.1}$$

where Δ_B^H is the anisotropic operator

$$\Delta_B^H u = \operatorname{div}(B'(H(\nabla u))\nabla H(\nabla u)), \tag{2.1.2}$$

H is a Finsler norm and B adheres to specific natural growth conditions detailed below.

If H is the standard Euclidean norm and $B(t) = \frac{t^p}{p}$, $p > 1$, the operator Δ_B^H reduces to the classical $p-$Laplacian operator.

Here we deal with the anisotropic case, i.e., the general case in which H is a Finsler norm (see the assumption (h_H) in Chapter 1, Section 1.2) and B satisfies

(h_B) (i) $B \in C^2((0, +\infty))$;

(ii) $B(0) = B'(0) = 0, \quad B(t), B'(t), B''(t) > 0 \quad \forall t \in (0, +\infty)$;

(iii) there exist $p > 1$, $\kappa \in [0, 1]$, $\gamma > 0$, $\Gamma > 0$ such that

$$\begin{aligned} \gamma(\kappa + t)^{p-2}t \leq& B'(t) \leq \Gamma(\kappa + t)^{p-2}t \quad \forall\, t > 0 \\ \gamma(p-1)(\kappa + t)^{p-2} \leq& B''(t) \leq \Gamma(p-1)(\kappa + t)^{p-2} \quad \forall\, t > 0\,. \end{aligned} \tag{2.1.3}$$

For $p > 1$, a function $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of (2.1.1) if

$$\int_\Omega B'(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle\, dx \;=\; \int_\Omega |u|^{q-1}u\varphi\, dx, \quad \forall \varphi \in C^1_c(\Omega)\,. \tag{2.1.4}$$

In [51] the authors proved that locally bounded weak solutions u to (2.1.1) are actually of class $C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z)$, where Z is the set of critical points of u and $\alpha \in (0, 1)$ (as in the isotropic case). These regularity properties will be used extensively without further mention. Additionally, [7, 35] provides new insights into the integrability of the second weak derivatives of the solutions.

For a historical overview of the problem, we refer the reader to the introduction.

Let us recall the following:

Definition 2.1.1 (see [57]). *We say that a weak solution u is stable if*

$$\begin{aligned} L_u(\varphi,\varphi) := &\int_\Omega B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2 \\ &+ \int_\Omega B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi,\nabla\varphi\rangle - q|u|^{q-1}v\varphi, \geq 0 \qquad \forall\varphi\in C_c^1(\Omega). \end{aligned} \tag{2.1.5}$$

Moreover, we say that a weak solution is stable outside a compact set $K\subset\Omega$ if (2.1.5) holds for every $\varphi\in C_c^1(\Omega\setminus K)$.

Remark 2.1.2. *We emphasize that the quantities involved in the definition of stability might not be well-defined on the critical set Z_u. The use of the following test-function $\varphi := |u|^{\frac{\beta-1}{2}}u\varphi^{p/2}$, with $\varphi\in C_c^1(\Omega)$, in (2.1.5), is enough for our purposes. Indeed if $u\neq 0$, by [35, Proposition 3.5] the Lebesgue measure of the critical set Z_u is zero. On the other hand $\varphi(x)=0$, for every $x\in\{u(x)=0\}$.*

Our main result is stated in the following theorem.

Theorem 2.1.3. *Let us assume that (h_H) and (h_B) hold and let $u\in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ be a weak stable solution of (2.1.1) with $p\geq 2$. If one of the following holds*

(i) $\Gamma\kappa^{p-2}(p-1)\geq 1$, $q>\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2$ *and*

$$N < g(q) := 2\frac{\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2(q-1)+2q+2\sqrt{q(q-\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2)}}{\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2(q-1)}$$

(ii) $0<\Gamma\kappa^{p-2}(p-1)<1$, $q>\kappa^{2-p}\gamma^{-1}$ *and*

$$N < f(q) := 2\frac{\kappa^{2-p}\gamma^{-1}(q-1)+2q+2\sqrt{q(q-\kappa^{2-p}\gamma^{-1})}}{\kappa^{2-p}\gamma^{-1}(q-1)}$$

(iii) $\kappa=0$, $q>\Gamma\gamma^{-1}(p-1)$ *and*

$$N < s(q) := p\frac{\Gamma\gamma^{-1}(p-1)(q-1)+2q+2\sqrt{q(q-\Gamma\gamma^{-1}(p-1))}}{\Gamma\gamma^{-1}(p-1)(q-(p-1))}$$

Then $u\equiv 0$.

We remark that under our structural assumptions, it also follows that $q>(p-1)\geq 1$.

Our result applies, in particular, to the case of the pure Finsler p-Laplacian, namely $B(t)=\frac{t^p}{p}$. In this case we have the following:

Corollary 2.1.4. *Let us assume that (h_H) holds and let u be a weak stable solution of*

$$-\operatorname{div}(H(\nabla u)^{p-1}\nabla H(\nabla u)) = |u|^{q-1}u \qquad \text{in } \mathbb{R}^N. \tag{2.1.6}$$

Assume that $p\geq 2$ and

$$\begin{cases} p-1<q<\infty & \text{if } N \leq \frac{p(p+3)}{p-1} \\ \\ p-1<q<q_c(N,p) & \text{if } N > \frac{p(p+3)}{p-1} \end{cases} \tag{2.1.7}$$

with

$$q_c(N,p) = \frac{((p-1)N-p)^2 + p^2(p-2) - p^2(p-1)N + 2p^2\sqrt{(p-1)(N-1)}}{(N-p)\left((p-1)N - p(p+3)\right)}. \tag{2.1.8}$$

Then, $u \equiv 0$.

Remark 2.1.5. *Let us stress the fact that the critical exponent $q_c(N,p)$ is the one found in [54, 85] when H is the standard Euclidean norm. Moreover, our result is sharp, see [54, 85].*

Exploiting the Pohozaev identity in the anisotropic setting, see [125], we shall also prove a Liouville-type result for solutions that are stable outside a compact set.

Theorem 2.1.6. *Let us assume that (h_H) and (h_B) hold, with $\kappa = 0$ in (h_B). Let $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ be a weak solution of (2.1.1) which is stable outside a compact set. Assume $p \geq 2$ and*

$$\begin{cases} \Gamma\gamma^{-1}(p-1) < q < \infty & \text{if } N \leq p\gamma\Gamma^{-1} \\ \\ \Gamma\gamma^{-1}(p-1) < q < \frac{N\left(p-\frac{\Gamma}{\gamma}\right)+p}{N\frac{\Gamma}{\gamma}-p}, & \text{if } N > p\gamma\Gamma^{-1}. \end{cases} \tag{2.1.9}$$

Then $u \equiv 0$.

Although the case $k \neq 0$ is non degenerate, in our framework it turns out to be quite involved, since the techniques developed in [54, 85] do not work in this case. This is due to the fact that, when k is not zero, we have a loss of the homogeneity proprieties of the stress field $B'(H(\nabla u))\nabla H(\nabla u)$. Indeed, in this case, $B'(t) \sim t^{p-1}$ for t large, while $B'(t) \sim t$ for t small. We also note that the parameter k plays a role in the statement of our Liouville-type theorem.

2.2 Proof of the main results

The main tool for the proof of Theorems 2.1.3 is the inequality stated in the following proposition.

Proposition 2.2.1. *Let Ω be a domain in $\mathbb{R}^N$, with $N \geq 2$, and u be a weak stable solution of (2.1.1) with $p \geq 2$. Then, if $\Gamma\kappa^{p-2}(p-1) \geq 1$ and $q > \Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2$, for any*

$$\beta \in \left[1, \frac{2q - \Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2 + 2\sqrt{q(q-\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2)}}{\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2}\right) \tag{2.2.1}$$

and for any integer

$$l > \max\left\{\frac{q+\beta}{q-(p-1)}, 2\right\} \tag{2.2.2}$$

there exist a positive constant $C = C(B, H, \delta, \varepsilon, q, \beta)$ such that

$$\int_\Omega |u|^{q+\beta}\varphi^{pl} \leq C\int_{\Omega_1} |\nabla\varphi|^{2\frac{q+\beta}{q-1}} + C\int_{\Omega_2} |\nabla\varphi|^{p\frac{q+\beta}{q-p+1}} \quad \forall\varphi \in C^1_c(\Omega), \quad 0 \leq \varphi \leq 1, \tag{2.2.3}$$

where $\Omega_1 := \Omega_1(\delta) := \{x \in \Omega \,:\, H(\nabla u(x)) < \delta\}$ *and* $\Omega_2 := \Omega_2(\delta) := \Omega \setminus \Omega_1$*, with* $\delta > 0$ *fixed.*

On the other hand, if $0 < \Gamma\kappa^{p-2}(p-1) < 1$ *and* $q > \kappa^{-p+2}\gamma^{-1}$*, for any*

$$\beta \in \left[1, \frac{2q - \kappa^{-p+2}\gamma^{-1} + 2\sqrt{q(q-\kappa^{-p+2}\gamma^{-1})}}{\kappa^{-p+2}\gamma^{-1}}\right) \tag{2.2.4}$$

and for any integer l *as in* (2.2.2)*, the inequality* (2.2.3) *still holds true.*

Remark 2.2.2. *The same result holds true if* u *is stable outside a compact set* $K \subset \Omega$*. In this case the test functions* φ *belong to* $C^1_c(\Omega \setminus K)$*.*

Proof. As already pointed out after Theorem 2.1.6, the case $\kappa > 0$ is quite involved. To overcome these difficulties we shall split the domain Ω into two subdomains, Ω_1 and Ω_2. We split the proof into several steps.

Step 1. *For any* $\beta \geq 1$ *and* $\varepsilon > 0$ *small enough there exists a positive constant* $\bar{C} = \bar{C}(B, H, \delta, \varepsilon)$ *such that*

$$\begin{aligned}
&\min\{\beta\gamma\kappa^{p-2} - \varepsilon^2, \frac{\beta\gamma\Gamma^{-1}}{p-1} - \varepsilon^2\}\cdot \\
&\cdot\left(\int_{\Omega_1} H(\nabla u)^2|u|^{\beta-1}\varphi^p + \int_{\Omega_2} B''(H(\nabla u)H(\nabla u)^2|u|^{\beta-1}\varphi^p\right) \\
&\leq \bar{C}\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2} + \bar{C}\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1} + \int_\Omega |u|^{q+\beta}\varphi^p
\end{aligned} \tag{2.2.5}$$

for any nonnegative $\varphi \in C^1_c(\Omega)$*.*

Let us consider

$$\phi = |u|^{\beta-1}u\varphi^p \in C^1_c(\Omega),$$

so that $\nabla\phi = p|u|^{\beta-1}u\varphi^{p-1}\nabla\varphi + \beta|u|^{\beta-1}\varphi^p\nabla u$. Taking ϕ as test function in (2.1.4), we get:

$$\begin{aligned}
&\beta\int_\Omega B'(H(\nabla u))\langle\nabla H(\nabla u), \nabla u\rangle|u|^{\beta-1}\varphi^p \\
&= -p\int_\Omega B'(H(\nabla u))\langle\nabla H(\nabla u), \nabla\varphi\rangle|u|^{\beta-1}u\varphi^{p-1} + \int_\Omega |u|^{q+\beta}\varphi^p.
\end{aligned} \tag{2.2.6}$$

We emphisize that, when the gradient vanishes, the following computations are well defined thanks to the structural hypothesis on $B(\cdot)$.

Combining (1.2.4), (1.2.6) and $(h_B) - (iii)$ with (2.2.6) we get:

$$\begin{aligned}
&\beta\int_\Omega B'(H(\nabla u))H(\nabla u)|u|^{\beta-1}\varphi^p \\
&\leq Mp\int_\Omega B'(H(\nabla u))|\nabla\varphi||u|^\beta\varphi^{p-1} + \int_\Omega |u|^{q+\beta}\varphi^p.
\end{aligned} \tag{2.2.7}$$

Under assumptions $(h_B) - (iii)$, there exist positive constants $C_1 = C_1(B, \delta) = \Gamma(\kappa + \delta)^{p-2}$, $\overline{C}_1 = \overline{C}_1(B, \delta) = \Gamma(p-1)(\kappa+\delta)^{p-2}$, $C_2 = C_2(B, \delta)$ and $\overline{C}_2 = \overline{C}_2(B, \delta)$ such that

$$\begin{aligned}
&B'(H(\nabla u)) \leq C_1 H(\nabla u) && \text{in } \Omega_1 \\
&B'(H(\nabla u)) \leq C_2 H(\nabla u)^{p-1} && \text{in } \Omega_2 \\
&B''(H(\nabla u)) \leq \overline{C}_1 && \text{in } \Omega_1 \\
&B''(H(\nabla u)) \leq \overline{C}_2 H(\nabla u)^{p-2} && \text{in } \Omega_2\,.
\end{aligned} \tag{2.2.8}$$

Moreover, we have that

$$\begin{aligned} \gamma\kappa^{p-2}H(\nabla u) &\leq B'(H(\nabla u)) \quad \text{in } \Omega \\ B''(H(\nabla u))H(\nabla u) &\leq \Gamma\gamma^{-1}(p-1)B'(H(\nabla u)) \quad \text{in } \Omega\,. \end{aligned} \tag{2.2.9}$$

Using (2.2.8) and (2.2.9) in (2.2.7) we get:

$$\begin{aligned} &\beta\gamma\kappa^{p-2}\int_{\Omega_1} H(\nabla u)^2|u|^{\beta-1}\varphi^p + \beta\frac{\gamma\Gamma^{-1}}{p-1}\int_{\Omega_2} B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p \\ &\leq MpC_1\int_{\Omega_1} H(\nabla u)|\nabla\varphi||u|^{\beta}\varphi^{p-1} + MpC_2\int_{\Omega_2} H(\nabla u)^{p-1}|\nabla\varphi||u|^{\beta}\varphi^{p-1} + \int_{\Omega}|u|^{q+\beta}\varphi^p. \end{aligned} \tag{2.2.10}$$

Now we estimate the first two terms of the right side of (2.2.10). By the standard weighted Young's inequality we get:

$$\int_{\Omega_1} H(\nabla u)|\nabla\varphi||u|^{\beta}\varphi^{p-1} \leq \varepsilon^2\int_{\Omega_1} H(\nabla u)^2|u|^{\beta-1}\varphi^p + \bar{C}''_{\varepsilon}\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2} \tag{2.2.11}$$

where $\bar{C}''_{\varepsilon}$ is a positive constant depending on ε.

Using again the weighted Young's inequality with exponents p and $\frac{p}{p-1}$ and recalling that

$$\gamma(p-1)H(\nabla u)^p \leq B''(H(\nabla u))H(\nabla u)^2 \quad \text{in } \Omega \tag{2.2.12}$$

we estimate the second term of (2.2.10):

$$\begin{aligned} &\int_{\Omega_2} H(\nabla u)^{p-1}|\nabla\varphi||u|^{\beta}\varphi^{p-1} \\ &\leq \varepsilon^2\gamma(p-1)\int_{\Omega_2} H(\nabla u)^p|u|^{\beta-1}\varphi^p + \bar{C}'_{\varepsilon}\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1} \\ &\leq \varepsilon^2\int_{\Omega_2} B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p + \bar{C}'_{\varepsilon}\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1}, \end{aligned} \tag{2.2.13}$$

where $\bar{C}'_{\varepsilon}$ is a positive constant depending on ε. By (2.2.13), (2.2.11) and (2.2.10) we have

$$\begin{aligned} &\min\left\{\beta\gamma\kappa^{p-2}-\varepsilon^2, \frac{\beta\gamma\Gamma^{-1}}{p-1}-\varepsilon^2\right\}\cdot \\ &\cdot\left(\int_{\Omega_1} H(\nabla u)^2|u|^{\beta-1}\varphi^p + \int_{\Omega_2} B''(H(\nabla u)H(\nabla u)^2|u|^{\beta-1}\varphi^p\right) \\ &\leq \bar{C}\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2} + \bar{C}\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1} + \int_{\Omega}|u|^{q+\beta}\varphi^p \end{aligned} \tag{2.2.14}$$

for any non negative $\varphi \in C^1_c(\Omega)$. Thus we have proved (2.2.5).

Step 2. *Let us fix $\delta, \epsilon > 0$ and set*

$$\alpha = \alpha(\epsilon,\delta) = q - \frac{\max\left\{\left(\frac{\beta+1}{2}\right)^2\Gamma(\kappa+\delta)^{p-2}+\epsilon^2, \left(\frac{\beta+1}{2}\right)^2+\epsilon^2\right\}}{\min\left\{\beta\gamma\kappa^{p-2}-\epsilon^2, \frac{\beta\gamma\Gamma^{-1}}{p-1}-\epsilon^2\right\}}. \tag{2.2.15}$$

Then there exist a positive constant $\tilde{C} = \tilde{C}(B, H, \delta, \epsilon, \beta)$ such that

$$\alpha \int_\Omega |u|^{q+\beta}\varphi^p \leq \tilde{C} \int_{\Omega_1} |\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2} + \tilde{C} \int_{\Omega_2} |\nabla\varphi|^p|u|^{\beta+p-1} \tag{2.2.16}$$

for any non-negative $\varphi \in C^1_c(\Omega)$.

To prove the claim, let us consider

$$\psi = |u|^{\frac{\beta-1}{2}} u\varphi^{\frac{p}{2}} \in C^1_c(\Omega)\,,$$

so that $\nabla\psi = \left(\frac{\beta+1}{2}\right)\varphi^{\frac{p}{2}}|u|^{\frac{\beta-1}{2}}\nabla u + \frac{p}{2}|u|^{\frac{\beta-1}{2}}u\varphi^{\frac{p}{2}-1}\nabla\varphi$. Recalling (1.2.9), we have

$$\int_\Omega B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u, \nabla\psi\rangle = \int_\Omega B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\psi, \nabla u\rangle = 0 \tag{2.2.17}$$

and therefore substituting ψ in the stability condition (2.1.5) and recalling the Euler's theorem (1.2.4) we obtain:

$$\begin{aligned} q\int_\Omega |u|^{q+\beta}\varphi^p \;\leq\; & \left(\frac{\beta+1}{2}\right)^2 \int_\Omega B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p \\ &+ \frac{p^2}{4}\int_\Omega B''(H(\nabla u))|\nabla H(\nabla u)|^2|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2} \\ &+ \left(\frac{\beta+1}{2}\right)p\int_\Omega B''(H(\nabla u))H(\nabla u)|\nabla\varphi||\nabla H(\nabla u)||u|^\beta\varphi^{p-1} \\ &+ \frac{p^2}{4}\int_\Omega B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla\varphi\rangle|u|^{\beta+1}\varphi^{p-2} \\ =\; & I_1 + I_2 + I_3 + I_4\,, \end{aligned} \tag{2.2.18}$$

By (2.2.8) and recalling that there exists $M > 0$ such that $|\nabla H(\xi)| \leq M$ for every $\xi \in \mathbb{R}^N \setminus \{0\}$, we have

$$I_2 \leq \frac{p^2}{4}M^2\overline{C}_1\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2} + \frac{p^2}{4}M^2\overline{C}_2\int_{\Omega_2}H(\nabla u)^{p-2}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2} \tag{2.2.19}$$

For[1] $p > 2$, we use Young's inequality with exponents $\frac{p}{2}$ and $\frac{p}{p-2}$ and (2.2.12) to get:

$$\begin{aligned} I_2 \leq \frac{p^2}{4}M^2\overline{C}_1\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2} &+ \frac{\varepsilon^2}{3}\int_{\Omega_2}B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p \\ &+ \tilde{C}_2\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1}\,, \end{aligned} \tag{2.2.20}$$

where $\tilde{C}_2 = \tilde{C}_2(B, H, \delta, \varepsilon)$ is a positive constant.

Now we estimate I_4. By (1.2.7) there exists $M' > 0$ such that $|D^2H(\xi)| \leq \frac{M'}{H(\xi)}$. By

[1] In the case $p = 2$, the estimate (2.2.20) follows immediately. Indeed in this case we do not need to use Young's inequality.

(2.2.8), (2.2.12) and Young's inequality with suitable exponents, we have

$$
\begin{aligned}
I_4 &\le \frac{p^2}{4}M'\int_\Omega \frac{B'(H(\nabla u))}{H(\nabla u)}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2}\\
&\le \frac{p^2}{4}M'C_1\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2}+\frac{p^2}{4}M'C_2\int_{\Omega_2}H(\nabla u)^{p-2}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2}\\
&\le \frac{p^2}{4}M'C_1\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2}\\
&+\frac{\epsilon^2}{3}\int_{\Omega_2}B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p+\tilde{C}_4\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1}
\end{aligned}
\tag{2.2.21}
$$

where $\tilde{C}_4=\tilde{C}_4(B,H,\delta,\epsilon)$ is a positive constant.

Similarly, we have:

$$
\begin{aligned}
I_3 &\le \varepsilon^2\int_{\Omega_1}H(\nabla u)^2|u|^{\beta-1}\varphi^p+\tilde{C}_3\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2}\\
&+\frac{\varepsilon^2}{3}\int_{\Omega_2}B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p+\tilde{C}_3\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1}
\end{aligned}
\tag{2.2.22}
$$

for a positive constant $\tilde{C}_3=\tilde{C}_3(B,H,\delta,\varepsilon,\beta)$.

Plugging (2.2.20), (2.2.21), (2.2.22), into (2.2.18), we get

$$
\begin{aligned}
q\int_\Omega |u|^{q+\beta}\varphi^p \le &\left(\frac{\beta+1}{2}\right)^2\int_\Omega B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p\\
&+\epsilon^2\int_{\Omega_1}H(\nabla u)^2|u|^{\beta-1}\varphi^p+\epsilon^2\int_{\Omega_2}B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p\\
&+\tilde{C}_1\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2}+\tilde{C}_1\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1},
\end{aligned}
\tag{2.2.23}
$$

where $\tilde{C}_1=\tilde{C}_1(B,H,\delta,\varepsilon,\beta)$ is a positive constant.

Now using the third estimate of (2.2.8), recalling that $\overline{C}_1=\Gamma(p-1)(\kappa+\delta)^{p-2}$, we have

$$
\begin{aligned}
q\int_\Omega |u|^{q+\beta}\varphi^p \le &\max\left\{\left(\frac{\beta+1}{2}\right)^2\Gamma(p-1)(\kappa+\delta)^{p-2}+\varepsilon^2,\left(\frac{\beta+1}{2}\right)^2+\varepsilon^2\right\}\\
&\left(\int_{\Omega_1}H(\nabla u)^2|u|^{\beta-1}\varphi^p+\int_{\Omega_2}B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p\right)\\
&+\tilde{C}_1\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2}+\tilde{C}_1\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1}.
\end{aligned}
\tag{2.2.24}
$$

Plugging (2.2.5) into (2.2.24), we have

$$
\alpha\int_\Omega |u|^{q+\beta}\varphi^p\le \tilde{C}\int_{\Omega_1}|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2}+\tilde{C}\int_{\Omega_2}|\nabla\varphi|^p|u|^{\beta+p-1}
\tag{2.2.25}
$$

for any non-negative $\varphi\in C_c^1(\Omega)$.

We note that

$$
\lim_{\varepsilon,\delta\to 0}\alpha(\varepsilon,\delta)=q-\frac{\max\left\{\left(\frac{\beta+1}{2}\right)^2\Gamma(p-1)\kappa^{p-2},\left(\frac{\beta+1}{2}\right)^2\right\}}{\min\left\{\beta\gamma\kappa^{p-2},\beta\frac{\gamma\Gamma-1}{p-1}\right\}},
\tag{2.2.26}
$$

and, if $\Gamma\kappa^{p-2}(p-1) \geq 1$, we have

$$\left(\frac{\beta+1}{2}\right)^2 \Gamma(p-1)\kappa^{p-2} \geq \left(\frac{\beta+1}{2}\right)^2 \quad \text{and} \quad \beta\frac{\gamma\Gamma^{-1}}{p-1} \leq \beta\gamma\kappa^{p-2}. \tag{2.2.27}$$

On the other hand, if $\Gamma\kappa^{p-2}(p-1) \leq 1$, we have

$$\left(\frac{\beta+1}{2}\right)^2 \Gamma(p-1)\kappa^{p-2} \leq \left(\frac{\beta+1}{2}\right)^2 \quad \text{and} \quad \beta\frac{\gamma\Gamma^{-1}}{p-1} \geq \beta\gamma\kappa^{p-2}. \tag{2.2.28}$$

Now the fact that

$$q - \frac{\max\left\{\left(\frac{\beta+1}{2}\right)^2 \Gamma(p-1)\kappa^{p-2}, \left(\frac{\beta+1}{2}\right)^2\right\}}{\min\left\{\beta\gamma\kappa^{p-2}, \beta\frac{\gamma\Gamma^{-1}}{p-1}\right\}} > 0$$

follows by the assumptions:

$$\beta \in \left[1, \frac{2q - \kappa^{-p+2}\gamma^{-1} + 2\sqrt{q(q-\kappa^{-p+2}\gamma^{-1})}}{\kappa^{-p+2}\gamma^{-1}}\right) \quad \text{if} \quad 0 < \Gamma\kappa^{p-2}(p-1) < 1\,,$$

$$\beta \in \left[1, \frac{2q - \Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2 + 2\sqrt{q(q-\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2)}}{\Gamma^2\kappa^{p-2}\gamma^{-1}(p-1)^2}\right) \quad \text{if}\ \Gamma\kappa^{p-2}(p-1) \geq 1.$$

Step 3. *End of the proof.* Set $\tilde{\varphi} = \varphi^l$, with $\varphi \in C_c^1(\Omega)$, $0 \leq \varphi \leq 1$ and let l be a positive integer. Substituting $\tilde{\varphi}$ in (2.2.16), we obtain:

$$\int_\Omega |u|^{q+\beta}\varphi^{pl} \leq \tilde{C}l^2 \int_{\Omega_1} |\nabla\varphi|^2|u|^{\beta+1}\varphi^{lp-2} + \tilde{C}l^p \int_{\Omega_2} |\nabla\varphi|^p|u|^{\beta+p-1}\varphi^{lp-p}\,. \tag{2.2.29}$$

Using Holder's inequality with exponents $\frac{q+\beta}{\beta+1}$ and $\frac{q+\beta}{q-1}$ in the integral over Ω_1, and exponents $\frac{q+\beta}{\beta+p-1}$ and $\frac{q+\beta}{q-p+1}$ in the integral over Ω_2 we get:

$$\begin{aligned}\int_\Omega |u|^{q+\beta}\varphi^{lp} \leq& \tilde{C}l^2\left(\int_{\Omega_1}\left(|u|^{\beta+1}\varphi^{lp-2}\right)^{\frac{q+\beta}{\beta+1}}\right)^{\frac{\beta+1}{q+\beta}}\left(\int_{\Omega_1}|\nabla\varphi|^{2\frac{q+\beta}{q-1}}\right)^{\frac{q-1}{q+\beta}} \\ &+\tilde{C}l^p\left(\int_{\Omega_2}\left(|u|^{\beta+p-1}\varphi^{lp-p}\right)^{\frac{q+\beta}{\beta+p-1}}\right)^{\frac{\beta+p-1}{q+\beta}}\left(\int_{\Omega_2}|\nabla\varphi|^{p\frac{q+\beta}{q+p-1}}\right)^{\frac{q-p+1}{q+\beta}}\,.\end{aligned} \tag{2.2.30}$$

By (2.2.2) we deduce that:

$$(lp-2)\left(\frac{q+\beta}{\beta+1}\right) \geq (lp-p)\left(\frac{q+\beta}{\beta+p-1}\right) \geq lp \tag{2.2.31}$$

Moreover, recalling (2.2.31), and since $0 \leq \varphi \leq 1$, (2.2.30) becomes:

$$\begin{aligned}\int_\Omega |u|^{q+\beta}\varphi^{lp} \leq& \tilde{C}l^2\left(\int_{\Omega_1}|u|^{q+\beta}\varphi^{lp}\right)^{\frac{\beta+1}{q+\beta}}\left(\int_{\Omega_1}|\nabla\varphi|^{2\frac{q+\beta}{q-1}}\right)^{\frac{q-1}{q+\beta}} \\ &+\tilde{C}l^p\left(\int_{\Omega_2}|u|^{q+\beta}\varphi^{lp}\right)^{\frac{\beta+p-1}{q+\beta}}\left(\int_{\Omega_2}|\nabla\varphi|^{p\frac{q+\beta}{q-p+1}}\right)^{\frac{q-p+1}{q+\beta}}\,.\end{aligned} \tag{2.2.32}$$

Multiplying both sides of (2.2.32) by $\left(\int_{\Omega_1}|u|^{q+\beta}\varphi^{lp}\right)^{-\frac{\beta+1}{q+\beta}}$, we get:

$$\begin{aligned}\left(\int_\Omega |u|^{q+\beta}\varphi^{lp}\right)^{\frac{q-1}{q+\beta}} &\le \tilde{C}l^2\left(\int_{\Omega_1}|\nabla\varphi|^{2\frac{q+\beta}{q-1}}\right)^{\frac{q-1}{q+\beta}}\\ &+\tilde{C}l^p\left(\int_\Omega |u|^{q+\beta}\varphi^{lp}\right)^{\frac{p-2}{q+\beta}}\left(\int_{\Omega_2}|\nabla\varphi|^{p\frac{q+\beta}{q-p+1}}\right)^{\frac{q-p+1}{q+\beta}}.\end{aligned} \tag{2.2.33}$$

Raising (2.2.33) to the exponent $\frac{q+\beta}{q-1}$, we get

$$\int_\Omega |u|^{q+\beta}\varphi^{pl}\le C'\int_{\Omega_1}|\nabla\varphi|^{2\frac{q+\beta}{q-1}}+C'\left(\int_\Omega |u|^{q+\beta}\varphi^{pl}\right)^{\frac{p-2}{q-1}}\left(\int_{\Omega_2}|\nabla\varphi|^{p\frac{q+\beta}{q-p+1}}\right)^{\frac{q-p+1}{q-1}} \tag{2.2.34}$$

where $C'=C'(B,H,\varepsilon,\delta,q,\beta)$ is a positive constant. Using a weighted Young inequality with exponent $\frac{q-1}{p-2}$ and $\frac{q-1}{q-p+1}$ we have

$$\int_\Omega |u|^{q+\beta}\varphi^{pl}\le C\int_{\Omega_1}|\nabla\varphi|^{2\frac{q+\beta}{q-1}}+C\int_{\Omega_2}|\nabla\varphi|^{p\frac{q+\beta}{q-p+1}} \tag{2.2.35}$$

where $C=C(B,H,\delta,\varepsilon,q,\beta)$ is a positive constant. □

The case $\kappa=0$ is simpler. Indeed, we do not need to split Ω into Ω_1 and Ω_2.

Proposition 2.2.3. *Let us assume that (h_H) and (h_B), with $\kappa=0$ in (h_B). Let u be a weak stable solution of (2.1.1) with $p\ge 2$ and $q>\Gamma\gamma^{-1}(p-1)$. Then, for any*

$$\beta\in\left[1,\frac{2q-\Gamma\gamma^{-1}(p-1)+2\sqrt{q(q-\Gamma\gamma^{-1}(p-1))}}{\Gamma\gamma^{-1}(p-1)}\right) \tag{2.2.36}$$

and any integer

$$l>\max\left\{\frac{q+\beta}{q-(p-1)},2\right\} \tag{2.2.37}$$

there exist a positive constant $C=C(B,H,\varepsilon,q,\beta)$ such that

$$\int_\Omega B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^{lp}+|u|^{q+\beta}\varphi^{lp}\le C\int_\Omega |\nabla\varphi|^{p\frac{q+\beta}{q-(p-1)}} \tag{2.2.38}$$

for every $\varphi\in C^1_c(\Omega)$, with $0\le\varphi\le 1$.

Proof. **Step 1.** *For any $\beta\ge 1$ and $\epsilon>0$ small enough there exist a positive constant $\overline{C}=\overline{C}(B,H,\delta,\epsilon)>0$ such that*

$$\left(\frac{\beta\gamma\Gamma^{-1}}{p-1}-\epsilon^2\right)\int_\Omega B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p\le\overline{C}\int_\Omega|\nabla\varphi|^p|u|^{\beta+p-1}+\int_\Omega|u|^{q+\beta}\varphi^p \tag{2.2.39}$$

for any non negative $\varphi\in C^1_c(\Omega)$.

As in the proof of the Proposition 2.2.1, let us consider $\phi=|u|^{\beta-1}u\varphi^p\in C^1_c(\Omega)$, so that

$$\nabla\phi=p|u|^{\beta-1}u\varphi^{p-1}\nabla\varphi+\beta|u|^{\beta-1}\varphi^p\nabla u.$$

Using ϕ as test function in (2.1.4), we get

$$
\begin{aligned}
&\beta \int_\Omega B'(H(\nabla u))H(\nabla u)|u|^{\beta-1}\varphi^p \\
&\le Mp \int_\Omega B'(H(\nabla u))|\nabla\varphi||u|^\beta\varphi^{p-1} + \int_\Omega |u|^{q+\beta}\varphi^p .
\end{aligned} \tag{2.2.40}
$$

By $(h_B)-(iii)$, we deduce that

$$
\frac{\gamma\Gamma^{-1}}{p-1}B''(H(\nabla u))H(\nabla u) \le B'(H(\nabla u)) \le \Gamma H(\nabla u)^{p-1}. \tag{2.2.41}
$$

By (2.2.41) and Young's inequality with exponent p and $\frac{p}{p-1}$, (2.2.40) becomes

$$
\begin{aligned}
&\beta\frac{\gamma\Gamma^{-1}}{p-1}\int_\Omega B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p \\
&\le \beta\int_\Omega B'(H(\nabla u))H(\nabla u)|u|^{\beta-1}\varphi^p \\
&\le Mp\int_\Omega B'(H(\nabla u))|\nabla\varphi||u|^\beta\varphi^{p-1} + \int_\Omega |u|^{q+\beta}\varphi^p \\
&\le \epsilon^2\gamma(p-1)\int_\Omega H(\nabla u)^p|u|^{\beta-1}\varphi^p + \overline{C}\int_\Omega |\nabla\varphi|^p|u|^{\beta+p-1} + \int_\Omega |u|^{q+\beta}\varphi^p \\
&\le \varepsilon^2\int_\Omega B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p + \overline{C}\int_\Omega |\nabla\varphi|^p|u|^{\beta+p-1} + \int_\Omega |u|^{q+\beta}\varphi^p
\end{aligned} \tag{2.2.42}
$$

where in the last inequality we used the fact that $(p-1)\gamma t^{p-2} \le B''(t)$.
Step 2. *Set*

$$
\alpha = \alpha(\varepsilon) = q - \left(\frac{(\beta+1)^2}{4} + \varepsilon^2\right)\left(\frac{\beta\gamma}{\Gamma(p-1)} - \varepsilon^2\right)^{-1}. \tag{2.2.43}
$$

There exist a positive constant $\tilde{C} = \tilde{C}(B, H, \delta, \varepsilon, \beta)$ such that

$$
\alpha\int_\Omega |u|^{q+\beta}\varphi^p \le \tilde{C}\int_\Omega |\nabla\varphi|^p|u|^{\beta+p-1} \tag{2.2.44}
$$

for any non-negative $\varphi \in C^1_c(\Omega)$.

To prove the claim, let us consider

$$
\psi = |u|^{\frac{\beta-1}{2}}u\varphi^{\frac{p}{2}} \in C^1_c(\Omega)\,,
$$

so that $\nabla\psi = \left(\frac{\beta+1}{2}\right)\varphi^{\frac{p}{2}}|u|^{\frac{\beta-1}{2}}\nabla u + \frac{p}{2}|u|^{\frac{\beta-1}{2}}u\varphi^{\frac{p}{2}-1}\nabla\varphi$. Substituting ψ in (2.1.5) and recalling (1.2.9) we obtain:

$$
\begin{aligned}
q\int_\Omega |u|^{q+\beta}\varphi^p \;\le\;& \left(\frac{\beta+1}{2}\right)^2\int_\Omega B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^p \\
&+ \frac{p^2}{4}\int_\Omega B''(H(\nabla u))|\nabla\varphi|^2|u|^{\beta+1}\varphi^{p-2}|\nabla H(\nabla u)|^2 \\
&+ \left(\frac{\beta+1}{2}\right)p\int_\Omega B''(H(\nabla u))H(\nabla u)|\nabla\varphi||\nabla H(\nabla u)||u|^\beta\varphi^{p-1} \\
&+ \frac{p^2}{4}\int_\Omega B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi,\nabla\varphi\rangle|u|^{\beta+1}\varphi^{p-2} \\
=\;& I_1 + I_2 + I_3 + I_4\,,
\end{aligned} \tag{2.2.45}
$$

Now, if $p > 2$, by (h_B) and Young's inequality with exponents $\frac{p}{2}$ and $\frac{p}{p-2}$ and we get

$$I_2 \leq \frac{\varepsilon^2}{3} \int_\Omega B''(H(\nabla u))H(\nabla u)^2 |u|^{\beta-1}\varphi^p + \tilde{C}_2 \int_\Omega |\nabla\varphi|^p |u|^{\beta+p-1}\,, \tag{2.2.46}$$

where $\tilde{C}_2 = \tilde{C}_2(B, H, \varepsilon)$ is a positive constant. For $p = 2$, inequality (2.2.46) is straightforward.

Now we estimate I_4. We remark that there exist a positive constant M' such that $|D^2H(\xi)| \leq \frac{M'}{H(\xi)}$. Using this fact, (2.2.41) and Young's inequality with exponent p and $\frac{p}{p-1}$ we get

$$I_4 \leq \frac{\varepsilon^2}{3} \int_\Omega B''(H(\nabla u))H(\nabla u)^2 |u|^{\beta-1}\varphi^p + \tilde{C}_4 \int_\Omega |\nabla\varphi|^p |u|^{\beta+p-1}\,, \tag{2.2.47}$$

with constant $\tilde{C}_4 = \tilde{C}_4(B, H, \epsilon) > 0$.

Similarly, we have

$$I_3 \leq \frac{\varepsilon^2}{3} \int_\Omega B''(H(\nabla u))H(\nabla u)^2 |u|^{\beta-1}\varphi^p + \tilde{C}_3 \int_\Omega |\nabla\varphi|^p |u|^{\beta+p-1}\,, \tag{2.2.48}$$

with constant $\tilde{C}_3 = \tilde{C}_3(B, H, \epsilon, \beta) > 0$.

Plugging (2.2.46), (2.2.47), and (2.2.48) into (2.2.45) we have

$$q \int_\Omega |u|^{q+\beta}\varphi^p \leq \left(\frac{(\beta+1)^2}{4} - \varepsilon^2\right) \int_\Omega B''(H(\nabla u))H(\nabla u)^2 |u|^{\beta-1}\varphi^p + \tilde{C}_1 \int_\Omega |\nabla\varphi|^p |u|^{\beta+p-1} \tag{2.2.49}$$

with $\tilde{C}_1 := \tilde{C}_2 + \tilde{C}_3 + \tilde{C}_4$. Plugging (2.2.39) into (2.2.49) we deduce

$$\alpha \int_\Omega |u|^{q+\beta}\varphi^p \leq \tilde{C} \int_\Omega |\nabla\varphi|^p |u|^{\beta+p-1} \tag{2.2.50}$$

where $\tilde{C} = \tilde{C}(B, H, \varepsilon, \beta)$ is a positive constant.

Since

$$\lim_{\epsilon\to 0} \alpha(\varepsilon) = q - \left(\frac{(\beta+1)^2}{4}\right)\left(\frac{\beta\gamma}{\Gamma(p-1)}\right)^{-1} \tag{2.2.51}$$

and

$$\beta \in \left[1, \frac{2q - \Gamma\gamma^{-1}(p-1) + 2\sqrt{q(q - \Gamma\gamma^{-1}(p-1))}}{\Gamma\gamma^{-1}(p-1)}\right) \tag{2.2.52}$$

we deduce that $q - \left(\frac{(\beta+1)^2}{4}\right)\left(\frac{\beta\gamma}{\Gamma(p-1)}\right)^{-1} > 0$.

Finally we observe that, (2.2.39) and (2.2.44) imply

$$\int_\Omega B''(H(\nabla u))H(\nabla u)^2 |u|^{\beta-1}\varphi^p \leq \hat{C} \int_\Omega |\nabla\varphi|^p |u|^{\beta+p-1} \tag{2.2.53}$$

where $\hat{C} = \hat{C}(B, H, \epsilon, \beta)$ is a positive constant.

Step 3. *End of the proof.* Set $\tilde{\varphi} = \varphi^l$, with $\varphi \in C_c^1(\Omega)$, $0 \leq \varphi \leq 1$ and let l be a positive integer. Substituting $\tilde{\varphi}$ in (2.2.44) we get

$$\begin{aligned}\int_\Omega |u|^{q+\beta}\varphi^{pl} &\leq \tilde{C}l^p \int_\Omega |\nabla\varphi|^p|u|^{\beta+p-1}\varphi^{lp-p} \\ &\leq \tilde{C}l^p \left(\int_\Omega (|u|^{p+\beta-1}\varphi^{lp-p})^{\frac{q+\beta}{\beta+p-1}}\right)^{\frac{\beta+p-1}{q+\beta}} \left(\int_\Omega |\nabla\varphi|^{p\frac{q+\beta}{q-p+1}}\right)^{\frac{q-p+1}{q+\beta}}.\end{aligned} \tag{2.2.54}$$

Now, by (2.2.37), we note that $(lp-p)\left(\frac{q+\beta}{\beta+p-1}\right) \geq lp$, and for $0 \leq \varphi \leq 1$, (2.2.54) becomes

$$\int_\Omega |u|^{q+\beta}\varphi^{pl} \leq \tilde{C}l^p \left(\int_\Omega |u|^{q+\beta}\varphi^{lp}\right)^{\frac{\beta+p-1}{q+\beta}} \left(\int_\Omega |\nabla\varphi|^{p\frac{q+\beta}{q-p+1}}\right)^{\frac{q-p+1}{q+\beta}} \tag{2.2.55}$$

and consequently

$$\int_\Omega |u|^{q+\beta}\varphi^{pl} \leq \tilde{C}l^p \int_\Omega |\nabla\varphi|^{p\frac{q+\beta}{q-p+1}}. \tag{2.2.56}$$

Now, substituting $\tilde{\varphi}$ in (2.2.53) we obtain

$$\begin{aligned}&\int_\Omega B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^{pl} \\ &\leq \hat{C}l^p \int_\Omega |\nabla\varphi|^p|u|^{\beta+p-1}\varphi^{pl-p} \\ &\leq \hat{C}l^p \left(\int_\Omega (|u|^{p+\beta-1}\varphi^{lp-p})^{\frac{q+\beta}{\beta+p-1}}\right)^{\frac{\beta+p-1}{q+\beta}} \left(\int_\Omega |\nabla\varphi|^{p\frac{q+\beta}{q-p+1}}\right)^{\frac{q-p+1}{q+\beta}}\end{aligned} \tag{2.2.57}$$

and using (2.2.56), we get

$$\int_\Omega B''(H(\nabla u))H(\nabla u)^2|u|^{\beta-1}\varphi^{pl} \leq \hat{C}' \int_\Omega |\nabla\varphi|^{p\frac{q+\beta}{q-p+1}} \tag{2.2.58}$$

where $\hat{C}' = \hat{C}'(\Gamma, \gamma, p, \epsilon, \beta)$ is a positive constant.
The conclusion follows now by (2.2.56) and (2.2.58).

□

We are now ready to prove our main result.

Proof of Theorem 2.1.3. For $R > 0$, let $\varphi_R \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that

$$\begin{cases} 0 \leq \varphi_R \leq 1 & \text{in } \mathbb{R}^N \\ \varphi_R \equiv 0 & \text{in } \mathbb{R}^N \setminus B(0,2R) \\ \varphi_R \equiv 1 & \text{in } B(0,R) \\ |\nabla\varphi_R| \leq \frac{2}{R} & \text{in } B(0,2R) \setminus B(0,R), \end{cases} \tag{2.2.59}$$

where $B(0,R)$ denotes the open ball centered at the origin and with radius R.
Case (i): $\Gamma\kappa^{p-2}(p-1) \geq 1$. From Proposition 2.2.1 we have:

$$\int_{\mathbb{R}^N} |u|^{q+\beta}\varphi_R^{pl} \leq CR^{N-2\frac{q+\beta}{q-1}} + CR^{N-p\frac{q+\beta}{q-p+1}}. \tag{2.2.60}$$

Next we prove that $N - 2\frac{q+\beta}{q-1} < 0$.

To this end, we denote by $\zeta := \Gamma^2\kappa^{p-2}\gamma^{-1}$ and for $t > \zeta(p-1)^2$, we set

$$\tilde{\beta}(t) := \frac{2t - \zeta(p-1)^2 + 2\sqrt{t(t-\zeta(p-1)^2)}}{\zeta(p-1)^2} \quad \text{and} \quad g(t) := 2\frac{t+\tilde{\beta}(t)}{t-1}. \tag{2.2.61}$$

Therefore if $N < g(q)$, we can always choose $\beta \in \left[1, \tilde{\beta}(q)\right)$ such that

$$N - 2\frac{q+\beta}{q-1} < 0 \tag{2.2.62}$$

Observing that

$$N - 2\frac{q+\beta}{q-1} > N - p\frac{q+\beta}{q-p+1}, \tag{2.2.63}$$

the thesis follows by letting $R \to \infty$ in (2.2.60).
Case (ii): $0 < \Gamma\kappa^{p-2}(p-1) < 1$. The proof is similar to the previous case but uses (2.2.4), that is

$$\tilde{\beta}(t) = \frac{2t - \eta + 2\sqrt{t(t-\eta)}}{\eta} \tag{2.2.64}$$

where $\eta := \kappa^{-p+2}\gamma^{-1}$.
Case (iii): $\kappa = 0$. From Proposition 2.2.38 we get

$$\int_{\mathbb{R}^N} B''(H(\nabla u))H(\nabla u)|u|^{\beta-1}\varphi_R^{lp} + |u|^{q+\beta}\varphi_R^{lp} \leq CR^{N-p\frac{q+\beta}{q-(p-1)}}. \tag{2.2.65}$$

As before, to conclude the proof it is enough to find β such that $N - p\frac{q+\beta}{q-(p-1)} < 0$. In this case we consider

$$\overline{\beta}(t) := \frac{2t - \theta(p-1) + 2\sqrt{t(t-\theta(p-1))}}{\theta(p-1)}, \tag{2.2.66}$$

where we have set $\theta := \Gamma\gamma^{-1}$. The proof can be concluded now as in the previous two cases.

$\square$

Proof of Corollary 2.1.4. The proof follows by case (iii) of Theorem 2.1.3.

$\square$

Proof of Theorem 2.1.6. Let us fix $R_0 > 0$ so that $K \subset B(0, R_0)$. For $R > R_0 + 3$, let $\psi_R \in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that

$$\begin{cases} \psi_R \equiv 0 & \text{if } |x| < R_0 + 1 \\ \psi_R \equiv 1 & \text{if } R_0 + 2 < |x| < R \\ \psi_R \equiv 0 & \text{if } |x| > 2R \\ |\nabla\psi_R| \leq C & \text{in } B(0, R_0+2) \setminus B(0, R_0+1) \\ |\nabla\psi_R| \leq \frac{C}{R} & \text{in } B(0, 2R) \setminus B(0, R). \end{cases} \tag{2.2.67}$$

From Proposition 2.2.3, substituting ψ_R in (2.2.38) we get

$$\int_{R_0+2<|x|<R} B''(H(\nabla u))H(\nabla u)^2 u^{\beta-1} + |u|^{q+\beta} \leq C + CR^{N-p\frac{q+\beta}{q-(p-1)}}. \tag{2.2.68}$$

Since $\frac{N\left(p-\frac{\Gamma}{\gamma}\right)+p}{N\frac{\Gamma}{\gamma}-p} \leq \frac{N(p-1)+p}{N-p}$ by (2.1.9), choosing $\beta = 1$ it follows

$$N - p\frac{q+\beta}{q-(p-1)} < 0. \tag{2.2.69}$$

For $R \to \infty$, recalling that $B(t) \leq C(p,\gamma,\Gamma)B''(t)t^2$ we have

$$B(H(\nabla u)) \in L^1(\mathbb{R}^N), \qquad u \in L^{q+1}(\mathbb{R}^N). \tag{2.2.70}$$

Now, using the Pohozaev identity in the anisotropic setting, see Theorem 1.3 in [125] for details, we get

$$\frac{N}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} = \int_{\mathbb{R}^N} NB(H(\nabla u)) - B'(H(\nabla u))H(\nabla u). \tag{2.2.71}$$

By (h_B) we get that

$$B(t) \leq \Gamma\frac{t^p}{p} \leq \frac{\Gamma}{\gamma}\frac{B'(t)t}{p}, \qquad \forall t \geq 0.$$

Using this inequality in (2.2.71), we have

$$\frac{N}{q+1}\int_{\mathbb{R}^N} |u|^{q+1} \leq \left(\frac{N\Gamma}{p\gamma} - 1\right)\int_{\mathbb{R}^N} B'(H(\nabla u))H(\nabla u). \tag{2.2.72}$$

Now if $N \leq p\Gamma^{-1}\gamma$, the conclusion follows immediately. Suppose $N > p\Gamma^{-1}\gamma$ and let us consider the test function $\phi = \varphi_R u$, where φ_R is defined in (2.2.59). Substituting ϕ in (2.1.4), we get

$$\begin{aligned}\int_{B(0,2R)} B'(H(\nabla u))H(\nabla u)\varphi_R + \int_{B(0,2R)} B'(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi_R\rangle u\,dx \\ = \int_{B(0,2R)} |u|^{q+1}\varphi_R\,dx.\end{aligned} \tag{2.2.73}$$

Using Holder inequality with exponents $\left(\frac{p}{p-1}, q+1, \frac{q+1-p}{p(q+1)}\right)$ we obtain

$$\begin{aligned}&\left|\int_{B(0,2R)} B'(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi_R\rangle u\,dx\right| \\ &\leq C\left(\int_{\mathbb{R}^N} H(\nabla u)^p\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N} |u|^{q+1}\right)^{\frac{1}{q+1}} R^{\frac{N(q+1-p)}{p(q+1)}-1}.\end{aligned} \tag{2.2.74}$$

Since (2.2.70) is in force, and $\frac{N(q+1-p)}{p(q+1)} - 1 < 0$ by (2.2.69), we can let $R \to +\infty$ in (2.2.73) and (2.2.74), to get

$$\int_{\mathbb{R}^N} |u|^{q+1} = \int_{\mathbb{R}^N} B'(H(\nabla u))H(\nabla u). \tag{2.2.75}$$

Combining (2.2.72) and (2.2.75) we get

$$\left(\frac{N}{q+1} - \frac{N\Gamma}{p\gamma} + 1\right)\int_{\mathbb{R}^N} |u|^{q+1} \leq 0. \tag{2.2.76}$$

Recalling (2.1.9)

$$\frac{N}{q+1} - \frac{N\Gamma}{p\gamma} + 1 > 0$$

and therefore we get the thesis.

□

Chapter 3

On a Poincaré type formula for stable solutions of quasilinear anisotropic elliptic equations

3.1 Introduction

In this chapter, we aim to establish rigidity results for weak solutions $u \in C^1(\mathbb{R}^N)$ of the following equation

$$- \operatorname{div} \left(B'(H(\nabla u))\nabla H(\nabla u) \right) = f(u) \quad \text{on } \mathbb{R}^n. \tag{3.1.1}$$

As usual the weak formulation of (3.1.1) is given by

$$\int_{\mathbb{R}^N} B'(H(\nabla u))\langle \nabla H(\nabla u), \nabla \varphi \rangle = \int_{\mathbb{R}^N} f(u)\varphi, \quad \varphi \in C^1_c(\mathbb{R}^N). \tag{3.1.2}$$

Hereafter, $H \in C^{3,\beta}_{loc}(\mathbb{R}^N \setminus \{0\})$ is a Finsler norm (see Section 1.2 for details), $f \in C^{1,\beta}_{loc}(0, +\infty)$ and B satisfies following assumptions

(h$_B$) (i) $B \in C^{3,\beta}_{loc}(0, +\infty) \cap C^1([0, \infty))$;

(ii) $B(0) = B'(0) = 0, \quad B(t), B'(t), B''(t) > 0 \quad \forall t \in (0, +\infty)$;

(iii) there exist $p > 1$, $\kappa \in [0, 1)$, $\gamma > 0$, $\Gamma > 0$ such that

$$\begin{aligned} \gamma(\kappa + t)^{p-2} &\le B'(t) \le \Gamma(\kappa + t)^{p-2} t \quad \forall\, t > 0 \\ \gamma(p-1)(\kappa + t)^{p-2} &\le B''(t) \le \Gamma(p-1)(\kappa + t)^{p-2} \quad \forall\, t > 0\,. \end{aligned} \tag{3.1.3}$$

In the isotropic framework, i.e. $H(\xi) = |\xi|$, and if $B(t) = \frac{t^p}{p}$, the operator at left-hand side of (3.1.1) boils down to the usual $p-$Laplace operator.

Note that under those assumptions the considered problem is well defined[1]. Under assumption (h_B)-(iii), by standard regularity results, see [67, 157, 52, 51], weak solutions of the equation (3.1.2) belong to $C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap C^2(\mathbb{R}^N \setminus Z_u)$, where Z_u is the critical set where the gradient of a solution vanishes.

We define the following matrix $A : \mathbb{R}^N \setminus \{0\} \to \operatorname{Mat}(N \times N)$ by

$$A_{ij}(\xi) := B''(H(\xi))\partial_i H(\xi)\partial_j H(\xi) + B'(H(\xi))\partial_{ij} H(\xi), \tag{3.1.4}$$

[1] The function $B \circ H$ is of class $C^1(\mathbb{R}^N)$, see [51, Lemma 2.3]

for $i, j = 1, ..., N$

For $k = 1, 2$, we set

$$\lambda_1(t) := B''(t), \quad \lambda_2(t) = \cdots = \lambda_N := \frac{B'(t)}{t}. \tag{3.1.5}$$

and, for any $t \in \mathbb{R}$,

$$F(t) := \int_0^t f(s)ds \tag{3.1.6}$$

and[2]

$$\Lambda_k(t) := \int_0^t \lambda_k(|s|)sds. \tag{3.1.7}$$

We remark that solutions of (3.1.2) are critical points of the functional

$$\int \Lambda_2(H(\nabla u)) - F(u)dx. \tag{3.1.8}$$

Now, we discuss the notion of stability for weak solutions to (3.1.1). This is a somewhat delicate issue due to the general nature of our framework. We address it in the following definition.

Definition 3.1.1. *Given a domain $\Omega \subset \mathbb{R}^N$, we say that a weak solution $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\{\nabla u \neq 0\})$ is stable if*

$$\begin{aligned} L_u(\varphi, \varphi) &= \int_\Omega \langle A(\nabla u)\nabla\varphi, \nabla\varphi\rangle - f'(u)\varphi^2 \\ &= \int_\Omega B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2 \\ &+ \int_\Omega B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla\varphi\rangle - f'(u)\varphi^2 \geq 0, \end{aligned} \tag{3.1.9}$$

for every $\varphi \in C^1_c(\Omega \setminus \{\nabla u = 0\})$.

Remark 3.1.2. *We note that the quantities involved in (3.1.9) are well-defined since the support of the test function φ is far from Z_u. Moreover, in the case $1 < p \leq 2$, if $\{\nabla u = 0\} = \emptyset$, we can consider test functions $\varphi \in C^1_c(\Omega)$. For $p > 2$, test functions $\varphi \in C^1_c(\Omega)$ can be considered if $\kappa = 0$. In these cases the integral in (3.1.9) is well-defined.*

In the Euclidean framework and in the case of $B(t) = 1$ and $f(t) = t - t^3$, the equation (3.1.1) becomes

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^N. \tag{3.1.10}$$

In [64], De Giorgi proposed the following problem:

Suppose that $u \in C^2(\mathbb{R}^N; [-1, 1])$ is a solution of (3.1.10), satisfying

$$\partial_{x_N} u > 0 \quad \textit{for any } x \in \mathbb{R}^N.$$

Is it true that u has a one dimensional symmetry, at least for $N \leq 8$?

The first result was obtained for $N = 2$ in [22, 99], and, subsequently, the case $N = 3$ was proved in [3, 5]. The conjecture fails for $N \geq 9$, see [65].

[2] Under assumptions (h_B), we note that $B'(t), B''(t)t \in C([0, +\infty))$. Therefore, the quantities Λ_k are well-defined.

To the best of our knowledge, the question is still open when $4 \leq N \leq 8$, though a positive answer holds under the additional assumption that

$$\lim_{x_N \to \pm\infty} u(x', x_N) = \pm 1, \tag{3.1.11}$$

for any $x' \in \mathbb{R}^{N-1}$, due to [138].

In the case of p-Laplace operator and non-uniformly elliptic operators, one dimensional symmetry for solutions has also been dealt with in [32, 83, 93, 89, 90, 160]. For further results on this topic, see also [17, 23, 88, 75].

Inspired by the works of [82, 89], we establish one-dimensional symmetry results for stable solutions to quasilinear anisotropic elliptic equations in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Our first result is the following:

Theorem 3.1.3. *Let $N = 2$. Let $u \in C^1(\mathbb{R}^2)$ be a weak stable solution of (3.1.1), with $\nabla u \in L^\infty(\mathbb{R}^2)$. Then there exist $u_0 : \mathbb{R} \to \mathbb{R}$ and $w \in S^1$ such that $u(x) = u_0(w \cdot x)$ for any $x \in \mathbb{R}^2$.*

We remark that, in the case $1 < p < 2$, the result is new even in the Euclidean case, since, unlike in [89], we did not assume $\{\nabla u = 0\} = \emptyset$.

The second symmetry result is the following

Theorem 3.1.4. *Let $N = 3$. Let $u \in C^1(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$ be a weak solution of (3.1.1). Suppose that*

$$\partial_{x_3} u > 0. \tag{3.1.12}$$

Then u has one dimensional symmetry in the full space $\mathbb{R}^3$, namely there exist $u_0 : \mathbb{R} \to \mathbb{R}$ and $w \in S^2$ such that $u(x) = u_0(w \cdot x)$ for any $x \in \mathbb{R}^3$.

We remark that we will prove Theorem 3.1.4 under weaker assumptions, namely: $\partial_{x_3} u \geq 0$, $\{\nabla u = 0\} = \emptyset$. Furthermore, the condition (3.1.12) implies the stability condition in (3.1.9) (see Lemma 3.4.1 in Section 3.4).

The techniques used in the proofs of Theorem 3.1.3 and Theorem 3.1.4 are based on a Poincaré type inequality represent an anisotropic modification introduced in [89, 150, 151]. In this context, we will extend a formula by Sternberg and Zumbrun that relates the second derivatives of a weak solution of equation (3.1.1) to the principal curvatures of the corresponding level set in the anisotropic setting.

The chapter is organized as follows. In Section 3.2, we will state some lemmas that will be useful for the proof of Theorem 3.1.3, which concludes the section. Section 3.3 contains some general preliminaries, needed for the proof of the Theorem 3.1.4. In Section 3.4 we give the proof of the Theorem 3.1.4.

3.2 Proof of Theorem 3.1.3

The aim of this section is to prove Theorem 3.1.3. For this purpose, we will prove several lemmas that are useful for the proof of Theorem 3.1.3.

Recalling (3.1.4), we start this section recalling a result that will be useful to us in the sequel (see [52, Theorem 1.5]).

Lemma 3.2.1 ([52]). *For any $\xi \in \mathbb{R}^N \setminus \{0\}$ the following matrix*

$$A(\xi) = B''(H(\xi))\nabla H(\xi) \otimes \nabla H(\xi) + B'(H(\xi))D^2 H(\xi) \tag{3.2.1}$$

is positive definite. In particular, there exist a constant C such that

$$A_{ij}(\xi)\eta_i\eta_j \geq C(\kappa + |\xi|)^{p-2}|\eta|^2, \quad \forall \xi \in \mathbb{R}^N \setminus \{0\}, \quad \forall \eta \in \mathbb{R}^N. \tag{3.2.2}$$

Now we prove the following

Lemma 3.2.2. *Let Ω be a domain in $\mathbb{R}^N$. Let $u \in C^1(\mathbb{R}^N)$ be a weak solution of (3.1.1). Then, $u_i := \frac{\partial u}{\partial x_i}$ is a solution of*

$$\begin{aligned}&\int_{\Omega\setminus Z_u} B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla \varphi\rangle \\ &+ B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla\varphi\rangle = \int_\Omega f'(u)u_i\varphi \quad \forall \varphi \in C^1_c(\Omega),\end{aligned} \tag{3.2.3}$$

for any $i = 1, ..., N$.

Moreover we have

$$\begin{aligned}&\int_{\Omega\cap\{\nabla u\neq 0\}} \sum_{i=1}^N B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla(u_i\varphi^2)\rangle \\ &+ \sum_{i=1}^N B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla(u_i\varphi^2)\rangle\, dx \\ &= \int_{\Omega\cap\{\nabla u\neq 0\}} f'(u)|\nabla u|^2\varphi^2\, dx \quad \forall\varphi\in C^1_c(\Omega).\end{aligned} \tag{3.2.4}$$

Proof. We recall that $B'(H(\nabla u))\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega)$ (see [8, 35, 125]). Using Stampacchia's theorem (see [112, Theorem 6.19]) we deduce that the distributional derivative of $B'(H(\nabla u))\nabla H(\nabla u)$ is given by

$$\partial_i\left(B'(H(\nabla u))\nabla H(\nabla u)\right) = \begin{cases} B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle \nabla H(\nabla u) & \\ +B'(H(\nabla u))D^2H(\nabla u)\nabla u_i & \text{if } x \in \{\nabla u \neq 0\}. \\ 0 & \text{if } x\in\{\nabla u = 0\},\end{cases} \tag{3.2.5}$$

for any $i = 1, ..., N$. Taking $\psi := \partial_i\varphi$ in (3.1.2), by (3.2.5) and integrating by parts, we get the (3.2.3).

Now we prove (3.2.4). Now, fix $\epsilon > 0$ and, for $\varphi \in C^1_c(\Omega)$, let us consider

$$\psi := G_\varepsilon(u_i)\varphi^2 \tag{3.2.6}$$

with $G_\epsilon \in C^\infty(\mathbb{R})$ such that $G_\epsilon(t) = t$ if $|t| \geq 2\epsilon$, $G_\epsilon(t) = 0$ if $|t| \leq \epsilon$ and $G'_\epsilon(t) \leq 3$ for any $t \in \mathbb{R}$. Using $G_\varepsilon(u_i)\varphi^2 \in C^1_c(\Omega)$ in (3.2.3) we have

$$\begin{aligned}0 &= \int_\Omega B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla(G_\epsilon(u_i)\varphi^2)\rangle \\ &\quad + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla(G_\epsilon(u_i)\varphi^2)\rangle - f'(u)u_iG_\epsilon(u_i)\varphi^2\, dx \\ &= \int_\Omega B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla u_i\rangle G'_\varepsilon(u_i)\varphi^2 \\ &\quad + B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla(\varphi^2)\rangle G_\epsilon(u_i) \\ &\quad + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla u_i\rangle G'_\varepsilon(u_i)\varphi^2 \\ &\quad + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla(\varphi^2)\rangle G_\epsilon(u_i) - f'(u)u_iG_\epsilon(u_i)\varphi^2\, dx.\end{aligned} \tag{3.2.7}$$

Since that ∇H is a continuous and $0-$homogeneous function, we infer there exist $M > 0$ such that

$$|\nabla H(\xi)| \leq M \qquad \forall \xi \in \mathbb{R}^N \setminus \{0\}. \tag{3.2.8}$$

By (3.2.8), and since $B'(H(\nabla u))\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega)$, we note that

$$\begin{aligned} &B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle \langle \nabla H(\nabla u), \nabla u_i\rangle G'_\varepsilon(u_i)\varphi^2 \\ &\leq 3M^2 B''(H(\nabla u))|\nabla u_i|^2\varphi^2 \in L^1(\Omega). \end{aligned} \tag{3.2.9}$$

Recalling that D^2H is a continuous and $-1-$homogeneous function, there exist a positive constant M' such that $D^2H(\xi) \leq \frac{M'}{H(\xi)}$ for all $\xi \in \mathbb{R}^N \setminus \{0\}$. Similarly, we get

$$\begin{aligned} &B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla u_i\rangle G'_\epsilon(u_i)\varphi^2 \\ &\leq 3M' \frac{B'(H(\nabla u))}{H(\nabla u)}|\nabla u_i|^2\varphi^2 \in L^1(\Omega). \end{aligned} \tag{3.2.10}$$

Consequently, by (3.2.9) and (3.2.10), passing $\epsilon \to 0$ in (3.2.7), using dominated convergence theorem and summing over i we get

$$\begin{aligned} &\int_{\Omega\cap\{\nabla u\neq 0\}} \sum_{i=1}^N B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla(u_i\varphi^2)\rangle \\ &+ \sum_{i=1}^N B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla(u_i\varphi^2)\rangle\, dx = \int_{\Omega\cap\{\nabla u\neq 0\}} f'(u)|\nabla u|^2\varphi^2\, dx. \end{aligned} \tag{3.2.11}$$

□

To simplify future computations, we now state a key result that will be useful later.

Lemma 3.2.3. *Let Ω be a domain in $\mathbb{R}^N$. Let $u \in C^1(\mathbb{R}^N)$ be a stable weak solution of (3.1.1). Then, we have*

$$\begin{aligned} &\int_{\Omega\cap\{\nabla u\neq 0\}} B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle^2\varphi^2 + B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2|\nabla u|^2 \\ &+ B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle\langle \nabla H(\nabla u), \nabla\varphi\rangle 2|\nabla u|\varphi \\ &+ B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle\varphi^2 + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla\varphi\rangle|\nabla u|^2 \\ &+ B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla\varphi\rangle|\nabla u|\varphi + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla|\nabla u|\rangle|\nabla u|\varphi \\ &- f'(u)|\nabla u|^2\varphi^2\, dx \geq 0 \qquad \forall \varphi \in C^1_c(\Omega). \end{aligned} \tag{3.2.12}$$

Proof. For $\epsilon > 0$, we consider

$$\psi := G_\epsilon(|\nabla u|)\varphi \tag{3.2.13}$$

where $\varphi \in C^1_c(\Omega)$ and $G_\epsilon \in C^\infty(\mathbb{R})$ such that $G_\epsilon(t) = t$ if $|t| \geq 2\epsilon$, $G_\epsilon(t) = 0$ if $|t| \leq \epsilon$ and $G'_\epsilon(t) \leq 3$ for any $t \in \mathbb{R}$.

We observe that $G_\epsilon(|\nabla u|)\varphi \in C^1_c(\Omega \setminus Z_u)$ is a good test function in (3.1.9). Substituting (3.2.13) in (3.1.9) we get

$$\begin{aligned}&\int_\Omega B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\left(G_\epsilon(|\nabla u|)\varphi\right)\rangle^2\\&+B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\left(G_\epsilon(|\nabla u|)\varphi\right), \nabla\left(G_\epsilon(|\nabla u|)\varphi\right)\rangle - f'(u)G_\epsilon(|\nabla u|)^2\varphi^2\,dx\ge 0.\end{aligned}\tag{3.2.14}$$

Since $\nabla\psi = G'_\epsilon(|\nabla u|)\nabla|\nabla u|\varphi + G_\epsilon(|\nabla u|)\nabla\varphi$, (3.2.14) becomes

$$\begin{aligned}0\le\int_\Omega & B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle^2 G'_\epsilon(|\nabla u|)^2\varphi^2\\&+B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2 G_\epsilon(|\nabla u|)^2\\&+B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle\langle \nabla H(\nabla u), \nabla\varphi\rangle 2G_\epsilon(|\nabla u|)G'_\epsilon(|\nabla u|)\varphi\\&+B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle G'_\epsilon(|\nabla u|)^2\varphi^2\\&+B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla\varphi\rangle G_\epsilon(|\nabla u|)^2\\&+B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla\varphi\rangle G_\epsilon(|\nabla u|)G'_\epsilon(|\nabla u|)\varphi\\&+B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla|\nabla u|\rangle G_\epsilon(|\nabla u|)G'_\epsilon(|\nabla u|)\varphi - f'(u)G_\epsilon(|\nabla u|)^2\varphi^2\,dx.\end{aligned}\tag{3.2.15}$$

We note that, for any $m>0$ and $\Psi\in L^1(\Omega)$, we have

$$\lim_{\epsilon\to 0}\int_\Omega G'_\epsilon(|\nabla u|)^m\Psi\,dx - \int_{\Omega\cap\{\nabla u\neq 0\}}\Psi\,dx \le \lim_{\epsilon\to 0}\int_{\Omega\cap\{\nabla u\neq 0\}} 3^m|\Psi|\chi_{\epsilon\le|\nabla u|\le 2\epsilon}\,dx = 0.\tag{3.2.16}$$

We now let $\epsilon\to 0$ in (3.2.15), using (3.2.16) and the dominated convergence theorem, proceeding as in the proof of the Lemma 3.2.2 we conclude that

$$\begin{aligned}0\le\int_{\Omega\cap\{\nabla u\neq 0\}} & B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle^2\varphi^2 + B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2|\nabla u|^2\\&+B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle\langle \nabla H(\nabla u), \nabla\varphi\rangle 2|\nabla u|\varphi\\&+B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle\varphi^2 + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla\varphi\rangle|\nabla u|^2\\&+B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla\varphi\rangle|\nabla u|\varphi + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla|\nabla u|\rangle|\nabla u|\varphi\\&-f'(u)|\nabla u|^2\varphi^2\,dx.\end{aligned}\tag{3.2.17}$$

□

Now we recall the definition of the tangential gradient with respect to a regular level set. For $u\in C^1(\mathbb{R}^N)$, we define the level set of u at x as

$$L_{u,x} := \{y\in\mathbb{R}^N \text{ such that } u(y)=u(x)\}.\tag{3.2.18}$$

Given $f\in C^1(B_r(x))$, for $r>0$, the tangential gradient denotes the orthogonal projection of the gradient along this level set, explicitly

$$\nabla_{L_{u,x}}f(x) := \nabla f(x) - \langle\nabla f(x), \frac{\nabla u(x)}{|\nabla u(x)|}\rangle\frac{\nabla u(x)}{|\nabla u(x)|}.\tag{3.2.19}$$

In the sequel we give an extension, in the anisotropic framework, of a formula obtained in [150, 151].

Lemma 3.2.4. *Let $\Omega \subset \mathbb{R}^N$ be an open set. Then for any function $u \in C^1(\Omega) \cap C^2(\{\nabla u \neq 0\})$ and any $x \in \{\nabla u \neq 0\}$, we have*

$$\begin{aligned}\sum_{i=1}^{N} \langle D^2 H(\nabla u)\nabla u_i, \nabla u_i\rangle - \langle D^2 H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle \\ \geq \frac{C_H}{H(\nabla u)} \sum_{i=1}^{N-1} k_{i,u}^2,\end{aligned} \tag{3.2.20}$$

where $k_{i,u}(x)$, for $i = 1, ..., N-1$, denote the principal curvature of the level set $L_{u,x}$ of u at x.

Proof. We suppose $\nabla u(x_0) \neq 0$, for some $x_0 \in \Omega$. For all x in a neighbourhood of x_0, we set $\tau_n(x) := \frac{\nabla u(x)}{|\nabla u(x)|}$ and we denote with $\{\tau_i\}_{i=1,...,N-1}$ an orthonormal basis for the tangent plane of the level set $\{x : u(x) = u(x_0)\}$ at x_0. We note that

$$\nabla u_i = \sum_{j=1}^{N} \langle \nabla u_i, \tau_j\rangle \tau_j, \tag{3.2.21}$$

for $i = 1, ..., N$. By (3.2.21), we get

$$\begin{aligned}&\sum_{i=1}^{N} \langle D^2 H(\nabla u)\nabla u_i, \nabla u_i\rangle - \langle D^2 H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle \\ &= \sum_{i,j,l=1}^{N} \langle D^2 H(\nabla u)\tau_j, \tau_l\rangle \langle \nabla u_i, \tau_j\rangle \langle \nabla u_i, \tau_l\rangle - \langle D^2 H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle.\end{aligned} \tag{3.2.22}$$

Since $D^2 H(\xi)\xi = 0$ for every $\xi \in \mathbb{R}^N \setminus \{0\}$, we have

$$\begin{aligned}&\sum_{i=1}^{N} \langle D^2 H(\nabla u)\nabla u_i, \nabla u_i\rangle - \langle D^2 H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle \\ &= \sum_{i=1}^{N} \sum_{j,l=1}^{N-1} \langle D^2 H(\nabla u)\tau_j, \tau_l\rangle \langle \nabla u_i, \tau_j\rangle \langle \nabla u_i, \tau_l\rangle - \langle D^2 H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle.\end{aligned} \tag{3.2.23}$$

Since τ_n is orthogonal to τ_j, for $j = 1, ..., N-1$, we deduce that

$$\langle \nabla u_i, \tau_j\rangle = \langle \partial_i \left(|\nabla u|\tau_n\right), \tau_j\rangle = |\nabla u| \langle (\tau_n)_{x_i}, \tau_j\rangle, \tag{3.2.24}$$

for $i = 1, ..., N$ and $j = 1, ..., N-1$.
From previous identity we get

$$\begin{aligned}&\sum_{i=1}^{N} \langle D^2 H(\nabla u)\nabla u_i, \nabla u_i\rangle - \langle D^2 H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle \\ &= \sum_{i=1}^{N-1} \sum_{j,l=1}^{N-1} |\nabla u|^2 \langle D^2 H(\nabla u)\tau_j, \tau_l\rangle \langle (\tau_n)_{x_i}, \tau_j\rangle \langle (\tau_n)_{x_i}, \tau_l\rangle \\ &+ \sum_{j,l=1}^{N-1} \langle D^2 H(\nabla u)\tau_j, \tau_l\rangle \langle \nabla u_N, \tau_j\rangle \langle \nabla u_N, \tau_l\rangle - \langle D^2 H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle.\end{aligned} \tag{3.2.25}$$

Without loss of generality, we now assume that $\tau_i(x_0)$ points in the direction of the coordinate x_i for $i = 1, ..., N$.

At the point $x = x_0$, we note that

$$\begin{aligned}\nabla|\nabla u|(x_0) &= \left(\sum_{p=1}^{N}\langle\nabla|\nabla u|, \tau_p\rangle\tau_p\right)(x_0)\\ &= \left(\sum_{p=1}^{N}\partial_p(|\nabla u|)\tau_p\right)(x_0) = \left(\sum_{p=1}^{N}\langle\nabla u_p, \tau_n\rangle\tau_p\right)(x_0).\end{aligned} \tag{3.2.26}$$

By (3.2.26), since D^2H is a symmetric matrix and $D^2H(\xi)\xi = 0$, $\forall\xi \in \mathbb{R}^N \setminus \{0\}$, we have

$$\begin{aligned}\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle(x_0) &= \left(\sum_{p,s=1}^{N-1}\langle D^2H(\nabla u)\tau_p, \tau_s\rangle\langle\nabla u_p, \tau_n\rangle\langle\nabla u_s, \tau_n\rangle\right)(x_0)\\ &= \left(\sum_{p,s=1}^{N-1}\langle D^2H(\nabla u)\tau_p, \tau_s\rangle\partial_N(u_p)\partial_N(u_s)\right)(x_0)\\ &= \left(\sum_{p,s=1}^{N-1}\langle D^2H(\nabla u)\tau_p, \tau_s\rangle\langle\nabla u_N, \tau_p\rangle\langle\nabla u_N, \tau_s\rangle\right)(x_0).\end{aligned} \tag{3.2.27}$$

By (3.2.25) and (3.2.27), we have at x_0

$$\begin{aligned}&\sum_{i=1}^{N}\langle D^2H(\nabla u)\nabla u_i, \nabla u_i\rangle - \langle D^2H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle\\ &= \sum_{i=1}^{N-1}\sum_{j,l=1}^{N-1}|\nabla u|^2\langle D^2H(\nabla u)\tau_j, \tau_l\rangle\langle(\tau_n)_{x_i}, \tau_j\rangle\langle(\tau_n)_{x_i}, \tau_l\rangle\\ &= \sum_{i=1}^{N-1}|\nabla u|^2\langle D^2H(\nabla u)\sum_{j=1}^{N-1}B_{ij,u}\tau_j, \sum_{l=1}^{N-1}B_{il,u}\tau_l\rangle\end{aligned} \tag{3.2.28}$$

where $B = (B_{ij,u})$ denotes the second fundamental form at x_0 associated with the level set $\{x : u(x) = u(x_0)\}$.

Since H is uniformly elliptic, see (1.2.1) and D^2H is -1-homogeneous function, there exist a constant $C_H > 0$, depending on H, such that

$$\begin{aligned}&\sum_{i=1}^{N-1}|\nabla u|^2\langle D^2H(\nabla u)\sum_{j=1}^{N-1}B_{ij,u}\tau_j, \sum_{l=1}^{N-1}B_{il,u}\tau_l\rangle\\ &\geq \frac{C_H}{H(\nabla u)}\sum_{i=1}^{N-1}\left|\sum_{j=1}^{N-1}B_{ij,u}\tau_j\right|^2 = \frac{C_H}{H(\nabla u)}\sum_{i,j=1}^{N-1}B_{ij,u}^2\\ &= \frac{C_H}{H(\nabla u)}\sum_{i=1}^{N-1}k_{i,u}^2,\end{aligned} \tag{3.2.29}$$

where $k_{i,u}(x)$, for $i = 1, ..., N-1$, denote the principal curvature of the level set L_{u,x_0} of u at x_0.

□

We are now ready to prove a key result for the proof of the Theorem 3.1.3.

Theorem 3.2.5. *Let $\Omega \subset \mathbb{R}^N$ be open. Let $u \in C^1(\Omega)$ be a weak stable solution of (3.1.1). For any $x \in \Omega \cap \{\nabla u \neq 0\}$ let $k_{i,u}(x)$, for $i = 1, ..., N-1$, denote the principal curvature of the level set $L_{u,x}$ of u at x. Then there exist a positive constant C_H, depending on H, such that*

$$\begin{aligned}&\int_{\Omega\cap\{\nabla u\neq 0\}} \left(\lambda_2(H(\nabla u))|\nabla u|^2 \sum_{i=1}^{N-1} k_{i,u}^2 + \lambda_1(H(\nabla u))\ \left|\nabla_{L_{u,x}} H(\nabla u)\right|^2\right) \varphi^2\, dx \\ &\leq C_H \int_\Omega \langle A(\nabla u)\nabla\varphi, \nabla\varphi\rangle |\nabla u|^2\, dx,\end{aligned} \tag{3.2.30}$$

for any $\varphi \in C_c^1(\Omega)$.

Proof. Using Lemma 3.2.2 and Lemma 3.2.3, in particular by (3.2.4) and (3.2.12) we deduce that

$$\begin{aligned}0 \leq \int_{\Omega\cap\{\nabla u\neq 0\}} & B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle^2\varphi^2 + B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2 |\nabla u|^2 \\ &+ B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle\langle \nabla H(\nabla u), \nabla\varphi\rangle 2|\nabla u|\varphi \\ &+ B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle\varphi^2 + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla\varphi\rangle|\nabla u|^2 \\ &+ B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla\varphi\rangle|\nabla u|\varphi + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla|\nabla u|\rangle|\nabla u|\varphi \\ &- \sum_{i=1}^N B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla(u_i\varphi^2)\rangle \\ &- \sum_{i=1}^N B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla(u_i\varphi^2)\rangle\, dx.\end{aligned} \tag{3.2.31}$$

Since $\nabla(u_i\varphi^2) = \nabla u_i\varphi^2 + 2\varphi u_i\nabla\varphi$, we have

$$
\begin{aligned}
0 \leq \int_{\Omega\cap\{\nabla u\neq 0\}} & \Big(B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle \\
& - \sum_{i=1}^{N} B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla u_i\rangle\Big)\varphi^2 \\
& + \left(B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle^2 - \sum_{i=1}^{N} B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle^2\right)\varphi^2 \\
& + \left(B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2 + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla\varphi\rangle\right)|\nabla u|^2 \\
& + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla\varphi\rangle|\nabla u|\varphi + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla|\nabla u|\rangle|\nabla u|\varphi \\
& + B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle\langle \nabla H(\nabla u), \nabla\varphi\rangle 2|\nabla u|\varphi \\
& - \sum_{i=1}^{N} B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla\varphi\rangle 2u_i\varphi \\
& - \sum_{i=1}^{N} B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla\varphi\rangle 2u_i\varphi \, dx.
\end{aligned} \tag{3.2.32}
$$

Since D^2H is a symmetric matrix and $|\nabla u|\nabla|\nabla u| = \sum_{i=1}^{N} \nabla u_i u_i$, we get

$$
\begin{aligned}
0 \leq \int_{\Omega\cap\{\nabla u\neq 0\}} & \Big(B'(H(\nabla u))\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle \\
& - \sum_{i=1}^{N} B'(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i, \nabla u_i\rangle\Big)\varphi^2 \\
& + \left(B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla|\nabla u|\rangle^2 - \sum_{i=1}^{N} B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle^2\right)\varphi^2 \\
& + \left(B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2 + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla\varphi\rangle\right)|\nabla u|^2 \, dx.
\end{aligned} \tag{3.2.33}
$$

We note that

$$
\left|\nabla_{L_{u,x}} H(\nabla u)\right|^2 = \sum_{i=1}^{N} \langle \nabla H(\nabla u), \nabla u_i\rangle^2 - \langle \nabla H(\nabla u), \nabla|\nabla u|\rangle^2. \tag{3.2.34}
$$

Therefore (3.2.33) becomes

$$
\begin{aligned}
0 \leq \int_{\Omega\cap\{\nabla u\neq 0\}} & B'(H(\nabla u))\varphi^2\left(\langle D^2H(\nabla u)\nabla|\nabla u|, \nabla|\nabla u|\rangle - \sum_{i=1}^{N}\langle D^2H(\nabla u)\nabla u_i, \nabla u_i\rangle\right) \\
& - B''(H(\nabla u))\left|\nabla_{L_{u,x}} H(\nabla u)\right|^2\varphi^2 \\
& + \left(B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2 + B'(H(\nabla u))\langle D^2H(\nabla u)\nabla\varphi, \nabla\varphi\rangle\right)|\nabla u|^2 \, dx.
\end{aligned} \tag{3.2.35}
$$

By Lemma 3.2.4, we get

$$
\begin{aligned}
& \int_{\Omega\cap\{\nabla u\neq 0\}} \frac{B'(H(\nabla u))}{H(\nabla u)}\varphi^2|\nabla u|^2 \sum_{i=1}^{N-1} k_{i,u}^2 + B''(H(\nabla u))\left|\nabla_{L_{u,x}} H(\nabla u)\right|^2\varphi^2 \, dx \\
& \leq C\int_{\Omega} \langle A(\nabla u)\nabla\varphi, \nabla\varphi\rangle\|\nabla u|^2, dx,
\end{aligned} \tag{3.2.36}
$$

where $k_{i,u}(x)$, for $i = 1, ..., N-1$, denote the principal curvature of the level set $L_{u,x}$ of u at x and C_H is a positive constant depending on H. We remark that the integral on the right-hand side of (3.2.36) is well defined also on $\{x \in \Omega : \nabla u(x) = 0\}$.

□

Using the previous result we give

Proof of Theorem 3.1.3. If $\nabla u(x) = 0$ for any $x \in \mathbb{R}^2$, the one-dimensional symmetry is trivial.

Since $|\nabla u|$ is bounded in $\mathbb{R}^2$ and $A(\nabla u)|\nabla u|^2 L^\infty_{loc}([0, +\infty))$ (by $(h_b)-(iii)$ and assumptions on H), by (3.2.30) we obtain

$$\begin{aligned} \int_{\{\nabla u \neq 0\}} & \frac{B'(H(\nabla u))}{H(\nabla u)} \varphi^2 |\nabla u|^2 k_{1,u}^2 + B''(H(\nabla u)) \left|\nabla_{L_{u,x}} H(\nabla u)\right|^2 \varphi^2 \\ & \leq C \int_{\mathbb{R}^2} |\nabla \varphi|^2 \end{aligned} \tag{3.2.37}$$

for suitable $C > 0$.

We fix $R > 0$, and let us consider the function

$$\varphi := \max \left\{ 0, \min \left\{ 1, \frac{\ln(R^2/|x|)}{\ln R} \right\} \right\}. \tag{3.2.38}$$

We remark that $\varphi \in W^{1,\infty}(\Omega)$, compactly supported. Therefore by density argument we can use φ in (3.2.37).

Substituting (3.2.38) in (3.2.37) we get

$$k_1(x) = 0 \tag{3.2.39}$$

and

$$\nabla_{L_{u,x}} H(\nabla u(x)) = 0 \tag{3.2.40}$$

for any $x \in \{\nabla u \neq 0\}$.

By (3.2.39) and (3.2.40), every a non-empty connected component $\overline{L}$ of $L_{u,x} \cap \{\nabla u \neq 0\}$ is a flat hyperplane (for all details see [89, Section 2.4]). Therefore u is constant on these hyperplane, since each of them lies on a level set. On the contrary, u is constant on any other possible hyperplane parallel to the ones of the above family, because the gradient is zero there. From this u has one-dimensional symmetry.

□

3.3 Preliminaries for the proof of Theorem 3.1.4

In the first part of this section we aim to classify the solutions of (3.1.1), in the case $N = 1$. The second part is devoted to the study of solutions that are monotone in one direction.

3.3.1 ODE analysis

We start this section analysing the following ODE

$$\left(B'(H(h'(t)))H'(h'(t))\right)' + f(h(t)) = 0 \tag{3.3.1}$$

for any $t \in \mathbb{R}$.

We consider weak solutions $h \in C^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$. Here, the function $H : \mathbb{R} \to \mathbb{R}$ is a Finsler norm, B satisfies the assumptions (h_B)-(iii) and f is a continuous function.

Remark 3.3.1. *Since a Finsler norm $H : \mathbb{R} \to \mathbb{R}$ is given by $H(\xi) = c|\xi|$, where c is a positive constant, the ODE (3.3.1) simplifies to the equation analyzed in [89]. For the sake of clarity, we revisit all the calculation related to the ODE from [89], this time using Finsler notation.*

We give a regularity result.

Lemma 3.3.2. *Suppose $\kappa = 0$ in (h_B)-(iii). Then we have $h \in C^{1,1/(p-1)}_{loc}(\mathbb{R}) \cap C^2(\{h' \neq 0\})$ if $p \geq 2$ and $h \in C^2(\mathbb{R})$ if $p < 2$.*

Otherwise, if $\kappa > 0$ in (h_B)-(iii), we have that $h \in C^{1,1}_{loc}(\mathbb{R}) \cap C^2(\{h' \neq 0\})$

Proof. Suppose $\kappa = 0$.

We set $\Phi(t) := B'(H(t))H'(t)$ and $\gamma(t) := B'(H(h'(t)))H'(h'(t))$ for any $t \in \mathbb{R}$. Under assumptions (h_B), we note that $\gamma \in C(\mathbb{R})$ and $\gamma'(t) = -f(h(t)) \in C(\mathbb{R})$ in the distributional sense, and hence $\gamma \in C^1(\mathbb{R})$ (see [112, Theorem 6.10]). We note also that $\Phi(h'(t)) = \gamma(t)$. We prove that Φ is invertible.

By (h_B)-(iii), (1.2.9) and using Euler's theorem (1.2.4), we deduce that

$$\begin{aligned} 0 =& \gamma H''(t)tH'(t)H(t)^{p-2} = \gamma H''(t)H(t)H(t)^{p-2} \\ &\leq B'(H(t))H''(t) \\ &\leq \Gamma H''(t)H(t)H(t)^{p-2} = \Gamma H''(t)tH'(t)H(t)^{p-2} = 0, \end{aligned} \tag{3.3.2}$$

for any $t > 0$. By (3.3.2) and (h_B)-(ii) we get

$$\begin{aligned} \Phi'(t) &= B''(H(t))H'(t)^2 + B'(H(t))H''(t) \\ &= B''(H(t))H'(t)^2 > 0 \end{aligned} \tag{3.3.3}$$

for any $t > 0$. Since Φ is odd, we get the invertibility of the function Φ. Moreover, since $t = \Phi\left(\Phi^{-1}(t)\right)$ and using (h_B)-(iii) we infer

$$C_1 t|t|^{(2-p)/(p-1)} \leq \Phi^{-1}(t) \leq C_2 t|t|^{(2-p)/(p-1)} \tag{3.3.4}$$

for any $t > 0$, where C_1 and C_2 are positive constants. Now, using the fact that $1 = \Phi'\left(\Phi^{-1}(t)\right)\left(\Phi^{-1}(t)\right)'$, by (h_B)-(iii) and (3.3.4), we get

$$\tilde{C}_1 t^{(2-p)/(p-1)} \leq \left(\Phi^{-1}(t)\right)' \leq \tilde{C}_2 t^{(2-p)/(p-1)} \tag{3.3.5}$$

for any $t > 0$, where $\tilde{C}_1, \tilde{C}_2$ are positive constants. Now we note that, since $\Phi \in C(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$, and $\Phi'(t) \neq 0$, for $t \neq 0$, then $\Phi^{-1}(t) \in C(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{0\})$. By (3.3.5) and using the fact that Φ^{-1} is odd, we deduce that $\Phi^{-1} \in C^1(\mathbb{R})$ if $p < 2$. Since $|a^q - b^q| \leq |a - b|^q$, for $a, b \geq 0$, $0 < q < 1$, by (3.3.4) and using the fact that Φ^{-1} is odd, we have that $\Phi^{-1} \in C^{0,1/(p-1)}(\mathbb{R}) \cap C^1(\mathbb{R} \setminus \{0\})$ if $p \geq 2$. Since $h' = \Psi^{-1} \circ \gamma$, with $\gamma \in C^1(\mathbb{R})$, we have the desired regularity.

The case $\kappa > 0$ reduces to the case $p = 2$. □

Lemma 3.3.3. *Let $L_1(t) := \Lambda_1(H(h'(t)))$. Then we have $L_1 \in C(\mathbb{R}) \cap C^1(\{h' \neq 0\})$ and that $L_1'(\bar{t}) = 0$ if $h'(\bar{t}) = 0$.*

Proof. Since $H \in C^2(\mathbb{R}\backslash\{0\}$ and $B''(t)t \in C(\mathbb{R})$, we deduce that $L_1 \in C(\mathbb{R}) \cap C^1(\{h' \neq 0\})$. Now we prove $L_1'(\bar{t}) = 0$ if $h'(\bar{t}) = 0$.

First we consider the case $\kappa = 0$ in (h_B)-(iii). If $\bar{t} \in \{h'(\bar{t}) = 0\}$, by $(h_B) - (iii)$ and (1.2.3) we have

$$\begin{aligned}&\left|\frac{|L_1(\bar{t}+s) - L_1(\bar{t})|}{s}\right| = \left|\frac{\int_0^{H(h'(t))} B''(l) l \, dl}{s}\right| \\ &\leq C(\Gamma, p, \alpha_2)\frac{|h'(\bar{t}+s)|^p}{s} = C(\Gamma, p, \alpha_2)\frac{|h'(\bar{t}+s) - h'(\bar{t})|^p}{s},\end{aligned} \tag{3.3.6}$$

where $C(\Gamma, p, \alpha_2)$ is a positive constant. By Lemma 3.3.2 we get

$$\left|\frac{|L_1(\bar{t}+s) - L_1(\bar{t})|}{s}\right| \leq C(B, H)|s|^{1/(p-1)}, \tag{3.3.7}$$

if $p \geq 2$, while

$$\left|\frac{|L_1(\bar{t}+s) - L_1(\bar{t})|}{s}\right| \leq C(B, H)|s|^{p-1}, \tag{3.3.8}$$

if $1 < p < 2$, where $C(B, H)$ is a positive constant. By sending $s \to 0$, we get that $L_1'(\bar{t}) = 0$.

Now we consider the case $\kappa > 0$. Since $\kappa > 0$ in (h_B)-(iii), (1.2.3), using the fact that $h \in W^{1,\infty}(\mathbb{R})$, by Lemma 3.3.2 we have

$$\begin{aligned}&\left|\frac{|L_1(\bar{t}+s) - L_1(\bar{t})|}{s}\right| = \left|\frac{\int_0^{H(h'(t))} B''(l) l \, dl}{s}\right| \\ &\leq C(\|B''\|_{L^\infty_{loc}((0,\infty))})\frac{|h'(\bar{t}+s)|^2}{s} \\ &= C(\|B''\|_{L^\infty_{loc}((0,\infty))})\frac{|h'(\bar{t}+s) - h'(\bar{t})|^2}{s} \leq C(\|B''\|_{L^\infty_{loc}((0,\infty))})|s|.\end{aligned} \tag{3.3.9}$$

By sending $s \to 0$ we get the thesis. □

Lemma 3.3.4. *Let h is a weak solution of* (3.3.1), *then we have*

$$\Lambda_1(H(h'(t)) + F(h(t)) = F(\inf h) = F(\sup h) \tag{3.3.10}$$

for any $t \in \mathbb{R}$.

Proof. First, we prove that

$$\Lambda_1(H(h'(t)) + F(h(t)) = \Lambda_1(H(h'(s))) + F(h(s)) \tag{3.3.11}$$

for any $t, s \in \mathbb{R}$.

Suppose $h \in C^2(t-\epsilon, t+\epsilon)$ with $h' \neq 0$ in $(t-\epsilon, t+\epsilon)$, for $\epsilon > 0$.

By (h_B)-(iii), (1.2.9) and using Euler's theorem (1.2.4), we deduce that

$$\begin{aligned}0 =&\gamma H''(h'(t))h'(t)H'(h'(t))(\kappa + H(h'(t)))^{p-2} = \gamma H''(h'(t))H(h'(t))(\kappa + H(h'(t)))^{p-2} \\ &\leq B'(H(h'(t)))H''(h'(t)) \\ &\leq \Gamma H''(h'(t))H(h'(t))(\kappa + H(h'(t)))^{p-2} \\ &= \Gamma H''(h'(t))h'(t)H'(h'(t))(\kappa + H(h'(t)))^{p-2} = 0.\end{aligned} \tag{3.3.12}$$

Since $h \in C^2(t-\epsilon, t+\epsilon)$, using (3.3.1) and (3.3.12), we have

$$\begin{aligned} 0 &= \left(B'(H(h'(t)))H'(h'(t))\right)' + f(h(t)) \\ &= B''(H(h'(t)))\left[H'(h'(t))\right]^2 h''(t) + B'(H(h'(t))H''(h'(t))h''(t) + f(h(t)) \\ &= B''(H(h'(t)))\left[H'(h'(t))\right]^2 h''(t) + f(h(t)). \end{aligned} \tag{3.3.13}$$

Using Euler's Theorem (1.2.4) and (3.3.13) we have

$$\begin{aligned} &\frac{d}{dt}\left(\Lambda_1(H(h'(t)) + F(h(t))\right) \\ &= B''(H(h'(t)))H(h'(t))H'(h'(t))h''(t) + f(h(t))h'(t) \\ &= \left[B''(H(h'(t)))[H'(h'(t))]^2 h''(t) + f(h(t))\right] h'(t) \\ &= 0. \end{aligned} \tag{3.3.14}$$

On the other hand, suppose $h'(\bar{t}) = 0$. From Lemma 3.3.3 we have

$$\frac{d}{dt}\left(\Lambda_1(H(h'(t))) + F(h(t))\right)(\bar{t}) = L_1'(\bar{t}) + f(h(\bar{t}))h'(\bar{t}) = 0. \tag{3.3.15}$$

By (3.3.14) and (3.3.15) we have (3.3.11).

Now we only consider $\inf h$, since $\sup h$ can be dealt with analogously. Firs we consider the case $\inf h = h(\bar{t})$, for some $\bar{t} \in \mathbb{R}$. Then by (3.3.11) we obtain

$$\Lambda_1(H(h'(t)) + F(h(t)) = \Lambda_1(0) + F(h(\bar{t})) = F(\inf h).$$

Now we suppose that $\inf h$ is not attained and we consider a sequence $t_n \to \infty$ such that $h(t_n) \to \inf h$. Let us consider $h_n(t) := h(t + t_n)$. By [51, 157] $\|h_n\|_{C^{1,\alpha}(\mathbb{R})}$ is finite[3], we have that h_n, up to subsequence, converges to a suitable $\tilde{h} \in C^1_{loc}(\mathbb{R})$. We note that $\tilde{h}$ is a weak solution of (3.3.1). Since

$$\tilde{h}(0) = \lim_n h(t_n) = \inf h \le \lim_n h(t + t_n) = \tilde{h}(t),$$

we have that $\inf \tilde{h} = \tilde{h}(0)$. So, by (3.3.11) we get

$$\begin{aligned} F(\inf h) &= F(\inf \tilde{h}) = \Lambda_1(H(\tilde{h}'(t))) + F(\tilde{h}(t)) \\ &= \lim_n \Lambda_1(H(h'(t + t_n))) + F(h(t + t_n)) \\ &= \lim_n \Lambda_1(H(h'(t))) + F(h(t)) \\ &= \Lambda_1(H(h'(t))) + F(h(t)). \end{aligned}$$

□

We remark that, as a consequence of Lemma 3.3.4,

$$F(\inf h) = F(\sup h) = F(h(t^*)), \tag{3.3.16}$$

for any $t^* \in \{h' = 0\}$.

From previous result follows the classification of the ODE in (3.3.1):

[3]Let $u \in W^{1,\infty}(\mathbb{R}^N)$ be a weak solution of (3.1.1). Then, given any $x_0 \in \mathbb{R}^N$ and $R \in (0,1)$, there exist $\alpha \in (0,1)$ and $C > 0$, depending only on $N, R, \|u\|_{L^\infty(\mathbb{R}^N))}, H, B$ such that

$$\|\nabla u\|_{L^\infty(\mathbb{R}^N))} \le C \quad \text{and}$$

$$|\nabla u(x) - \nabla u(y)| \le CR^{-\alpha}|x-y|^\alpha,$$

for any $x, y \in B_R(x_0)$. For more details see [51, Proposition 3.1] and [157]

Lemma 3.3.5. *One of the following holds:*

(i) h *is constant*

(ii) $\{h' = 0\} = \emptyset$

(iii) $h'(t) \neq 0$ *for any* t *in a bounded interval* (β_1, β_2) *with* $h'(\beta_1) = h'(\beta_2) = 0$ *and*

$$F(h(\beta_1)) = F(h(\beta_2)) = F(\inf h) = F(\sup h). \tag{3.3.17}$$

(iv) $h'(t) \neq 0$ *for any* t *in a unbounded interval either* $(-\infty, \beta)$ *or* $(\beta, +\infty)$ *with* $h'(\beta) = 0$ *and*

$$F(h(\beta)) = F(\inf h) = F(\sup h). \tag{3.3.18}$$

Proof. Assume that condition (i) is not satisfied. Then, there exists some $t^* \in \mathbb{R}$ such that $h'(t^*) \neq 0$. Take an interval I around t^*, that is as large as possible, ensuring $h'(t) \neq 0$, for $t \in I$. If $I = \mathbb{R}$, then we fall into case (ii).

If I is bounded, with endpoint $\beta_1 < \beta_2$, then $h'(\beta_1) = h'(\beta_2) = 0$, which puts us in case (iii), and (3.3.17) follows by (3.3.16).

If I is unbounded in one direction, meaning I is either $(\beta, +\infty)$ or $(-\infty, \beta)$, then $h'(\beta) = 0$, placing us in case (iv). Hence, (3.3.18) follows by (3.3.16). □

3.3.2 Behavior of the one-dimensional profiles at infinity

Given $v : \mathbb{R}^N \to \mathbb{R}$, we set $v^t(x) := v(x', x_N + t)$, for any $x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R}$ and $t \in R$. We now prove a Poincarè type formula, in one dimension less.

Theorem 3.3.6. *Let* $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ *be a stable weak solution of* (3.1.1)*. Suppose* $\partial_{x_N} u \geq 0$*. We set*

$$\begin{aligned}
\underline{u}(x') &:= \lim_{t \to -\infty} u(x', x_N + t) \\
\overline{u}(x') &:= \lim_{t \to +\infty} u(x', x_N + t) \\
\overline{H}(x') &= \underline{H}(x') := H(x', 0) \\
\tilde{A}_{ij}(\xi) &:= A_{ij}(\xi) \qquad i, j = 1, ..., N-1.
\end{aligned} \tag{3.3.19}$$

Then $\underline{u}, \overline{u} \in C^1(\mathbb{R}^{N-1})$ *are weak solutions of* (3.1.1) *in* $\mathbb{R}^{N-1}$*. Moreover, we have*

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1} \cap \{\nabla \overline{u} \neq 0\}} \lambda_2(\overline{H}(\nabla \overline{u})) \varphi^2 |\nabla \overline{u}|^2 \sum_{i=1}^{N-2} k^2_{i,\overline{u}} + \lambda_1(\overline{H}(\nabla \overline{u})) \left| \nabla_{L_{\overline{u},x'}} \overline{H}(\nabla \overline{u}) \right|^2 \varphi^2 \\
&\leq C \int_{\mathbb{R}^{N-1}} \langle \tilde{A}(\nabla \overline{u}) \nabla \varphi, \nabla \varphi \rangle |\nabla \overline{u}|^2
\end{aligned} \tag{3.3.20}$$

for any $\varphi \in C^1_c(\mathbb{R}^{N-1})$ *and* C *is a positive constant depending on* $\overline{H}$ *(and analogous claim holds for* $\underline{u}$*.)*

Proof. We note that $\underline{u}$ and $\overline{u}$ are well defined since u is monotone in the direction x_N and bounded. Since $u \in W^{1,\infty}(\mathbb{R}^N)$, by [51, Proposition 3.1] and [157, Theorem 1],

$$\|u_t\|_{C^{1,\alpha}(\mathbb{R}^N)} \leq C,$$

where C is a positve constant not depending on t. From Ascoli-Arzela Theorem we deduce that u^t tends to $\overline{u}$, for $t \to +\infty$, in the norm $\|\cdot\|_{C^{1,\alpha'}_{loc}(\mathbb{R}^N)}$, with $\alpha' < \alpha$. (same claim for $\underline{u}$.)

We now fix $\epsilon > 0$ and we set $A_\epsilon := \{(x', x_N) \in \mathbb{R} \times \mathbb{R}^{N-1}$ such that $|\nabla \overline{u}| \geq \epsilon\}$. We consider a compact set $K \subset\subset A_\varepsilon$. Since u_t converges to $\overline{u}$ in the norm $C^1(K)$, we deduce that $|\nabla u(x', x_N + t)| \geq \varepsilon/2$, for t sufficiently large. Since $u \in C^{2,\alpha}_{loc}(\mathbb{R}^N \setminus \{\nabla u \neq 0\})$, by Shauder estimates we obtain

$$\|u_t\|_{C^{2,\alpha}(K)} \leq C,$$

where C is a positive constant not depending on t. From Ascoli theorem u^t tends to $\overline{u}$, for $t \to +\infty$, in the norm $\|\cdot\|_{C^{2,\alpha'}_{loc}(A_\epsilon)}$ (same claim for $\underline{u}$.)

Let us consider $\varphi := \varphi_1(x')\varphi_2(x_N)$ with $\varphi_1 \in C^1_c(\mathbb{R}^{N-1})$ and $\varphi \in C^1_c(\mathbb{R})$ satisfying

$$\int_{\mathbb{R}} \varphi_2(x_N) dx_N = 1. \tag{3.3.21}$$

We note that u_t is a weak solution of (3.1.1), that is

$$\int_{\mathbb{R}^N} B'(H(\nabla u_t))\langle \nabla H(\nabla u_t), \nabla \varphi\rangle \, dx = \int_{\mathbb{R}^N} f(u_t)\varphi \, dx \tag{3.3.22}$$

Substituting φ in (3.3.22), passing to the limit, we get

$$\int_{\mathbb{R}^N} B'(H(\nabla \overline{u}))\langle \nabla H(\nabla \overline{u}), \nabla \left(\varphi_1(x')\varphi_2(x_N)\right)\rangle \, dx' \, dx_N = \int_{\mathbb{R}^N} f(\overline{u})\varphi_1(x')\varphi_2(x_N) \, dx' \, dx_N. \tag{3.3.23}$$

By (3.3.21) and recalling (3.3.19) we have

$$\begin{aligned}&\int_{\mathbb{R}^{N-1}} B'(\overline{H}(\nabla \overline{u}))\langle \nabla \overline{H}(\nabla \overline{u}), \nabla \varphi_1\rangle dx' \\ &+ \int_{\mathbb{R}^N} B'(\overline{H}(\nabla \overline{u}(x')))\partial_{x_N} H(\nabla \overline{u}(x')))\varphi_2'(x_N)\varphi_1(x') \, dx' \, dx_N = \int_{\mathbb{R}^{N-1}} f(\overline{u})\varphi_1 \, dx'.\end{aligned} \tag{3.3.24}$$

Since φ_2 is compactly supported in $\mathbb{R}$, we deduce

$$\int_{\mathbb{R}} \varphi_2'(s) \, ds = 0.$$

By Fubini Theorem we get

$$\int_{\mathbb{R}^{N-1}} B'(\overline{H}(\nabla \overline{u}))\langle \nabla \overline{H}(\nabla \overline{u}), \nabla \varphi_1\rangle \, dx' = \int_{\mathbb{R}^{N-1}} f(\overline{u})\varphi_1 \, dx', \tag{3.3.25}$$

for any $\varphi_1 \in C^1_c(\mathbb{R}^{N-1})$, and the same for $\underline{u}$.

Since $u(x', x_N + t) \to \overline{u}$, for $t \to +\infty$, in the norm $\|\cdot\|_{C^{2,\alpha'}_{loc}(A_\epsilon)}$, and recalling the definition of the tangential gradient (3.2.19) we deduce

$$\begin{aligned}&\lim_{t\to+\infty} \nabla_{L_{u,(x',x_N+t)}} H(\nabla u)(x', x_N + t) \\ &= \lim_{t\to+\infty} D^2u(x', x_N + t)\nabla H(\nabla u(x', x_N + t)) \\ &- \langle D^2u(x', x_N + t)\nabla H(\nabla u(x', x_N + t)), \frac{\nabla u(x', x_N + t)}{|\nabla u(x', x_N + t)|}\rangle \frac{\nabla u(x', x_N + t)}{|\nabla u(x', x_N + t)|} \\ &= \nabla_{L_{\overline{u},x'}} \overline{H}(\nabla \overline{u}))(x'),\end{aligned} \tag{3.3.26}$$

in $x' \in \{|\nabla \overline{u}| \geq \varepsilon\}$.

By [150, 151] we recall that

$$\begin{aligned}&\sum_j |\nabla u_j(x', x_N+t)| - \left|\nabla_{L_{u,(x',x_N+t)}} |\nabla u(x', x_N+t)|\right|^2 \\ &- |\nabla |\nabla u(x', x_N+t)||^2 = |\nabla u(x', x_N+t)|^2 \sum_l k^2_{l,u^t}\end{aligned} \tag{3.3.27}$$

where $k_{1,u^t}, ..., k_{(N-1),u^t}$ denote the principal curvatures of $L_{u,(x',x_N+t)}$.

Using (3.3.26) with $H = |\cdot|$ and by (3.3.27) we obtain

$$|\nabla \overline{u}(x')|^2 \sum_{l=1}^{N-2} k_{l,\overline{u}}(x') = \lim_{t\to+\infty} |\nabla u(x', x_N+t)|^2 \sum_l k^2_{l,u^t}, \tag{3.3.28}$$

where we denote with $k_{i,\overline{u}}(x')$ the $i-$th principal curvature of $L_{\overline{u},x'}$ at $x' \in L_{\overline{u},x'} \cap \{|\nabla \overline{u}| \geq \epsilon\}$.

We now consider $\varphi := \varphi_1(x')\varphi_2(x_N)$ with $\varphi_1 \in C^1_c(\mathbb{R}^{N-1})$ and $\varphi_2 \in C^1_c(\mathbb{R})$. In particular, we take $\varphi_2 := \sqrt{\mu}\psi(\mu x_N)$, where $\mu > 0$ and $\psi \in C^1_c(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \psi^2(x_N)dx_N = 1 \quad \text{and} \quad \int_{\mathbb{R}} \varphi^2(x_N)dx_N = 1. \tag{3.3.29}$$

We remark that, by Theorem 3.2.5, (3.2.30) holds for u_t. Using (3.3.26), (3.3.28), (3.3.29) and (3.2.30), we have

$$\begin{aligned}&\int_{\{\nabla\overline{u}\geq\epsilon\}} \frac{B'(\overline{H}(\nabla\overline{u}))}{\overline{H}(\nabla\overline{u})}\varphi_1^2|\nabla\overline{u}|^2 \sum_{i=1}^{N-2} k^2_{i,\overline{u}} + B''(\overline{H}(\nabla\overline{u}))\left|\nabla_{L_{\overline{u},x'}}\overline{H}(\nabla\overline{u})\right|^2 \varphi_1^2\, dx' \\ &= \int_{\mathbb{R}}\int_{\{\nabla\overline{u}\geq\epsilon\}} \frac{B'(\overline{H}(\nabla\overline{u}))}{\overline{H}(\nabla\overline{u})}\varphi_1^2|\nabla\overline{u}|^2 \sum_{i=1}^{N-2} k^2_{i,\overline{u}} + B''(\overline{H}(\nabla\overline{u}))\left|\nabla_{L_{\overline{u},x'}}\overline{H}(\nabla\overline{u})\right|^2 \varphi_1^2\varphi_2^2\, dx'\, dx_N \\ &= \lim_{t\to+\infty}\int_{\mathbb{R}}\int_{\{\nabla\overline{u}\geq\epsilon\}} \frac{B'(H(\nabla u^t))}{H(\nabla u^t)}\varphi_1^2|\nabla u^t|^2 \sum_{i=1}^{N-1} k^2_{i,u^t} \\ &\qquad\qquad + B''(H(\nabla u^t))\left|\nabla_{L_{u^t,(x',x_N+t)}} H(\nabla u^t)\right|^2 \varphi_1^2\varphi_2^2\, dx'\, dx_N \\ &\leq C_H \lim_{t\to+\infty}\int_{\mathbb{R}^N} \langle A(\nabla u^t)\nabla(\varphi_1\varphi_2), \nabla(\varphi_1\varphi_2)\rangle|\nabla u^t|^2 \\ &= \int_{\mathbb{R}^N} \langle A(\nabla\overline{u})\nabla\varphi, \nabla\varphi\rangle|\nabla\overline{u}|^2\, dx'\, dx_N.\end{aligned} \tag{3.3.30}$$

where C_H is a positive constant depending on H.

We note that

$$\begin{aligned}\langle A(\nabla\overline{u})\nabla\varphi, \nabla\varphi\rangle &= \langle \tilde{A}(\nabla\overline{u})\nabla_{x'}\varphi, \nabla_{x'}\varphi\rangle \\ &\quad + \sum_{i=1}^{N} A_{in}(\nabla\overline{u})\partial_{x_i}\varphi\partial_{x_N}\varphi \\ &\quad + \sum_{j=1}^{N-1} A_{nj}(\nabla\overline{u})\partial_{x_j}\varphi\partial_{x_N}\varphi,\end{aligned} \tag{3.3.31}$$

where $\tilde{A}_{ij} = A_{ij}$, for $i, j = 1, ..., N-1$ and $\nabla_{x'}$ is the gradient of the variables $x_1, ..., x_{N-1}$. Now we prove that

$$\int_{\mathbb{R}^N} |\nabla \overline{u}|^2 \sum_{i=1}^{N} A_{in}(\nabla \overline{u}) \partial_{x_i} \varphi \partial_{x_N} \varphi \, dx = \int_{\mathbb{R}^N} |\nabla \overline{u}|^2 \sum_{j=1}^{N-1} A_{nj}(\nabla \overline{u}) \partial_{x_j} \varphi \partial_{x_N} \varphi \, dx = 0. \quad (3.3.32)$$

Since $u \in W^{1,\infty}(\mathbb{R}^N)$ and by definition of φ we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \sum_{j=1}^{N-1} |\nabla \overline{u}|^2 A_{nj}(\nabla \overline{u}) \partial_{x_j} \varphi \partial_{x_N} \varphi \, dx \\ &\leq C \int_{\mathbb{R}^N} \sum_{j=1}^{N-1} \partial_{x_j} \varphi_1(x') \varphi_2(x_N) \partial_{x_N} \varphi_2(x_N) \varphi_1(x') \, dx' \, dx_N \\ &= C \sum_{j=1}^{N-1} \int_{\mathbb{R}^{N-1}} \partial_{x_j} \varphi_1(x') \varphi_1(x') \, dx' \int_{\mathbb{R}} \partial_{x_N} \varphi_2(x_N) \varphi_2(x_N) \, dx_N \\ &\leq C \int_{\mathbb{R}} \partial_{x_N} \varphi_2(x_N) \varphi_2(x_N) \, dx_N \\ &\leq C \int_{\mathbb{R}} \mu^2 \psi(\mu x_N) \psi'(\mu x_N) \, dx_N \\ &= C\mu \int_{\mathbb{R}} \psi(y_N) \psi'(y_N) \, dy_N \to 0, \end{aligned} \quad (3.3.33)$$

for $\mu \to 0$ and C is a positive constant depending on u, H and B. In a similar way we deduce that

$$\int_{\mathbb{R}^N} |\nabla \overline{u}|^2 \sum_{i=1}^{N} A_{in}(\nabla \overline{u}) \partial_{x_i} \varphi \partial_{x_N} \varphi \, dx = 0, \quad (3.3.34)$$

hence (3.3.32) holds.

Taking ϵ arbitrarily small in (3.3.30), using (3.3.31), (3.3.32) and (3.3.29), we have

$$\begin{aligned} &\int_{\{\nabla \overline{u} \neq 0\}} \frac{B'(\overline{H}(\nabla \overline{u}))}{\overline{H}(\nabla \overline{u})} \varphi_1^2 |\nabla \overline{u}|^2 \sum_{i=1}^{N-2} k_{i,\overline{u}}^2 + B''(\overline{H}(\nabla \overline{u})) \left| \nabla_{L_{\overline{u},x'}} \overline{H}(\nabla \overline{u}) \right|^2 \varphi_1^2 dx' \\ &\leq C \int_{\mathbb{R}^N} \langle \tilde{A}(\nabla \overline{u}) \nabla_{x'} \varphi, \nabla_{x'} \varphi \rangle |\nabla \overline{u}|^2 \, dx' \, dx_N \\ &= C \int_{\mathbb{R}^N} \langle \tilde{A}(\nabla \overline{u}) \nabla \varphi_1, \nabla \varphi_1 \rangle |\nabla \overline{u}|^2 \, dx', \end{aligned} \quad (3.3.35)$$

where C is a positive constant depending on $\overline{H}$.

□

We now recall a result (see [51, Theorem 1.1]) of a pointwise energy bound in an anisotropic setting. Similar estimates were given in [32, 58].

Lemma 3.3.7 (see [51]). *Let $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ a weak solution of (3.1.1). We set*

$$c_u := \sup_{t \in [\inf u, \sup u]} F(t). \quad (3.3.36)$$

Then we have

$$\Lambda_1(H(\nabla u(x))) = B'(H(\nabla u(x)))H(\nabla u(x)) - B(H(\nabla u(x))) \leq c_u - F(u(x)). \quad (3.3.37)$$

Remark 3.3.8. *If the set where the gradient of u vanishes is empty then follows that $c_u = \max\{F(\inf u), F(\sup u)\}$. Indeed, suppose by contradiction that $c_u = F(t)$, for some $t \in (\inf u, \sup u)$. Then there exist $\hat{x}$ such that $u(\hat{x}) = t$, and by previous Lemma $\Lambda_1(H(\nabla u(\hat{x}))) = 0$, against the assumption that the set of the gradient of u is empty.*

From here we suppose, for simplicity, that

$$c_u = F(\inf u). \tag{3.3.38}$$

We now are ready to give the following result of classification

Lemma 3.3.9. *Let $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ be a stable weak solution of (3.1.1). Suppose that $\{\nabla u = 0\} = \emptyset$ and $\partial_{x_N} u \geq 0$. Let $\underline{u}$ as in (3.3.19) and suppose that exist $\underline{w} \in S^{N-2}$ and $\underline{h} : \mathbb{R} \to \mathbb{R}$ such that $\underline{u}(x') = \underline{h}(x' \cdot \underline{w})$, for any $x' \in \mathbb{R}^{N-1}$. Then one of the following possibilities holds:*

(i) $\underline{h}$ *is constant,*

(ii) $\{\underline{h}' = 0\} = \emptyset$,

(iii) There exist $\beta \in \mathbb{R}$ such that $\underline{h}'(t) < 0$ for $t < \beta$ and $\underline{h}(t) = \inf u$ for $t \geq \beta$,

(iv) There exist $\beta \in \mathbb{R}$ such that $\underline{h}'(t) > 0$ for $t > \beta$ and $\underline{h}(t) = \inf u$ for $t \leq \beta$,

(v) There exist β_1 and β_2 such that $\underline{h}'(t) < 0$ for $t < \beta_1$, $\underline{h}'(t) > 0$ for $t > \beta_2$ and $\underline{h}(t) = \inf u$ for $\beta_1 \leq t \leq \beta_2$.

Proof. We note that from Theorem 3.3.6, $\underline{u}$ is a weak solution of (3.1.1) in $\mathbb{R}^{N-1}$. Now we will prove, since $\underline{h}(t) = \underline{u}(\underline{w}t)$, that $\underline{h}$ satisfies

$$\left(B'(\tilde{H}(\underline{h}'(t)))\tilde{H}'(\underline{h}'(t))\right)' + f(\underline{h}(t)) = 0 \tag{3.3.39}$$

where we set $\tilde{H}(t) = \underline{H}(t\underline{w})$.
Indeed, we consider the following change of variables given by $(t, y') = \psi^{-1}(x_1, ..., x_{N-1}) = (\langle t, \underline{w}\rangle, \langle t, w^{1,\perp}\rangle, ..., \langle t, w^{N-2,\perp}\rangle)$, where $\underline{w}, w^{1,\perp}, ..., w^{N-2,\perp}$ is a basis of $\mathbb{R}^{N-1}$. Since $\underline{u}$ is a weak solution of (3.1.1), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} B'(\underline{H}(\nabla \underline{u}))\langle \nabla \underline{H}(\nabla \underline{u}), \nabla \varphi_1\rangle \, dx' = \int_{\mathbb{R}^{N-1}} f(\underline{u})\varphi_1 \, dx' \\
&= \int_{\mathbb{R}^{N-1}} B'(\tilde{H}(\underline{h}'(t)))\langle \nabla \underline{H}(\underline{h}'(t)w), \nabla \varphi_1(\psi(t, y'))\rangle \, dt' \, dy' \\
&= \int_{\mathbb{R}^{N-1}} f(\underline{h}(t))\varphi_1(\psi(t, y')) \, dt' \, dy',
\end{aligned} \tag{3.3.40}$$

for any $\varphi_1 \in C^1_c(\mathbb{R}^{N-1})$.

Now we set $\tilde{\varphi}(t, y') = \varphi_1(\psi(t, y'))$, hence $\nabla\tilde{\varphi}(t, y') = D\psi(t, y')\nabla\varphi_1(\psi(t, y'))$, where

$D\psi(t,y')$ denotes the Jacobian matrix of the function ψ. Therefore (3.3.40) becomes

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} B'(\tilde{H}(h'(t)))\langle\nabla\underline{H}(\underline{h}'(t)\underline{w}),\nabla\varphi_1(\psi(t,y'))\rangle\,dt'\,dy' \\
&=\int_{\mathbb{R}^{N-1}} f(\underline{h}(t))\varphi_1(\psi(t,y'))\,dt'\,dy' \\
&=\int_{\mathbb{R}^{N-1}} B'(\tilde{H}(\underline{h}'(t)))\langle\nabla\underline{H}(\underline{h}'(t)\underline{w}),(D\psi(t,y'))^{-1}\nabla\tilde{\varphi}(t,y')\rangle\,dt'\,dy' \\
&=\int_{\mathbb{R}^{N-1}} f(\underline{h}(t))\tilde{\varphi}(t,y')\,dt'\,dy' \qquad (3.3.41)\\
&=\int_{\mathbb{R}^{N-1}} B'(\tilde{H}(\underline{h}'(t)))\langle\nabla\underline{H}(\underline{h}'(t)\underline{w}),\underline{w}\rangle\partial_t\tilde{\varphi}(t,y')\,dt'\,dy' \\
&+\int_{\mathbb{R}^{N-1}} B'(\tilde{H}(\underline{h}'(t)))\langle\nabla\underline{H}(\underline{h}'(t)\underline{w}),(\hat{D}\psi(t,y'))^{-1}\nabla_{y'}\tilde{\varphi}(t,y')\rangle\,dt'\,dy' \\
&=\int_{\mathbb{R}^{N-1}} f(\underline{h}(t))\tilde{\varphi}(t,y')\,dt'\,dy',
\end{aligned}$$

where we set $(\hat{D}\psi(t,y'))^{-1})_{ij} := (D\psi(t,y'))^{-1})_{ij}$, for $i=1,\ldots,N-1$, $j=2,\ldots,N-1$ and $\nabla_{y'}$ denotes the gradient of the variables y'. Now we consider $\tilde{\varphi}(t,y')=\tilde{\varphi}_1(t)\tilde{\varphi}_2(y')$, with $\tilde{\varphi}_1\in C^1_c(\mathbb{R})$ and $\tilde{\varphi}_2\in C^1_c(\mathbb{R}^{N-2})$, satisfying

$$\int_{R^{N-2}}\tilde{\varphi}_2(y')\,dy'=1 \qquad \text{and} \qquad \int_{R^{N-2}}|\nabla\tilde{\varphi}_2(y')|\,dy'=0. \tag{3.3.42}$$

By (3.3.41) and (3.3.42) we have that

$$\int_{\mathbb{R}} B'(\tilde{H}(\underline{h}'(t)))\tilde{H}'(\underline{h}'(t))\tilde{\varphi}_1'(t)\,dt=\int_{\mathbb{R}} f(\underline{h}(t))\tilde{\varphi}_1(t)\,dt \tag{3.3.43}$$

for all $\tilde{\varphi}_1\in C^1_c(\mathbb{R})$.

Additionally, we have:

$$\underline{h}(t)<\sup u \tag{3.3.44}$$

for any $t\in\mathbb{R}$ because if $\underline{h}(t)=\sup u$ for some t, since $\partial_{x_N}u\geq 0$, then

$$\sup u=\underline{h}(t)=\underline{u}(\underline{w}t)=\lim_{s\to-\infty}u(\underline{w}t,s)\leq u(\underline{w}t,0)\leq\sup u,$$

so

$$\sup u=u(\underline{w}t,0),$$

which implies $\nabla u(\underline{w}t,0)=0$, contradicting the fact that $\{\nabla u=0\}=\emptyset$. Next, we use the classification from Lemma 3.3.5: if $\underline{h}$ satisfies conditions (i) or (ii) of the Lemma 3.3.5, we fall into cases (i) or (ii), and we are done. Now, we show that case (iii) of Lemma 3.3.5 is impossible in this situation. Indeed, if case (iii) were to hold, then by the fact that $\partial_{x_N}u\geq 0$ and $\{\nabla u=0\}=\emptyset$, we have

$$F(\underline{h}(\beta_1))=F(\underline{h}(\beta_2))=F(\inf\underline{h})=F(\inf\underline{u})=F(\inf u).$$

By (3.3.38) we obtain

$$F(\underline{h}(\beta_1))=F(\underline{h}(\beta_2))=c_u,$$

so from Remark 3.3.8 we deduce that

$$\underline{h}(\beta_1),\underline{h}(\beta_2)\in\{\inf u,\sup u\}.$$

By (3.3.44), we get that $\underline{h}(\beta_1) = \underline{h}(\beta_2) = \inf u$, which contradicts the fact that $\underline{h}$ is strictly monotone in (β_1, β_2).

Thus, the only remaining possibility is that $\underline{h}$ satisfies condition (iv) of Lemma 3.3.5. By changing t to $-t$ if necessary, we can consider the case where the interval in (iv) of Lemma 3.3.5 is of the form $(-\infty, \beta)$. By the fact that $\partial_{x_N} u \geq 0$, $\{\nabla u = 0\} = \emptyset$, and by (3.3.38) we have

$$F(\underline{h}(\beta)) = F(\inf \underline{u}) = F(\inf u) = c_u.$$

Thus by Remark 3.3.8 and (3.3.44) we get $\underline{h}(\beta) = \inf u$. Now, if $\underline{h}(t) = \inf u$ for $t \geq \beta$, then we are in case (iii). Thus, we can assume there exists some $t^* > \beta$ such that $\underline{h}(t^*) > \inf u$. This implies there must be some $t' > \beta$ where $h'(t') > 0$. Consequently, there exists an interval (β_2, β_3), as large as possible, with $\beta_2 \geq \beta =: \beta_1$ and $\beta_3 \in (\beta_2, +\infty) \cup \{+\infty\}$ such that $h'(t) > 0$ for any $t \in (\beta_2, \beta_3)$.

If $\beta_3 \neq +\infty$ then $h'(\beta_2) = h'(\beta_3) = 0$, and by (3.3.16), we would revert to case (iii) of Lemma 3.3.5, which we have already shown to be impossible.

Therefore, $\beta_3 = +\infty$. Similarly, $\underline{h}(t)$ must equal to $\inf u$ in $[\beta_1, \beta_2]$, because if not, there would exist an interval $(\beta_1', \beta_2') \subset [\beta_1, \beta_2]$ such that $\underline{h}'(t) \neq 0$ in (β_1', β_2') and $\underline{h}'(\beta_1') = \underline{h}'(\beta_2') = 0$, leading us back to the impossible case (iii). This shows that $\underline{h}'(t) \neq 0$, for $t < \beta_1$ and $t > \beta_2$ and $\underline{h}(t) = \inf u$ for $t \in [\beta_1, \beta_2]$ resulting in case (v).

□

Given $v : \mathbb{R}^N \to \mathbb{R}$, with $|\nabla v| \in L^\infty(\mathbb{R}^N)$, we consider, for $R > 0$, the energy

$$E_R(v) := \int_{B_R} \Lambda_2(H(\nabla v)) - F(v).$$

We give the following result that controls the energy of a solution v with the one v^t, up to a term of order R^{N-1}.

Lemma 3.3.10. *Let $u \in C^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ be a weak solution of (3.1.1), with $\partial_{x_N} u \geq 0$ and $\{\nabla u = 0\} = \emptyset$. Then exists $C > 0$, depending only on N, u, H and B such that*

$$E_R(u) \leq E_R(u^t) + CR^{N-1} \tag{3.3.45}$$

for any $t \in \mathbb{R}$.

Proof. Since $u \in C^2(\mathbb{R}^N)$, we have that u, therefore u^t, is a strong solution of (3.1.1). Integrating by parts we have

$$\begin{aligned}
\partial_t E_R(u^t) &= \int_{B_R} B'(H(\nabla u^t))\langle \nabla H(\nabla u^t), \partial_t(\nabla u^t)\rangle - f(u^t)\partial_t u^t \\
&= \int_{B_R} B'(H(\nabla u^t))\langle \nabla H(\nabla u^t), \partial_t(\nabla u^t)\rangle + \operatorname{div}\left(B'(H(\nabla u^t))\nabla H(\nabla u^t)\right)\partial_t u^t \\
&= \int_{\partial B_R} B'(H(\nabla u^t))\partial_t u^t \langle \nabla H(\nabla u^t), \eta\rangle d\sigma,
\end{aligned} \tag{3.3.46}$$

where η is the exterior normal of B_R.

Since ∇H is a $0-$homogeneous function and $B'(t) \in L^{\infty}_{loc}([0,+\infty))$, by (3.3.46), and since $\partial_{x_N} u \geq 0$, we get

$$\begin{aligned} E_R(u^t) - E_R(u) &= \int_0^t \partial_s E_R(u^s) ds \geq -C \int_{\partial B_R} \int_0^t \partial_s u^s d\sigma ds \\ &= -\tilde{C} \int_{\partial B_r} (u^t - u) d\sigma \geq -2\tilde{C} \|u\|_{L^\infty(\mathbb{R}^N)} |\partial B_r| \\ &\geq -CR^{N-1}, \end{aligned} \tag{3.3.47}$$

where C is a positive constant depending on B, H, N and u. □

Lemma 3.3.11. *Let $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ be a stable weak solution of (3.1.1). Suppose that $\{\nabla u = 0\} = \emptyset$ and $\partial_{x_N} u \geq 0$. Let $\underline{u}$ has one-dimensional symmetry. Then*

$$\int_{B_R} \Lambda_2(H(\nabla u)) - F(u) + c_u dx \leq CR^{N-1} \tag{3.3.48}$$

where C is a positive constant depending on B, H, N and u

Proof. We suppose that there exist $\underline{w} \in S^{N-2}$ and $\underline{h} : \mathbb{R} \to \mathbb{R}$ such that $\underline{u}(x') = \underline{h}(x' \cdot w)$, for any $x' \in \mathbb{R}^{N-1}$. Recalling (3.3.19), we set $\tilde{H}(t) := \underline{H}(tw) := H(tw, 0)$.

First, we will show that

$$\int_{\mathbb{R}} \Lambda_2(\tilde{H}(\underline{h}'(t))) - F(\underline{h}(t)) + c_u dt < +\infty. \tag{3.3.49}$$

Indeed, since $\partial_{x_N} u \geq 0$, we remark that $c_u = F(\inf u) = F(\inf \underline{h})$. By Lemma 3.3.4, by (h_B)-(iii) and by (3.3.38), we get

$$\begin{aligned} &\int_{-\infty}^{+\infty} c_u - F(\underline{h}(t))\, dt = \int_{-\infty}^{+\infty} \Lambda_1(\tilde{H}(\underline{h}'(t)))\, dt \\ &= \int_{-\infty}^{+\infty} \int_0^{\tilde{H}(\underline{h}'(t))} B''(s) s\, ds\, dt \leq \hat{C} \int_{-\infty}^{+\infty} \int_0^{\tilde{H}(\underline{h}'(t))} B'(s)\, ds\, dt \\ &= \hat{C} \int_{-\infty}^{+\infty} \Lambda_2(\tilde{H}(\underline{h}'(t)))\, dt, \end{aligned} \tag{3.3.50}$$

where $\hat{C}$ is a positive constant depending on B. Now, (3.3.49) is proved, if

$$\int_{-\infty}^{+\infty} \Lambda_2(\tilde{H}(\underline{h}'(t)))\, dt < +\infty. \tag{3.3.51}$$

To prove this, we consider different cases in Lemma 3.3.9. We consider case (v) in Lemma 3.3.9, the other cases are similar. Since $\underline{h} \in W^{1,\infty}(\mathbb{R})$ and $B'(t) \in L^{\infty}_{loc}[0,+\infty)$, using the fact that H is a norm equivalent to Euclidean one (see (1.2.3)) we get

$$\begin{aligned} &\int_{-\infty}^{+\infty} \Lambda_2(\tilde{H}(\underline{h}'(t)))\, dt = \int_{-\infty}^{+\infty} \int_0^{\tilde{H}(\underline{h}'(t))} B'(s)\, ds \\ &\leq \|B'\|_{L^\infty([0,\tilde{H}(\underline{h}'(t))])} \alpha_2 \int_{-\infty}^{+\infty} |\underline{h}'(t)|\, dt \\ &= \|B'\|_{L^\infty([0,\tilde{H}(\underline{h}'(t))])} \alpha_2 \left(\lim_{\alpha \to -\infty} \int_\alpha^{\beta_1} -h'(t) + \lim_{\beta \to \infty} \int_{\beta_2}^{\beta} h'(t) \right) \\ &= \|B'\|_{L^\infty([0,\tilde{H}(\underline{h}'(t))])} \alpha_2 \left(\lim_{\alpha \to -\infty} h(\alpha) - h(\beta_1) + \lim_{\beta \to \infty} h(\beta) - h(\beta_2) \right) \\ &\leq 4 \|B'\|_{L^\infty([0,\tilde{H}(\underline{h}'(t))])} \alpha_2 \|u\|_{L^\infty(\mathbb{R}^N)} < +\infty, \end{aligned} \tag{3.3.52}$$

and (3.3.49) is done.

Therefore

$$
\begin{aligned}
&\int_{B_R} \Lambda_2(\underline{H}(\nabla \underline{u})) - F(\underline{u}) + c_u dx \\
&= \int_{B_R} \Lambda_2(\tilde{H}(\underline{h}'(t))) - F(h(t)) + c_u dx'' dt dx_N \leq \tilde{C} R^{N-1},
\end{aligned}
\tag{3.3.53}
$$

where $\tilde{C}$ is a positive constant depending on B, H, and u. By Lemma 3.3.10 and (3.3.53) we get

$$
\begin{aligned}
&\int_{B_R} \Lambda_2(H(\nabla u)) - F(u) + c_u dx = E_R(u) + c_u|B_R| \\
&\leq \lim_{t \to +\infty} E_R(u^t) + c_u|B_R| + CR^{N-1} \\
&\leq \int_{B_R} \Lambda_2(\underline{H}(\nabla \underline{u})) - F(\underline{u}) + c_u dx + CR^{N-1} \leq CR^{N-1},
\end{aligned}
\tag{3.3.54}
$$

where C is a positive constant depending on B, H, N and u.

□

3.4 Proof of Theorem 3.1.4

We start this section, showing that the weak solutions of (3.1.1) that are monotone in one variable are stable.

Lemma 3.4.1. *Let $u \in C^1(\mathbb{R}^N)$, be a weak solution of (3.1.1). Suppose $\partial_{x_N} u \geq 0$. Then u is stable.*

Proof. Fixed $\delta > 0$, let us consider

$$
\psi := \frac{\varphi^2}{u_N + \delta}
\tag{3.4.1}
$$

for $\varphi \in C^1_c(\mathbb{R}^N \setminus \{\nabla u = 0\})$. Since $\partial_{x_N} u \geq 0$, $\psi \in C^1_c(\mathbb{R}^N \setminus \{\nabla u = 0\})$ and using (3.4.1) in (3.2.3) we get

$$
\begin{aligned}
&\int_{\mathbb{R}^N} B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_N\rangle \langle \nabla H(\nabla u), \nabla \varphi\rangle \frac{2\varphi}{u_N + \delta} \\
&- B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_N\rangle \langle \nabla H(\nabla u), \nabla u_N\rangle \frac{\varphi^2}{(u_N + \delta)^2} \\
&+ B'(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_N, \nabla \varphi\rangle \frac{2\varphi}{u_N + \delta} \\
&- B'(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_N, \nabla u_N\rangle \frac{\varphi^2}{(u_N + \delta)^2} = \int_{\mathbb{R}^N} f'(u) u_N \frac{\varphi^2}{u_N + \delta}.
\end{aligned}
\tag{3.4.2}
$$

Recalling the definition of the matrix A (see (3.1.4)), we deduce

$$
\begin{aligned}
0 = \int_{\mathbb{R}^N} & B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_N\rangle\langle \nabla H(\nabla u), \nabla \varphi\rangle \frac{2\varphi}{u_N+\delta} \\
& - B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_N\rangle\langle \nabla H(\nabla u), \nabla u_N\rangle \frac{\varphi^2}{(u_N+\delta)^2} \\
& + B'(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_N, \nabla \varphi\rangle \frac{2\varphi}{u_N+\delta} \\
& - B'(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_N, \nabla u_N\rangle \frac{\varphi^2}{(u_N+\delta)^2} - f'(u)u_N\frac{\varphi^2}{u_N+\delta} \\
= \int_{\mathbb{R}^N} & -\langle A(\nabla u)\left(\nabla\varphi - \frac{\varphi}{u_N+\delta}\nabla u_N\right), \left(\nabla\varphi\frac{\varphi}{u_N+\delta}\nabla u_N\right)\rangle \\
& \quad + \langle A(\nabla u)\nabla\varphi, \varphi\rangle - f'(u)u_N\frac{\varphi^2}{u_N+\delta}.
\end{aligned}
\tag{3.4.3}
$$

Since A is definite positive by Lemma 3.2.1, for δ sufficiently small we obtain

$$
\int_{\mathbb{R}^N} \langle A(\nabla u)\nabla\varphi, \varphi\rangle - f'(u)\varphi^2 \geq 0, \tag{3.4.4}
$$

and therefore the thesis.

$\square$

Given a function $v \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$, let us consider the graph of a function $Y(x) := (x, v(x)) \in \mathbb{R}^N \times \mathbb{R}$. The following result is useful for the proof of Theorem 3.1.4.

Lemma 3.4.2. *Suppose $v \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ satisfies*

$$
\int_{|Y(x)|\leq r} B'(H(\nabla v))H(\nabla v)dx \leq Cr^2 \tag{3.4.5}
$$

for r sufficiently large and C positive constant. Then we have

$$
\int_{\sqrt{r}\leq|Y|\leq r} \frac{B'(H(\nabla v))H(\nabla v)}{|Y|^2} \leq C\ln r \tag{3.4.6}
$$

for r sufficiently large and $C > 0$.

Proof. Since

$$
2\int_{|Y|}^{r} s^{-3}\, ds = |Y|^{-2} - r^{-2}, \tag{3.4.7}
$$

using the Fubini theorem and by (3.4.5), we get

$$
\begin{aligned}
& \int_{\sqrt{r}\leq|Y|\leq r} B'(H(\nabla v))H(\nabla v)(|Y|^{-2} - r^{-2}) \\
& = 2\int_{|Y|}^{r}\int_{\sqrt{r}\leq|Y|\leq r} B'(H(\nabla v))H(\nabla v)s^{-3}\, ds\, dx \\
& = 2\int_{\sqrt{r}}^{r} s^{-3}\int_{\sqrt{r}\leq|Y|\leq s} B'(H(\nabla v))H(\nabla v)dxds \\
& \leq C\int_{\sqrt{r}}^{r} s^{-1}ds \leq C\ln r
\end{aligned}
\tag{3.4.8}
$$

if r is large enough. By (3.4.8), using the fact that $|Y|^{-2} = (|Y|^{-2} - r^{-2}) + r^{-2}$ and (3.4.5) we have

$$\begin{aligned}&\int_{\sqrt{r}\le|Y|\le r} B'(H(\nabla v))H(\nabla v)|Y|^{-2}\\ &\le \int_{\sqrt{r}\le|Y|\le r} B'(H(\nabla v))H(\nabla v)r^{-2} + C\ln r\\ &\le C + C\ln r,\end{aligned} \tag{3.4.9}$$

where C is a positive constant.

□

Proof of Theorem 3.1.4. Since $\underline{u}$ and $\overline{u}$ are functions on $\mathbb{R}^2$ using (3.3.20), we have

$$k_{1,\underline{u}} = k_{1,\overline{u}} = 0 = \nabla_{L_{\underline{u},x'}}\underline{H}(\nabla\underline{u}) = \nabla_{L_{\overline{u},x'}}\overline{H}(\nabla\overline{u}).$$

Therefore $\underline{u}$ and $\overline{u}$ have one-dimensional symmetry.

Now we prove that u satisfies (3.4.5). By (3.1.7) and by assumptions (h_B)-(iii) we remark that

$$B'(t)t - \Lambda_2(t) = \Lambda_1(t), \qquad \text{and} \qquad \Lambda_1(t) \le C(B)\Lambda_2(t) \tag{3.4.10}$$

for any $t \ge 0$, and where $C(B)$ is a positive constant depending on B. Using Lemma 3.3.11 and by (3.4.10), we get

$$\begin{aligned}&\int_{|(x,u(x))|\le R} B'(H(\nabla u))H(\nabla u)\\ &\le \int_{B_R} B'(H(\nabla u))H(\nabla u) \le C(B)\int_{B_R}\Lambda_2(H(\nabla u))\\ &\le C(B)\int_{B_R}\Lambda_2(H(\nabla u)) - F(u) + c_u \le C(B,H,N,u)R^2,\end{aligned} \tag{3.4.11}$$

where $C(B,H,N,u)$ is a positive constant depending on B,H,N and u.

Since $\nabla u \in L^\infty(\mathbb{R}^N)$, we have

$$H(\nabla u)^4 B'(H(\nabla u)) \le C(u,H)B'(H(\nabla u))H(\nabla u), \tag{3.4.12}$$

where $C(u,H)$ is a positive constant depending on u and H.

Fixed $R > 0$, we now consider the function

$$\varphi_R(x) := \begin{cases} 1 & \text{if } |Y| < \sqrt{R} \\ \frac{2\ln(R/|Y|)}{\ln R} & \text{if } \sqrt{R} < |Y| < R \\ 0 & \text{if } |Y| \ge R. \end{cases} \tag{3.4.13}$$

By construction and using the fact that H is a norm equivalent to euclidean norm we have

$$|\nabla\varphi_R(x)|^2 \le \tilde{C}_H\frac{|x|^2 + u(x)^2H(\nabla u)^2}{|Y|^4(\ln R)^2}, \tag{3.4.14}$$

where $\tilde{C}_H$ is a positive constant depending on H.

Since H is 1-homogeneous function, there exist positive constants $M, \overline{M}$ such that

$$|\nabla H(\xi)| \le M, \qquad |D^2H(\xi)| \le \frac{\overline{M}}{H(\xi)} \quad \forall\xi \in \mathbb{R}^N \setminus \{0\}. \tag{3.4.15}$$

Taking (3.4.13) in (3.2.30), by (3.4.11), (3.4.12), (3.4.14), (3.4.15), by Lemma 3.4.2 and by assumptions (h_B)-(iii), we get

$$
\begin{aligned}
&\int_{\{\nabla u \neq 0\} \cap B_R} \frac{B'(H(\nabla u))}{H(\nabla u))} |\nabla u|^2 \sum_{i=1}^{N-1} k_{i,u}^2 + B''(H(\nabla u)) \left|\nabla_{L_{u,x}} H(\nabla u)\right|^2 \\
&\leq C_H \int_{\mathbb{R}^3} \langle A(\nabla u)\nabla\varphi, \nabla\varphi\rangle |\nabla u|^2 \\
&= C_H \int_{\mathbb{R}^3} \left(B''(H(\nabla u))\langle \nabla H(\nabla u), \nabla\varphi\rangle^2 + B'(H(\nabla u))\langle D^2 H(\nabla u)\nabla\varphi, \nabla\varphi\rangle\right) |\nabla u|^2 \\
&\leq C_H \int_{\mathbb{R}^3} \left(M^2 B''(H(\nabla u)) + \overline{M}\frac{B'(H(\nabla u))}{H(\nabla u)}\right) |\nabla\varphi|^2 |\nabla u|^2 \\
&\leq C(H,B) \int_{B_R \setminus B_{\sqrt{R}}} \frac{H(\nabla u)^2 B'(H(\nabla u))|x|^2}{|Y|^4 (\ln R)^2} + C(H,B) \int_{B_R \setminus B_{\sqrt{R}}} \frac{H(\nabla u)^4 B'(H(\nabla u))|u|^2}{|Y|^4 (\ln R)^2} \\
&\leq C(H,B) \int_{B_R \setminus B_{\sqrt{R}}} \frac{H(\nabla u) B'(H(\nabla u))}{|Y|^2 (\ln R)^2} + C(H,B,u) \int_{B_R \setminus B_{\sqrt{R}}} \frac{H(\nabla u) B'(H(\nabla u))}{|Y|^2 \ln R^2} \\
&\leq C(H,B,N,u) \frac{\ln R}{(\ln R)^2},
\end{aligned}
\tag{3.4.16}
$$

where $C(H,B,N,u)$ is a positive constant depending on H, B, N and u. For R arbitrarily large we obtain that u possesses one-dimensional symmetry. □

Chapter 4

Asymptotic behaviour of solutions to the anisotropic doubly critical equation

4.1 Main Results

This chapter is devoted to the study of the following anisotropic doubly critical problem

$$\begin{cases} -\Delta_p^H u - \dfrac{\gamma}{H^\circ(x)^p} u^{p-1} = u^{p^*-1} & \text{in } \mathbb{R}^N \\ u > 0 & \text{in } \mathbb{R}^N \\ u \in \mathcal{D}^{1,p}(\mathbb{R}^N), \end{cases} \tag{$\mathcal{P}_H$}$$

where $1 < p < N$, $p^* := Np/(N-p)$ is the Sobolev critical exponent, $0 \leq \gamma < C_H := ((N-p)/p)^p$ is the Hardy constant and

$$\Delta_p^H u := \operatorname{div}(H^{p-1}(\nabla u)\nabla H(\nabla u)), \tag{4.1.1}$$

where Δ_p^H is the so-called anisotropic p-Laplacian or Finsler p-Laplacian. We point out that H is a Finsler type norm and H° is its the dual norm (H satisfies assumptions (h_H), see Chapter 1, in particular we refer to Section 1.2 for further details). In particular, when $H(\xi) = |\xi| = H^\circ(\xi)$ the Finsler type p-Laplacian coincides with the classical p-Laplacian, and, hence it is singular when $1 < p < 2$ and degenerate when $p > 2$. Then, according with standard regularity theory [67, 157] and the regularity results in the anisotropic framework [7, 35], we say that any solution of ($\mathcal{P}_H$) has to be understood in the weak distributional meaning, i.e. $u \in D^{1,p}(\mathbb{R}^N)$ satisfies the following integral equality

$$\int_{\mathbb{R}^N} \left(H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla\varphi\rangle - \frac{\gamma}{H^\circ(x)^p} u^{p-1}\varphi \right) dx = \int_{\mathbb{R}^N} u^{p^*-1}\varphi \, dx \quad \forall \varphi \in C_c^\infty(\mathbb{R}^N). \tag{4.1.2}$$

The study of critical problems has been a rich field of investigation in the context of both Euclidean and anisotropic frameworks. For the Euclidean framework, when we consider $H(\xi) = |\xi| = H^\circ(\xi)$ in ($\mathcal{P}_H$), we deal with

$$-\Delta_p u - \frac{\gamma}{|x|^p} u^{p-1} = u^{p^*-1} \qquad \text{in } \mathbb{R}^N. \tag{4.1.3}$$

In the seminal paper by Caffarelli, Gidas, and Spruck [33], any positive solution to (4.1.3) with $p = 2$, $N \geq 3$, and $\gamma = 0$ was classified. Prior to this, Gidas, Ni, and Nirenberg

[96] obtained a similar result under stronger decay assumptions for solutions. Additionally, the celebrated work of Gidas and Spruck [97] provided a complete answer in the subcritical case, proving Liouville-type theorems.

In the quasilinear framework, the situation is much more involved due to the nonlinear nature of the operator. Recently, a classification result of positive solutions to to (4.1.3) with $p > 2$, $\gamma = 0$ and $u \in \mathcal{D}^{1,p}(\mathbb{R}^N) := \{u \in L^{p^*}(\mathbb{R}^N) \mid \nabla u \in L^p(\mathbb{R}^N)\}$ was obtained in [140]. The proof of this result is based on a refined version of the well-known moving plane method of Alexandrov-Serrin [4, 144] and on some a priori estimates of the solutions and their gradients, proved in [163].

To be more precise, we note that the classification result of positive solution to the Sobolev critical quasilinear equation with finite energy started in [55] in the case, and then was extended in [163] for every $1 < p < 2$. Subsequently the full case was obtained in [140].

Recently, new partial results on the classification of positive solutions without a priori assumptions on the energy of solutions were presented in [36, 135, 164].

In the anisotropic setting, Ciraolo, Figalli, and Roncoroni [49] obtained a complete classification result for positive solutions to ($\mathcal{P}_H$) with $\gamma = 0$ using different techniques that do not require the moving plane method, which is inapplicable in the anisotropic context due to the lack of invariance.

When $\gamma \neq 0$ the situation is significantly different. In the seminal paper by Terracini [155], the classification result for positive solutions to (4.1.3) was established for the first time in the case $p = 2$. The author proved the existence of solutions using a minimization argument based on the concentration and compactness principle. Subsequently, she proved that any solution to this problem is radial and radially decreasing about the origin by combining the moving-plane technique and the use of the Kelvin transformation, in the same spirit as [33].

The case where $p \neq 2$, $\gamma \neq 0$ is much more complex and is addressed in [134]. The techniques used in this case are mainly based on a fine asymptotic analysis at infinity and refined versions of the moving plane procedure. Additionally, some asymptotic estimates proved in [166, 167] are also employed.

Our aim is to prove some decay estimates for positive weak solutions to ($\mathcal{P}_H$) in the anisotropic framework $1 < p < N$ and $\gamma \neq 0$. More precisely, our first main result is the following:

Theorem 4.1.1. *Let $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a weak solution of ($\mathcal{P}_H$) with $1 < p < N$, $0 \leq \gamma < C_H$. Then there exist positive constants $0 < R_1 < 1 < R_2$ depending on N, p, γ and u, such that*

$$\frac{c_1}{[H^\circ(x)]^{\mu_1}} \leq u(x) \leq \frac{C_1}{[H^\circ(x)]^{\mu_1}} \qquad x \in \mathcal{B}_{R_1}^{H^\circ}, \tag{4.1.4}$$

and

$$\frac{c_2}{[H^\circ(x)]^{\mu_2}} \leq u(x) \leq \frac{C_2}{[H^\circ(x)]^{\mu_2}} \qquad x \in (\mathcal{B}_{R_2}^{H^\circ})^c, \tag{4.1.5}$$

where μ_1, μ_2 are the solutions of

$$\mu^{p-2}[(p-1)\mu^2 - (N-p)\mu] + \gamma = 0, \tag{4.1.6}$$

C_1, C_2 are positive constants depending on N, p, γ, H and u, c_1 is a positive constant depending on $N, p, \gamma, H, R_1, \mu_1$ and u, c_2 is a positive constant depending on $N, p, \gamma, H, R_2, \mu_2$ and u.

Remark 4.1.2. *In the following we shall assume that* $\mu_1 < \mu_2$ *and it is easy to see that*

$$0 \leq \mu_1 < \frac{N-p}{p} < \mu_2 \leq \frac{N-p}{p-1}.$$

furthermore $\mathcal{B}_R^{H^\circ}$ *is the dual anisotropic ball also known as Frank diagram (see Section 1.2 for further details).*

In the proof we will also exploit some clever ideas from [166] facing the difficulties of the anisotropic issue. A different approach is in fact needed for the study of the asymptotic behaviour of the gradient. In particular, the fact that the moving plane plane technique cannot be applied, a crucial point is given by the following classification result:

Theorem 4.1.3. *Let* $v \in C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{0\})$ *be a positive weak solution of the equation*

$$-\Delta_p^H v - \frac{\gamma}{[H^\circ(x)]^p} v^{p-1} = 0 \text{ in } \mathbb{R}^N \setminus \{0\}, \tag{4.1.7}$$

where $0 \leq \gamma < C_H$. *Assume that there exist two positive constants* C *and* c *such that*

$$\frac{c}{[H^\circ(x)]^{\mu_i}} \leq v(x) \leq \frac{C}{[H^\circ(x)]^{\mu_i}} \ \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{4.1.8}$$

where μ_i $(i = 1, 2)$ *are the roots of* (4.1.6) *and suppose that there exists a positive constant* $\hat{C}$ *such that*

$$|\nabla v(x)| \leq \frac{\hat{C}}{[H^\circ(x)]^{\mu_i+1}} \quad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{4.1.9}$$

then

$$v(x) = \frac{\bar{c}}{[H^\circ(x)]^{\mu_i}}, \tag{4.1.10}$$

for some $\bar{c} > 0$.

Theorem 4.1.3 is new and interesting in itself. The proof is very much different than the ones available in the euclidean case $H(\xi) = |\xi|$. Here we shall exploit it to deduce the precise asymptotic estimates for the gradient. More precisely we have the following:

Theorem 4.1.4. *Let* $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ *be a weak solution of* $(\mathcal{P}_H)$ *with* $1 < p < N$, *and* $0 \leq \gamma < C_H$. *Then there exist positive constants* $\tilde{c}, \tilde{C}$ *depending on* N, p, γ, H *and* u *such that*

$$\frac{\tilde{c}}{[H^\circ(x)]^{\mu_1+1}} \leq |\nabla u(x)| \leq \frac{\tilde{C}}{[H^\circ(x)]^{\mu_1+1}} \qquad x \in \mathcal{B}_{R_1}^{H^\circ}, \tag{4.1.11}$$

and

$$\frac{\tilde{c}}{[H^\circ(x)]^{\mu_2+1}} \leq |\nabla u(x)| \leq \frac{\tilde{C}}{[H^\circ(x)]^{\mu_2+1}} \qquad x \in (\mathcal{B}_{R_2}^{H^\circ})^c, \tag{4.1.12}$$

where μ_1, μ_2 *are roots of* (4.1.6) *as in Theorem 4.1.1, and* $0 < R_1 < 1 < R_2$ *are constants depending on* N, p, γ *and* u.

The Chapter is structured as follows:

- In Section 4.2 we state some technical lemmas that will be crucial in the proof of the main results.
- In Section 4.3 we prove some preliminary estimates, elliptic estimates and weak comparison principles in bounded and exterior domains that will be essential in the proof of Theorem 4.1.1.

- In Section 4.4 we give the proof of decay estimates of solutions to ($\mathcal{P}_H$) near the origin and at infinity, i.e. we prove Theorem 4.1.1. The, using this result we also prove decay estimates for the gradient of positive weak solutions to ($\mathcal{P}_H$) near the origin and at infinity, i.e. we prove Theorem 4.1.4.
- Although the existence of solutions can be easy deduced in the radial-anisotropic setting, in the Section 4.5 we show that problem ($\mathcal{P}_H$) admits at least a positive solution $u \in D^{1,p}(\mathbb{R}^N)$ that minimizes the Hardy-Sobolev anisotropic inequality. This result follows using classical arguments (see also [155]).

4.2 Preliminaries

In this section, we will present results that will be useful in the following parts.

We state now the Hardy inequality for the anisotropic operator $\Delta_p^H u$, defined in (4.1.1). We refer to [161, Proposition 7.5].

Theorem 4.2.1 (Hardy inequality)**.** *For any H satisfying the assumption (h_H) and any $u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ and $1 < p < N$,*

$$C_H \int_{\mathbb{R}^N} \frac{|u|^p}{H^\circ(x)^p}\, dx \leq \int_{\mathbb{R}^N} H^p(\nabla u)\, dx, \tag{4.2.1}$$

where $C_H = ((N-p)/p)^p$ is optimal.

Now we prove a technical lemma that will be very important in the proof of the asymptotic estimates.

Lemma 4.2.2. *Let $p > 1$ and $a, b \geq 0$. Then, for all $\delta > 0$ there exist $\mathcal{C}_\delta > 0$ such that*

$$a^p \geq \frac{1}{1+2^{p+1}\delta}(a+b)^p - \mathcal{C}_\delta b^p. \tag{4.2.2}$$

Proof. Let us consider $p > 1$ as follows:

$$p = \lfloor p \rfloor + \{p\},$$

where $\lfloor \cdot \rfloor$ is the floor function and $\{\cdot\}$ is the mantissa function. Without loss of generality we assume that $\{p\} \neq 0$ and, moreover, we set $m := \lfloor p \rfloor$. Hence, we have

$$(a+b)^p = (a+b)^{\{p\}}(a+b)^m = (a+b)^{\{p\}} \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k = (\star) \tag{4.2.3}$$

Noticing that $0 \leq \{p\} < 1$ it follows that

$$(a+b)^{\{p\}} \leq a^{\{p\}} + b^{\{p\}}.$$

Using this inequality in (4.2.3), we deduce

$$\begin{aligned} (\star) \leq& a^{\{p\}} \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k + b^{\{p\}} \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k \\ =& a^p + b^p + \sum_{k=1}^{m} \binom{m}{k} a^{p-k} b^k + \sum_{k=0}^{m-1} \binom{m}{k} a^{m-k} b^{k+\{p\}} = (*), \end{aligned} \tag{4.2.4}$$

where we used the fact that $p = m + \{p\}$. Now, we can apply the weighted Young's inequality to each member of the first sum with conjugate exponents $(p/(p-k), p/k)$ and to each member of the second sum with conjugate exponents $(p/(m-k), p/(k+\{p\}))$ as follows

$$\begin{aligned} a^{p-k}b^k &\le \frac{p-k}{p}\delta a^p + \frac{k}{p}\mathcal{C}_\delta b^p \le \delta a^p + \mathcal{C}_\delta b^p, \\ a^{m-k}b^{k+\{p\}} &\le \delta a^p + \mathcal{C}_\delta b^p. \end{aligned} \tag{4.2.5}$$

Hence, using this estimate we deduce

$$\begin{aligned} (*) \le& a^p + b^p + (\delta a^p + \mathcal{C}_\delta b^p)\sum_{k=1}^{m}\binom{m}{k} + (\delta a^p + \mathcal{C}_\delta b^p)\sum_{k=0}^{m-1}\binom{m}{k} \\ &\le \left(1+2^{p+1}\delta\right)a^p + \mathcal{C}_\delta b^p, \end{aligned} \tag{4.2.6}$$

where we renamed $\mathcal{C}_\delta := \left(1 + 2^{p+1}\mathcal{C}_\delta\right)$. Collecting (4.2.3), (4.2.4) and (4.2.6), we deduce that

$$a^p \ge \frac{1}{1+2^{p+1}\delta}(a+b)^p - \mathcal{C}_\delta b^p, \tag{4.2.7}$$

with $\mathcal{C}_\delta := \mathcal{C}_\delta/(1+2^{p+1}\delta)$, and hence the thesis (4.2.2). □

Finally, we recall a lemma (see Lemma 4.19 in [104]) that will be very useful in the proofs of our results.

Lemma 4.2.3 ([104])**.** *Let $\mathcal{L}$ and g be two nondecreasing functions on the interval $(0, \bar{R}]$, for some $\bar{R} > 0$. Suppose that it holds*

$$\mathcal{L}(\tau R) \le \sigma\mathcal{L}(R) + g(R) \quad \textit{for all } R \le \bar{R},$$

for some $0 < \sigma, \tau < 1$. Then, for any $\mu \in (0,1)$ and $R \le \bar{R}$ we have

$$\mathcal{L}(R) \le \frac{1}{\sigma}\left(\frac{R}{\bar{R}}\right)^{\alpha}\mathcal{L}(\bar{R}) + \frac{1}{1-\sigma}g(\bar{R}^\mu R^{1-\mu})$$

where $\alpha = \alpha(\sigma,\tau,\mu) = (1-\mu)\log\sigma/\log\tau$.

4.3 Preliminary asymptotic estimates and comparison principles

The aim of this section is to prove some preliminary estimates that will be crucial in the proofs of the main results.

Lemma 4.3.1. *There exists a positive constant τ depending only on N, p and γ such that for any $\bar{R} > 0$ and for any solution u to problem ($\mathcal{P}_H$) satisfying*

$$\|u\|_{L^{p^*}(\mathcal{B}^{H^\circ}_{\bar{R}})} + \|u\|_{L^{p^*}(\mathbb{R}^N \setminus \mathcal{B}^{H^\circ}_{1/\bar{R}})} \le \tau, \tag{4.3.1}$$

there exists a positive constant $\mathcal{C}$ depending only on N, p, γ and $\overline{R}$ such that

$$\|u\|_{L^{p^*}(\mathcal{B}^{H^\circ}_{R})} \le \mathcal{C}R^{\sigma_1} \quad \textit{for } R \le \bar{R}, \tag{4.3.2}$$

and that

$$\|u\|_{L^{p^*}((\mathcal{B}^{H^\circ}_{R})^c)} \le \frac{\mathcal{C}}{R^{\sigma_2}} \quad \textit{for } R \ge \frac{1}{\bar{R}}, \tag{4.3.3}$$

where σ_1, σ_2 are two positive constants depending on N, p and γ.

Proof. We start proving (4.3.2). To this aim let us consider $R > 0$ and a cut-off function $\eta \in C_c^\infty(\mathbb{R}^N)$ such that

$$\begin{cases} 0 \le \eta \le 1 & \text{in } \mathbb{R}^N \\ \eta \equiv 0 & \text{in } (\mathcal{B}_R^{H^\circ})^c \\ \eta \equiv 1 & \text{in } \mathcal{B}_{R/2}^{H^\circ} \\ |\nabla \eta| \le \frac{4}{R} & \text{in } \mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}. \end{cases} \tag{4.3.4}$$

By density argument it is possible to put $\varphi = \eta^p u$ as test function in (4.1.2), so that we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} p\eta^{p-1} u H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \eta\rangle \, dx + \int_{\mathbb{R}^N} \eta^p H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla u\rangle \, dx \\ &= \int_{\mathbb{R}^N} \frac{\gamma}{H^\circ(x)^p} u^{p-1}\eta^p u \, dx + \int_{\mathbb{R}^N} u^{p^*-1}\eta^p u \, dx. \end{aligned} \tag{4.3.5}$$

First of all, using Euler's Theorem (1.2.4), the 0-homogeneity of ∇H (1.2.6) and Schwarz's inequality, equation (4.3.5) becomes

$$\begin{aligned} &\int_{\mathbb{R}^N} H^p(\nabla u)\eta^p \, dx = \int_{\mathbb{R}^N} H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla u\rangle \eta^p \, dx \\ &\le p\, M \int_{\mathbb{R}^N} H^{p-1}(\nabla u)|\nabla \eta|\eta^{p-1} u \, dx + \int_{\mathbb{R}^N} \frac{\gamma}{H^\circ(x)^p} u^p \eta^p \, dx + \int_{\mathbb{R}^N} u^{p^*}\eta^p \, dx \end{aligned} \tag{4.3.6}$$

Recalling that H is 1-homogeneous function, using the weighted Young's inequality $ab \le \varepsilon a^{\frac{p}{p-1}} + \mathcal{C}_\varepsilon b^p$ on the first term of the right hand side of (4.3.6), for any $0 < \varepsilon < 1$ we have

$$\begin{aligned} \int_{\mathbb{R}^N} H^p(\eta \nabla u) \, dx \le \varepsilon \int_{\mathbb{R}^N} H^p(\eta \nabla u) \, dx + C(p, M, \varepsilon) \int_{\mathbb{R}^N} |\nabla \eta|^p u^p \, dx \\ + \int_{\mathbb{R}^N} \frac{\gamma}{H^\circ(x)^p} u^p \eta^p \, dx + \int_{\mathbb{R}^N} u^{p^*}\eta^p \, dx, \end{aligned} \tag{4.3.7}$$

where $C(p, M, \varepsilon) := (pM)^p \mathcal{C}_\varepsilon$. Now, noticing that $\nabla(\eta u) = u\nabla \eta + \eta \nabla u$, by the triangular inequality, we deduce that for every $p > 1$ it holds

$$H^p(\nabla(\eta u)) = H^p(u\nabla \eta + \eta \nabla u) \le [H(u\nabla \eta) + H(\eta \nabla u)]^p. \tag{4.3.8}$$

Thanks to (4.3.8) and applying Lemma 4.2.2 with $a = H(\eta \nabla u)$ and $b = H(u\nabla \eta)$, we deduce that

$$\int_{\mathbb{R}^N} H^p(\eta \nabla u) \, dx \ge \frac{1}{1 + 2^{p+1}\delta} \int_{\mathbb{R}^N} H^p(\nabla(\eta u)) \, dx - \mathcal{C}_\delta \int_{\mathbb{R}^N} H^p(u\nabla \eta) \, dx \tag{4.3.9}$$

Using (4.3.9) in (4.3.7) we obtain

$$\begin{aligned} \frac{1-\varepsilon}{1 + 2^{p+1}\delta} \int_{\mathbb{R}^N} H^p(\nabla(\eta u)) \, dx \le \mathcal{C}_\delta \int_{\mathbb{R}^N} H^p(u\nabla \eta) \, dx + C(p, M, \varepsilon) \int_{\mathbb{R}^N} |\nabla \eta|^p u^p \, dx \\ + \gamma \int_{\mathbb{R}^N} \frac{(u\eta)^p}{H^\circ(x)^p} \, dx + \int_{\mathbb{R}^N} u^{p^*}\eta^p \, dx \end{aligned} \tag{4.3.10}$$

Now, applying the anisotropic Hardy inequality (see Theorem 4.2.1 or [161]) and (1.2.3) we have

$$\begin{aligned} \frac{1-\varepsilon}{1 + 2^{p+1}\delta} \int_{\mathbb{R}^N} H^p(\nabla(\eta u)) \, dx \le (\alpha_2^p \mathcal{C}_\delta + C(p, M, \varepsilon)) \int_{\mathbb{R}^N} |\nabla \eta|^p u^p \, dx \\ + \frac{\gamma}{C_H} \int_{\mathbb{R}^N} H^p(\nabla(\eta u)) \, dx + \int_{\mathbb{R}^N} u^{p^*}\eta^p \, dx. \end{aligned} \tag{4.3.11}$$

Let us fix $\varepsilon, \delta > 0$ sufficiently small such that $\mathcal{C}_1 := (1-\varepsilon)/(1+2^{p+1}\delta) - \gamma/C_H > 0$, so that

$$\mathcal{C}_1 \int_{\mathbb{R}^N} H^p(\nabla(\eta u))\, dx \le \mathcal{C}_2 \int_{\mathbb{R}^N} |\nabla \eta|^p u^p\, dx + \int_{\mathbb{R}^N} u^{p^*} \eta^p\, dx, \tag{4.3.12}$$

where $\mathcal{C}_2 := C(p, M, \varepsilon) + \alpha_2^p \mathcal{C}_\delta$. By (1.2.3) we have

$$\alpha_1^p \mathcal{C}_1 \int_{\mathbb{R}^N} |\nabla(\eta u)|^p\, dx \le \mathcal{C}_2 \int_{\mathbb{R}^N} |\nabla \eta|^p u^p\, dx + \int_{\mathbb{R}^N} u^{p^*} \eta^p\, dx. \tag{4.3.13}$$

Now, using the Sobolev inequality in the left hand side of (4.3.13), the Hölder inequality and (4.3.4) in the right hand side, we obtain

$$\begin{aligned}
\alpha_1^p \mathcal{C}_1 \mathcal{C}_S^p \left(\int_{\mathbb{R}^N} |\eta u|^{p^*}\, dx \right)^{\frac{p}{p^*}} &\le \alpha_1^p \mathcal{C}_1 \int_{\mathbb{R}^N} |\nabla(\eta u)|^p\, dx \\
&\le \mathcal{C}_2 \int_{\mathbb{R}^N} |\nabla \eta|^p u^p\, dx + \int_{\mathbb{R}^N} u^{p^*-p} (\eta u)^p\, dx \\
&\le \mathcal{C}_2 \left(\int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} |\nabla \eta|^N\, dx \right)^{\frac{p}{N}} \left(\int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} u^{p^*}\, dx \right)^{\frac{p}{p^*}} \\
&\quad + \left(\int_{\mathcal{B}_R^{H^\circ}} u^{p^*}\, dx \right)^{\frac{p}{N}} \left(\int_{\mathbb{R}^N} (\eta u)^{p^*}\, dx \right)^{\frac{p}{p^*}} \\
&\le \mathcal{C}_2 C(p, N) \left(\int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} u^{p^*}\, dx \right)^{\frac{p}{p^*}} \\
&\quad + \|u\|_{p^*}^{p^*-p} \left(\int_{\mathbb{R}^N} (\eta u)^{p^*}\, dx \right)^{\frac{p}{p^*}},
\end{aligned} \tag{4.3.14}$$

hence we deduce

$$\begin{aligned}
\left(\int_{\mathbb{R}^N} |\eta u|^{p^*}\, dx \right)^{\frac{p}{p^*}} &\le \frac{\mathcal{C}_2 C(p, N)}{\alpha_1^p \mathcal{C}_1 \mathcal{C}_S^p} \left(\int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} u^{p^*}\, dx \right)^{\frac{p}{p^*}} \\
&\quad + \frac{\|u\|_{L^{p^*}(\mathcal{B}_R^{H^\circ})}^{p^*-p}}{\alpha_1^p \mathcal{C}_1 \mathcal{C}_S^p} \left(\int_{\mathbb{R}^N} (\eta u)^{p^*}\, dx \right)^{\frac{p}{p^*}},
\end{aligned} \tag{4.3.15}$$

where $C(p, N)$ is a positive constant depending on p and N.

Setting

$$\tau = \left(\frac{\alpha_1^p \mathcal{C}_1 \mathcal{C}_S^p}{2} \right)^{\frac{1}{p^*-p}},$$

and choosing $\overline{R} > 0$ sufficiently small such that (4.3.1) holds, then $\|u\|_{L^{p^*}(\mathcal{B}_R^{H^\circ})}^{p^*-p} / (\alpha_1^p \mathcal{C}_1 \mathcal{C}_S^p) \le 1/2$ for all $0 < R \le \overline{R}$. Hence we obtain that

$$\int_{\mathcal{B}_{R/2}^{H^\circ}} u^{p^*}\, dx \le \int_{\mathbb{R}^N} |\eta u|^{p^*}\, dx \le \bar{C} \int_{\mathcal{B}_R^{H^\circ} \setminus \mathcal{B}_{R/2}^{H^\circ}} u^{p^*}\, dx \qquad \forall\, 0 < R \le \bar{R},$$

where $\bar{C} := \left((2\mathcal{C}_2 C(p, N)) / (\alpha_1^p \mathcal{C}_1 \mathcal{C}_S^p) \right)^{\frac{p^*}{p}}$ and it depends only on N, p and γ. Denoting with $\mathcal{L}(R) := \displaystyle\int_{\mathcal{B}_R^{H^\circ}} u^{p^*}\, dx$ for $0 < R \le \bar{R}$, we get that

$$\mathcal{L}(R/2) \le \vartheta \mathcal{L}(R) \qquad \forall\, 0 < R \le \bar{R},$$

where $\vartheta = \bar{C}/(\bar{C}+1) \in (0,1)$, depends only on N, p and γ. Now, by Lemma 4.2.3 it follows that

$$\mathcal{L}(R) \le \frac{1}{\vartheta}\mathcal{L}(\bar{R})\left(\frac{R}{\bar{R}}\right)^{\sigma_1'} \qquad \forall\, 0 < R \le \bar{R},$$

where $\sigma_1' = \frac{1}{2}\log(1/\vartheta)/\log 2$ depends only on ϑ, Now (4.3.2) follows by setting $\sigma_1 = \sigma_1'/p^*$ and $\mathcal{C} = (\vartheta^{-1}\overline{R}^{-\sigma_1'}\mathcal{L}(\bar{R}))^{1/p^*}$. In a similar way, we can deduce (4.3.3). □

Now, we denote by $\mathcal{A}_R = \mathcal{B}_{8R}^{H^\circ} \setminus \mathcal{B}_{R/8}^{H^\circ}$ and $\mathcal{D}_R = \mathcal{B}_{4R}^{H^\circ} \setminus \mathcal{B}_{R/4}^{H^\circ}$ for $R > 0$.

Lemma 4.3.2. *Let $t \in (p^*, N/\mu_1)$. There exists a positive constant $\sigma = \sigma(N,p,\gamma,t)$ such that for any solution u to problem ($\mathcal{P}_H$) and for any $\bar{R} > 0$ satisfying the following inequality*

$$\|u\|_{L^{p^*}(B_{\bar{R}})} + \|u\|_{L^{p^*}(\mathbb{R}^N \setminus B_{1/\bar{R}})} \le \sigma, \tag{4.3.16}$$

then

$$\left(\fint_{\mathcal{D}_R} u^t\,dx\right)^{\frac{1}{t}} \le \mathcal{C}\left(\fint_{\mathcal{A}_R} u^{p^*}\,dx\right)^{\frac{1}{p^*}} \qquad \forall R < \overline{R}/8 \text{ or } R > 8/\overline{R}, \tag{4.3.17}$$

where $\fint_{\mathcal{D}_R} u^t\,dx = \dfrac{1}{|\mathcal{D}_R|}\displaystyle\int_{\mathcal{D}_R} u^t\,dx$ and $\mathcal{C}$ is a positive constant depending only on $N, p, \gamma, \overline{R}$ and t.

Proof. It is easy to see that, setting $\hat{u}(x) = u(Rx)$, for $R > 0$,

$$-\Delta_p^H \hat{u} - \gamma\frac{\hat{u}^{p-1}}{H^\circ(x)^p} = R^p\hat{u}^{p^*-1} \qquad \text{in } \mathcal{A}_1.$$

Let $m \ge 1$ and set

$$\mathcal{A} := \mathcal{A}_1 \cap \{\hat{u} < m\} \quad \text{and} \quad \mathcal{B} := \mathcal{A}_1 \cap \{\hat{u} \ge m\}. \tag{4.3.18}$$

Hence we can consider the weak formulation of the last equation as follows

$$\begin{aligned} I_1 + I_2 :=& \int_{\mathcal{A}} H^{p-1}(\nabla\hat{u})\langle\nabla H(\nabla\hat{u}),\nabla\varphi\rangle\,dx + \int_{\mathcal{B}} H^{p-1}(\nabla\hat{u})\langle\nabla H(\nabla\hat{u}),\nabla\varphi\rangle\,dx \\ =& \int_{\mathcal{A}_1} \gamma\frac{\hat{u}^{p-1}}{H^\circ(x)^p}\varphi\,dx + R^p\int_{\mathcal{A}_1}\hat{u}^{p^*-1}\varphi\,dx \qquad \forall\varphi \in C_c^\infty(\mathcal{A}_1). \end{aligned} \tag{4.3.19}$$

Let define $\hat{u}_m := \min(\hat{u}, m)$ for $m \ge 1$. By density argument, for any $\eta \in C_c^\infty(\mathcal{A}_1)$ it is possible to choose $\varphi = \eta^p\hat{u}_m^{p(s-1)}\hat{u}$, with $s \ge 1$, as test function in (4.3.19), so that, using (4.3.18) and (1.2.4), we can compute I_1 and I_2

$$\begin{aligned} I_1 =& \int_{\mathcal{A}} H^{p-1}(\nabla\hat{u})\langle\nabla H(\nabla\hat{u}), p\eta^{p-1}\hat{u}^{p(s-1)+1}\nabla\eta\rangle\,dx \\ &+ [p(s-1)+1]\int_{\mathcal{A}} H^{p-1}(\nabla\hat{u})\langle\nabla H(\nabla\hat{u}), \eta^p\hat{u}^{p(s-1)}\nabla\hat{u}\rangle\,dx \\ =& p\int_{\mathcal{A}} H^{p-1}(\nabla\hat{u})\langle\nabla H(\nabla\hat{u}),\nabla\eta\rangle\eta^{p-1}\hat{u}^{p(s-1)+1}\,dx \\ &+ [p(s-1)+1]\int_{\mathcal{A}} H^p(\nabla\hat{u})\eta^p\hat{u}^{p(s-1)}\,dx. \end{aligned} \tag{4.3.20}$$

In the same way, we obtain

$$
\begin{aligned}
I_2 =& \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), p\eta^{p-1} m^{p(s-1)} \hat{u} \nabla \eta \rangle \, dx \\
&+ \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), \eta^p m^{p(s-1)} \nabla \hat{u} \rangle \, dx \\
=& p \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \langle \nabla H(\nabla \hat{u}), \nabla \eta \rangle \eta^{p-1} m^{p(s-1)} \hat{u} \, dx + \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} \, dx.
\end{aligned}
\tag{4.3.21}
$$

Collecting both (4.3.20), (4.3.21), using the Scwharz's inequality and recalling (1.2.6) we obtain

$$
\begin{aligned}
&[p(s-1)+1] \int_{\mathcal{A}} H^p(\nabla \hat{u}) \eta^p \hat{u}^{p(s-1)} \, dx + \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} \, dx \\
\leq& pM \left[\int_{\mathcal{A}} H^{p-1}(\nabla \hat{u}) \eta^{p-1} \hat{u}^{p(s-1)+1} |\nabla \eta| \, dx + \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \eta^{p-1} m^{p(s-1)} |\nabla \eta| \hat{u} \, dx \right] \\
&+ \int_{\mathcal{A}_1} \gamma \frac{\hat{u}^{p-1}}{H^\circ(x)^p} \varphi \, dx + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-1} \varphi \, dx \\
=& pM \left[\int_{\mathcal{A}} H^{p-1}(\nabla \hat{u}) \eta^{p-1} \hat{u}^{(p-1)(s-1)} \hat{u}^s |\nabla \eta| \, dx + \int_{\mathcal{B}} H^{p-1}(\nabla \hat{u}) \eta^{p-1} m^{(p-1)(s-1)} m^{s-1} |\nabla \eta| \hat{u} \, dx \right] \\
&+ \int_{\mathcal{A}_1} \gamma \frac{\hat{u}^{p-1}}{H^\circ(x)^p} \varphi \, dx + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-1} \varphi \, dx.
\end{aligned}
\tag{4.3.22}
$$

Now we can apply the weighted Young's inequality to the first two terms in the right hand side of (4.3.22) with conjugate exponent $(p/(p-1), p)$ in order to obtain

$$
\begin{aligned}
&[p(s-1)+1] \int_{\mathcal{A}} H^p(\nabla \hat{u}) \eta^p \hat{u}^{p(s-1)} \, dx + \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} \, dx \\
&\quad \leq \varepsilon_1 [p(s-1)+1] \int_{\mathcal{A}} H^p(\nabla \hat{u}) \eta^p \hat{u}^{p(s-1)} \, dx + \mathcal{C}_{\varepsilon_1}(p, s, M) \int_{\mathcal{A}} \hat{u}^{ps} |\nabla \eta|^p \, dx \\
&\quad + \varepsilon_2 \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} \, dx + \mathcal{C}_{\varepsilon_2}(p, M) \int_{\mathcal{B}} |\nabla \eta|^p m^{p(s-1)} \hat{u}^p \, dx \\
&\quad + \int_{\mathcal{A}_1} \gamma \frac{\hat{u}^{p-1}}{H^\circ(x)^p} \varphi \, dx + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-1} \varphi \, dx,
\end{aligned}
\tag{4.3.23}
$$

where $\mathcal{C}_{\varepsilon_1}(p, s, M)$ and $\mathcal{C}_{\varepsilon_2}(p, M)$ are two positive constants. Hence we obtain

$$
\begin{aligned}
&(1-\varepsilon_1)[p(s-1)+1] \int_{\mathcal{A}} H^p(\eta \hat{u}_m^{s-1} \nabla \hat{u}) \, dx + (1-\varepsilon_2) \int_{\mathcal{B}} H^p(\eta \hat{u}_m^{s-1} \nabla \hat{u}) \, dx \\
=& (1-\varepsilon_1)[p(s-1)+1] \int_{\mathcal{A}} H^p(\nabla \hat{u}) \eta^p \hat{u}^{p(s-1)} \, dx + (1-\varepsilon_2) \int_{\mathcal{B}} H^p(\nabla \hat{u}) \eta^p m^{p(s-1)} \, dx \\
\leq& \mathcal{C}_{\varepsilon_1}(p, s, M) \int_{\mathcal{A}} |\nabla \eta|^p \hat{u}^{p(s-1)} \hat{u}^p \, dx + \mathcal{C}_{\varepsilon_2}(p, M) \int_{\mathcal{B}} |\nabla \eta|^p m^{p(s-1)} \hat{u}^p \, dx \\
&+ \int_{\mathcal{A}_1} \frac{\gamma}{H^\circ(x)^p} \hat{u}^{p-1} \eta^p \hat{u}_m^{p(s-1)} \hat{u} \, dx + R^p \int_{\mathcal{A}_1} \hat{u}^{p^*-1} \eta^p \hat{u}_m^{p(s-1)} \hat{u} \, dx.
\end{aligned}
\tag{4.3.24}
$$

Thanks to (4.3.8) and applying Lemma 4.2.2, with $a = H(\eta\hat{u}^{s-1}\nabla\hat{u})$ and $b = H(\hat{u}^s\nabla\eta)$, we deduce that there hold the following inequalities in the sets $\mathcal{A}$ and $\mathcal{B}$ respectively:

$$\begin{aligned}\int_{\mathcal{A}} H^p(\eta\hat{u}_m^{s-1}\nabla\hat{u})\,dx &= \int_{\mathcal{A}} H^p(\eta\hat{u}^{s-1}\nabla\hat{u})\,dx \\ &\geq \frac{1}{1+2^{p+1}\delta_1}\cdot\frac{1}{s^p}\int_{\mathcal{A}} H^p(\nabla(\eta\hat{u}^s))\,dx - \mathcal{C}_{\delta_1}\int_{\mathcal{A}} H^p(\hat{u}^s\nabla\eta)\,dx \\ &= \frac{1}{1+2^{p+1}\delta_1}\cdot\frac{1}{s^p}\int_{\mathcal{A}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx - \mathcal{C}_{\delta_1}\int_{\mathcal{A}} H^p(\hat{u}_m^{s-1}\hat{u}\nabla\eta)\,dx\end{aligned} \tag{4.3.25}$$

$$\begin{aligned}\int_{\mathcal{B}} H^p(\eta\hat{u}_m^{s-1}\nabla\hat{u})\,dx &= \int_{\mathcal{B}} H^p(\eta m^{s-1}\nabla\hat{u})\,dx \\ &\geq \frac{1}{1+2^{p+1}\delta_2}\int_{\mathcal{B}} H^p(\nabla(\eta m^{s-1}\hat{u}))\,dx - \mathcal{C}_{\delta_2}\int_{\mathcal{B}} H^p(m^{s-1}\hat{u}\nabla\eta)\,dx \\ &= \frac{1}{1+2^{p+1}\delta_2}\int_{\mathcal{B}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx - \mathcal{C}_{\delta_2}\int_{\mathcal{B}} H^p(\hat{u}_m^{s-1}\hat{u}\nabla\eta)\,dx\end{aligned} \tag{4.3.26}$$

By (4.3.25) and (4.3.26), we obtain

$$\begin{aligned}&\frac{1-\varepsilon_1}{1+2^{p+1}\delta_1}\cdot\frac{p(s-1)+1}{s^p}\int_{\mathcal{A}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx + \frac{1-\varepsilon_2}{1+2^{p+1}\delta_2}\int_{\mathcal{B}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx \\ \leq&\mathcal{C}_{\delta_1}(\varepsilon_1,p,s)\int_{\mathcal{A}} H^p(\hat{u}_m^{s-1}\hat{u}\nabla\eta)\,dx + \mathcal{C}_{\delta_2}(\varepsilon_2)\int_{\mathcal{B}} H^p(\hat{u}_m^{s-1}\hat{u}\nabla\eta)\,dx \\ &+\mathcal{C}_{\varepsilon_1}(p,s,M)\int_{\mathcal{A}} |\nabla\eta|^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx + \mathcal{C}_{\varepsilon_2}(p,M)\int_{\mathcal{B}} |\nabla\eta|^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx \\ &+\int_{\mathcal{A}_1}\frac{\gamma}{H^\circ(x)^p}\hat{u}^{p-1}\eta^p\hat{u}_m^{p(s-1)}\hat{u}\,dx + R^p\int_{\mathcal{A}_1}\hat{u}^{p^*-p}\eta^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx.\end{aligned} \tag{4.3.27}$$

Using (1.2.3), we deduce that

$$\begin{aligned}&\frac{1-\varepsilon_1}{1+2^{p+1}\delta_1}\cdot\frac{p(s-1)+1}{s^p}\int_{\mathcal{A}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx + \frac{1-\varepsilon_2}{1+2^{p+1}\delta_2}\int_{\mathcal{B}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx \\ \leq&\int_{\mathcal{A}_1}\frac{\gamma}{H^\circ(x)^p}\hat{u}^{p-1}\eta^p\hat{u}_m^{p(s-1)}\hat{u}\,dx + \hat{\mathcal{C}}\int_{\mathcal{A}_1}|\nabla\eta|^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx + R^p\int_{\mathcal{A}_1}\hat{u}^{p^*-p}(\eta\hat{u}_m^{s-1}\hat{u})^p\,dx,\end{aligned} \tag{4.3.28}$$

where $\hat{\mathcal{C}}$ depends on $\delta_1,\delta_2,\varepsilon_1,\varepsilon_2,p,s,M$. Using Hardy's and Hölder's inequality in the right hand side of (4.3.28), and the definition of the sets $\mathcal{A}$ and $\mathcal{B}$, we obtain

$$\begin{aligned}&\frac{1-\varepsilon_1}{1+2^{p+1}\delta_1}\cdot\frac{p(s-1)+1}{s^p}\int_{\mathcal{A}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx + \frac{1-\varepsilon_2}{1+2^{p+1}\delta_2}\int_{\mathcal{B}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx \\ \leq&\frac{\gamma}{C_H}\int_{\mathcal{A}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx + \frac{\gamma}{C_H}\int_{\mathcal{B}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx \\ &+\hat{\mathcal{C}}\int_{\mathcal{A}_1}|\nabla\eta|^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx + \left(\int_{\mathcal{A}_1}\hat{u}^{p^*}R^N\,dx\right)^{\frac{p^*-p}{p^*}}\left(\int_{\mathcal{A}_1}(\eta\hat{u}_m^{s-1}\hat{u})^{p\chi}\,dx\right)^{\frac{1}{\chi}},\end{aligned} \tag{4.3.29}$$

where $\chi = p^*/p$. Finally, we deduce

$$\begin{aligned}
&\left(\frac{1-\varepsilon_1}{1+2^{p+1}\delta_1}\cdot\frac{p(s-1)+1}{s^p}-\frac{\gamma}{C_H}\right)\int_{\mathcal{A}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx\\
&\qquad+\left(\frac{1-\varepsilon_2}{1+2^{p+1}\delta_2}-\frac{\gamma}{C_H}\right)\int_{\mathcal{B}} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx\\
&\leq \hat{\mathcal{C}}\int_{\mathcal{A}_1}|\nabla\eta|^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx+\|u\|_{L^{p^*}(\mathcal{A}_R)}^{p^*-p}\left(\int_{\mathcal{A}_1}(\eta\hat{u}_m^{s-1}\hat{u})^{p\chi}\,dx\right)^{\frac{1}{\chi}}.
\end{aligned} \tag{4.3.30}$$

Noticing that $(p(s-1)+1)/s^p > \gamma/C_H$ for all $s \in ((N-p)/(p\mu_2),(N-p)(p\mu_1))$, we can fix $\delta_1, \varepsilon_1 > 0$ sufficiently small such that

$$\frac{1-\varepsilon_1}{1+2^{p+1}\delta_1}\cdot\frac{p(s-1)+1}{s^p}-\frac{\gamma}{C_H}>0,$$

and $\delta_2, \varepsilon_2 > 0$ sufficiently small such that

$$\frac{1-\varepsilon_2}{1+2^{p+1}\delta_2}-\frac{\gamma}{C_H}>0.$$

Hence by (4.3.30) we get

$$\bar{\mathcal{C}}\int_{\mathcal{A}_1} H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx \leq \hat{\mathcal{C}}\int_{\mathcal{A}_1}|\nabla\eta|^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx+\|u\|_{L^{p^*}(\mathcal{A}_R)}^{p^*-p}\left(\int_{\mathcal{A}_1}(\eta\hat{u}_m^{s-1}\hat{u})^{p\chi}\,dx\right)^{\frac{1}{\chi}}, \tag{4.3.31}$$

where $\bar{\mathcal{C}}$ is a positive constant depending on $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2, p, s, \gamma, C_H$. In conclusion, by (1.2.3) and Sobolev inequality we have

$$\begin{aligned}
\alpha_1^p\bar{\mathcal{C}}\mathcal{C}_S^p\left(\int_{\mathcal{A}_1}(\eta\hat{u}_m^{s-1}\hat{u})^{p\chi}\,dx\right)^{\frac{1}{\chi}} &\leq \alpha_1^p\bar{\mathcal{C}}\int_{\mathcal{A}_1}|\nabla(\eta\hat{u}_m^{s-1}\hat{u})|^p\,dx \leq \bar{\mathcal{C}}\int_{\mathcal{A}_1}H^p(\nabla(\eta\hat{u}_m^{s-1}\hat{u}))\,dx\\
&\leq \hat{\mathcal{C}}\int_{\mathcal{A}_1}|\nabla\eta|^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx\\
&\quad+\|u\|_{L^{p^*}(\mathcal{A}_R)}^{p^*-p}\left(\int_{\mathcal{A}_1}(\eta\hat{u}_m^{s-1}\hat{u})^{p\chi}\,dx\right)^{\frac{1}{\chi}}.
\end{aligned} \tag{4.3.32}$$

In order to apply the Moser's iteration method we need to rewrite the last inequality as follows

$$\left(\int_{\mathcal{A}_1}(\eta\hat{u}_m^{s-1}\hat{u})^{p\chi}\,dx\right)^{\frac{1}{\chi}} \leq \mathcal{C}_1\int_{\mathcal{A}_1}|\nabla\eta|^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx+\mathcal{C}_2\|u\|_{L^{p^*}(\mathcal{A}_R)}^{p^*-p}\left(\int_{\mathcal{A}_1}(\eta\hat{u}_m^{s-1}\hat{u})^{p\chi}\,dx\right)^{\frac{1}{\chi}}, \tag{4.3.33}$$

where $\mathcal{C}_1 := \hat{\mathcal{C}}/(\alpha_1^p\bar{\mathcal{C}}\mathcal{C}_S^p)$, $\mathcal{C}_2 := 1/(\alpha_1^p\bar{\mathcal{C}}\mathcal{C}_S^p)$ and $\chi = p^*/p$.

Fix $t \in (p^*, N/\mu_1)$ and $k \in \mathbb{N}$ so that $p\chi^k \leq t \leq p\chi^{k+1}$. Then there exist positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$ such that (4.3.33) holds for all $1 \leq s \leq \min\{(N-p)/(p\mu_1), \chi^k\}$. Now, we set $\sigma = (1/(2\mathcal{C}_2))^{\frac{1}{p^*-p}}$ and choosing $\overline{R}$ sufficiently small such that (4.3.16) holds, we get

$$\left(\int_{\mathcal{A}_1}(\eta\hat{u}_m^{s-1}\hat{u})^{p\chi}\,dx\right)^{\frac{1}{\chi}} \leq \mathcal{C}\int_{\mathcal{A}_1}|\nabla\eta|^p\hat{u}_m^{p(s-1)}\hat{u}^p\,dx \tag{4.3.34}$$

for all $1 \le s \le \min\{(N-p)/(p\mu_1), \chi^k\}$ and $\mathcal{C} = \mathcal{C}_1/2$. Applying Moser's iteration method (see e.g. [104, 166] for further details), we conclude, after finitely many times of iteration,

$$\left(\fint_{\mathcal{D}_1} |\hat{u}|^t \, dx\right)^{\frac{1}{t}} \le \mathcal{C}\left(\fint_{\mathcal{A}_1} |\hat{u}|^{p^*} \, dx\right)^{\frac{1}{p^*}}. \tag{4.3.35}$$

for any $t \in (p^*, N/\mu_1)$. This proves (4.3.17).

□

Let us now prove the following:

Theorem 4.3.3. *Let u be a weak solution of $(\mathcal{P}_H)$. Then there exists a positive constant $C = C(N, p, \gamma, u)$ such that*

$$|u(x)| \le \frac{C}{[H^\circ(x)]^{\frac{N-p}{p} - \sigma_1}} \qquad \text{in } \mathcal{B}_{R_1}^{H^\circ},$$

and that

$$|u(x)| \le \frac{C}{[H^\circ(x)]^{\frac{N-p}{p} + \sigma_2}} \qquad \text{in } (\mathcal{B}_{R_2}^{H^\circ})^c,$$

where σ_1, σ_2 are givem in Lemma 4.3.1 and $R_1, R_2 > 0$ are constants depending on N, p, γ and u.

Proof. Let us fix $t := (p^* + N/\mu_1)/2 \in (p^*, N/\mu_1)$ as in Lemma 4.3.2 and $\kappa := \min\{\tau, \sigma\}$, where τ and σ are respectively as in Lemma 4.3.1 and Lemma 4.3.2. Let $\bar{R} > 0$ such that (4.3.1) holds for κ and let us consider $\hat{u}(x) = u(Rx)$, for $R > 0$ fixed. We note that $\hat{u}$ satisfies the equation

$$-\Delta_p^H \hat{u} + c(x)\hat{u}^{p-1} = 0 \qquad \text{in } \mathcal{D}_1,$$

where

$$c(x) = -\frac{\gamma}{H^\circ(x)^p} - R^p \hat{u}^{p^*-p}(x).$$

We note that $H^\circ(x)^{-p}$ is bounded in $\mathcal{D}_1$ and $V_R(x) := R^p \hat{u}^{p^*-p}(x) \in L^q(\mathcal{D}_1)$ with $q = t/(p^*-p) > N/p$ due to Lemma 4.3.2. Hence, as in the proof of [143, Theorem 1] a classical Moser iteration argument yields

$$\sup_{\mathcal{B}_r^{H^\circ}(x)} \hat{u} \le \mathbf{C}\left(\fint_{\mathcal{B}_{2r}^{H^\circ}(x)} \hat{u}^p \, dx\right)^{\frac{1}{p}} \tag{4.3.36}$$

for any ball $\mathcal{B}_{2r}^{H^\circ}(x) \subset \mathcal{D}_1$, where $\mathbf{C} = \mathbf{C}(N, p, \gamma, \|V_R\|_{L^q(\mathcal{D}_1)})$. We claim that $\|V_R\|_{L^q(\mathcal{D}_1)}$ is uniformly bounded with respect to R. Indeed from Lemma 4.3.2, since

$$p - \frac{N}{q} + N\frac{p^*-p}{t} - N\frac{p^*-p}{p^*} = 0,$$

we get

$$
\begin{aligned}
\left(\int_{\mathcal{D}_1} V_R^q\,dx\right)^{\frac{1}{q}} &= R^{p-\frac{N}{q}}\left(\int_{\mathcal{D}_R} u^{(p^*-p)q}\,dx\right)^{\frac{1}{q}} \\
&\leq \mathcal{C}R^{p-\frac{N}{q}+N\frac{p^*-p}{t}}\left(\fint_{\mathcal{D}_R} u^t\,dx\right)^{\frac{p^*-p}{t}} \\
&\leq \mathcal{C}R^{p-\frac{N}{q}+N\frac{p^*-p}{t}-N\frac{p^*-p}{p^*}}\left(\int_{\mathcal{A}_R} u^t\,dx\right)^{\frac{p^*-p}{p^*}} \\
&\leq \mathcal{C}\left(\int_{\mathbb{R}^N} u^{p^*}\,dx\right)^{\frac{p^*-p}{p^*}} \qquad \forall 0<R\leq\frac{\overline{R}}{8} \text{ or } R\geq\frac{8}{\overline{R}},
\end{aligned}
\tag{4.3.37}
$$

where $\mathcal{C}$ is a positive constant depending on N, p, γ, q and $\overline{R}$.

Using a covering argument we deduce that

$$
\sup_{\mathcal{B}_2^{H^\circ}\setminus\mathcal{B}_1^{H^\circ}} \hat{u} \leq \mathbf{C}\left(\fint_{\mathcal{D}_1} \hat{u}^p\,dx\right)^{\frac{1}{p}} \tag{4.3.38}
$$

Noticing that $\hat{u}(x)=u(Rx)$, by (4.3.38) we obtain that

$$
\sup_{\mathcal{B}_{2R}^{H^\circ}\setminus\mathcal{B}_R^{H^\circ}} u \leq \mathbf{C}\left(\fint_{\mathcal{D}_R} u^p\,dx\right)^{\frac{1}{p}} \tag{4.3.39}
$$

for each $0<R\leq\bar{R}/8$ or $R\geq 8/\bar{R}$. By applying the Hölder's inequality in (4.3.39), we get

$$
\sup_{\mathcal{B}_{2R}^{H^\circ}\setminus\mathcal{B}_R^{H^\circ}} u \leq \mathbf{C}\left(\fint_{\mathcal{D}_R} u^p\,dx\right)^{\frac{1}{p}} \leq \mathbf{C}\left(\fint_{\mathcal{D}_R} u^{p^*}\,dx\right)^{\frac{1}{p^*}} = \mathbf{C}\|u\|_{L^{p^*}(\mathcal{D}_R)}R^{\frac{p-N}{p}}, \tag{4.3.40}
$$

for each $0<R\leq\bar{R}/8$ or $R\geq 8/\bar{R}$ and $\mathbf{C}$ depends only on $N, p, \gamma, q, \bar{R}$ and $\|u\|_{L^{p^*}(\mathbb{R}^N)}$.

Now we note that, since $\mathcal{A}_R\subset\mathcal{B}_{\bar{R}}^{H^\circ}$ for any $0<R\leq\bar{R}/8$ and $\mathcal{A}_R\subset(\mathcal{B}_{1/\bar{R}}^{H^\circ})^c$ for any $R\geq 8/\bar{R}$, there exist, by Lemma 4.3.1, $\sigma_1,\sigma_2>0$ depending only on N,p,γ such that

$$
\|u\|_{L^{p^*}(\mathcal{B}_R^{H^\circ})} \leq \mathcal{C}R^{\sigma_1} \quad \text{for } 0<R\leq\frac{\bar{R}}{8}
$$

and that

$$
\|u\|_{L^{p^*}((\mathcal{B}_R^{H^\circ})^c)} \leq \frac{\mathcal{C}}{R^{\sigma_2}} \quad \text{for } R\geq 8/\bar{R}.
$$

Now, if we set $R_1=\bar{R}/8$ and $R_2=8/\bar{R}$, by (4.3.40) we get the thesis.

□

The next result is devoted to showing the existence of some special supersolutions of our problem, in order to perform a comparison between them and the solutions of the doubly critical equation ($\mathcal{P}_H$).

Proposition 4.3.4. *Given two constants $A>0$ and $\alpha<p$, there exist constants $0<\varepsilon,\delta<1$, depending on N,p,γ,A,α, such that*

$$
v(H^\circ(x)) = \frac{1-\delta[H^\circ(x)]^\varepsilon}{[H^\circ(x)]^{\mu_1}} \in \mathcal{D}^{1,p}(\mathcal{B}_{R_1}^{H^\circ}) \tag{4.3.41}
$$

is a positive supersolution to equation

$$-\Delta_p^H v - \frac{\gamma}{H^\circ(x)^p} v^{p-1} = g(x)v^{p-1} \quad \textit{in } \mathcal{B}_{R_1}^{H^\circ}, \tag{4.3.42}$$

for some positive constant $0 < R_1 < 1$ *depending only on* N, p, γ, A *and* α*, where* $g(x)$ *is a positive function that belongs to* $L^{\frac{N}{p}}(\mathcal{B}_{R_1}^{H^\circ})$ *such that*

$$g(x) \geq A[H^\circ(x)]^{-\alpha} \quad \textit{in } \mathcal{B}_{R_1}^{H^\circ}. \tag{4.3.43}$$

In a similar way, given $A > 0$ *and* $\alpha > p$*, there exist* $0 < \varepsilon, \delta < 1$ *such that*

$$v(H^\circ(x)) = \frac{1 - \delta[H^\circ(x)]^{-\varepsilon}}{[H^\circ(x)]^{\mu_2}} \in \mathcal{D}^{1,p}((\mathcal{B}_{R_2}^{H^\circ})^c) \tag{4.3.44}$$

is a positive supersolution to equation

$$-\Delta_p^H v - \frac{\gamma}{H^\circ(x)^p} v^{p-1} = g(x)v^{p-1} \quad \textit{in } (\mathcal{B}_{R_2}^{H^\circ})^c, \tag{4.3.45}$$

for some positive constant $R_2 > 1$ *depending only on* N, p, γ *and* α*, where* $g(x)$ *is a positive function that belongs to* $L^{\frac{N}{p}}((\mathcal{B}_{R_2}^{H^\circ})^c)$ *such that*

$$g(x) \geq A[H^\circ(x)]^{-\alpha} \quad \textit{in } (\mathcal{B}_{R_2}^{H^\circ})^c. \tag{4.3.46}$$

Proof. Let us consider $\mu, \delta, \varepsilon > 0$ and let us define the function

$$u(x) = v(H^\circ(x)) = \frac{1 - \delta[H^\circ(x)]^{\varepsilon}}{[H^\circ(x)]^{\mu}}.$$

It is easy to deduce that

$$\nabla u(x) = s(H^\circ(x))\nabla H^\circ(x), \tag{4.3.47}$$

where $s(t) := t^{-\mu-1}[-\mu + \delta(\mu - \varepsilon)t^\varepsilon]$. Using (1.2.10), we now compute

$$\begin{aligned} -\Delta_p^H u &= -\operatorname{div}\left(H^{p-1}(\nabla u)\nabla H(\nabla u)\right) = -\operatorname{div}\left(|s(H^\circ(x))|^{p-1}\operatorname{sign}(s)\nabla H(\nabla H^\circ(x))\right) \\ &= -(p-1)|s(H^\circ(x))|^{p-2}s'(H^\circ(x))\langle \nabla H^\circ(x), \nabla H(\nabla H^\circ(x))\rangle \\ &\quad - |s(H^\circ(x))|^{p-1}\operatorname{sign}(s)\operatorname{div}(\nabla H(\nabla H^\circ(x))) \\ &= -(p-1)|s(H^\circ(x))|^{p-2}s'(H^\circ(x)) - (N-1)\frac{|s(H^\circ(x))|^{p-2}s(H^\circ(x))}{H^\circ(x)}, \end{aligned} \tag{4.3.48}$$

where in the last line we used the fact that $\langle \nabla H^\circ(x), \nabla H(\nabla H^\circ(x))\rangle = 1$ and

$$\operatorname{div}(\nabla H(\nabla H^\circ(x))) = \frac{N-1}{H^\circ(x)}$$

due to (1.2.10) and (1.2.11).

Making standard computations on the right hand side of (4.3.48), one can deduce

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p}|u|^{p-2}u = g(H^\circ(x))|u|^{p-2}u, \tag{4.3.49}$$

where

$$g(t) = \frac{h(t)}{|1 - \delta t^\varepsilon|^{p-2}(1 - \delta t^\varepsilon)t^p},$$

and

$$\begin{aligned} h(t) := & - |\mu - \delta(\mu-\varepsilon)t^{\varepsilon}|^{p-2} \cdot \\ & \cdot \left\{\mu^2(p-1) - (N-p)\mu - (p-1)(\mu-\varepsilon)^2\delta t^{\varepsilon} + (N-p)(\mu-\varepsilon)\delta t^{\varepsilon}\right\} \\ & - \gamma|1-\delta t^{\varepsilon}|^{p-2}(1-\delta t^{\varepsilon}), \qquad \text{for } t \in [0,+\infty). \end{aligned} \tag{4.3.50}$$

We note that $h(0) = -|\mu|^{p-2}[\mu^2(p-1) - \mu(N-p)] - \gamma$. Hence, using the definition of μ_1 and μ_2, we deduce that $h(0) = 0$ when $\mu = \mu_1$ and $\mu = \mu_2$. Also we have $h'(0) > 0$ if $\mu = \mu_1$, $\varepsilon > 0$ or $\mu = \mu_2$, $\varepsilon < 0$. This implies there exist $0 < \delta_h < 1$ such that

$$2h'(0)t \geq h(t) \geq \frac{1}{2}h'(0)t > 0 \qquad \forall t \in (0, \delta_h). \tag{4.3.51}$$

We set $\delta = \min\{\delta_h, 1/2\}$, $\varepsilon = (p-\alpha)/2$ and

$$R_0 = \left\{1, \left(\frac{\delta}{2A}h'(0)\right)^{1/(p-\alpha-\varepsilon)}\right\}.$$

It easy to check that $v(H^\circ(x)) = (1-\delta[H^\circ(x)]^{\varepsilon})[H^\circ(x)]^{-\mu_1} \in \mathcal{D}^{1,p}(\mathcal{B}^{H^\circ}_{R_1})$ is positive, which thanks to (4.3.51), $g(x) \in L^{\frac{N}{p}}(\mathcal{B}^{H^\circ}_{R_1})$ and it satisfies (4.3.43). The other case is similar.

□

Now, we consider the following equation

$$-\Delta^H_p w - \frac{\gamma}{H^\circ(x)^p}w^{p-1} = f(x)w^{p-1} \qquad \text{in } \Omega, \tag{4.3.52}$$

where Ω is an open subset of $\mathbb{R}^N$, $w > 0$ and $w \in \mathcal{D}^{1,p}(\Omega)$. Let us start with a comparison principle in bounded domains.

The first result is given by the following pointwise estimate, in the same spirit of [134, 166].

Proposition 4.3.5. *Let u, v two weakly differentiable strictly positive functions on a domain Ω*[1]*. We have that:*

(i) *if $p \geq 2$, then*

$$\begin{aligned} & H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla\left(u - \frac{v^p}{u^p}u\right)\rangle + H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla\left(v - \frac{u^p}{v^p}v\right)\rangle \\ & \geq C_p(u^p+v^p)H^p(\nabla(\ln u - \ln v)), \end{aligned} \tag{4.3.53}$$

for some positive constant C_p depending only on p;

(ii) *if $1 < p < 2$, then*

$$\begin{aligned} & H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla\left(u - \frac{v^p}{u^p}u\right)\rangle + H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla\left(v - \frac{u^p}{v^p}v\right)\rangle \\ & \geq C_p(u^p+v^p)\left[H(\nabla \ln u) + H(\nabla \ln v)\right]^{p-2} H^2(\nabla(\ln u - \ln v)), \end{aligned} \tag{4.3.54}$$

for some positive constant C_p depending only on p.

[1] We mean that $u, v \geq C > 0$ a.e. in Ω.

Proof. Let u, v two weakly differentiable positive functions and consider the following

$$\psi_1 := u - \frac{v^p}{u^p}u \qquad \text{and} \qquad \psi_2 := v - \frac{u^p}{v^p}v.$$

Then, thanks to the Euler's theorem for 1-homogeneous functions, and since ∇H is 0-homogeneus, we deduce that

$$\begin{aligned}
&H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1\rangle = \\
&\qquad = H^p(\nabla u) - p\frac{v^{p-1}}{u^{p-1}}H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla v\rangle + (p-1)\frac{v^p}{u^p}H^p(\nabla u) \\
&\qquad = H^p(\nabla u) - v^p\left[(1-p)H^p\left(\frac{\nabla u}{u}\right) + pH^{p-1}\left(\frac{\nabla u}{u}\right)\langle \nabla H(\nabla u), \frac{\nabla v}{v}\rangle\right] \\
&\qquad = H^p(\nabla u) - v^p\left[H^p(\nabla \ln u) + pH^{p-1}(\nabla \ln u)\langle \nabla H(\nabla \ln u), \nabla \ln v - \nabla \ln u\rangle\right]
\end{aligned} \tag{4.3.55}$$

and

$$\begin{aligned}
&H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2\rangle = \\
&\qquad = H^p(\nabla v) - p\frac{u^{p-1}}{v^{p-1}}H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla u\rangle + (p-1)\frac{u^p}{v^p}H^p(\nabla v) \\
&\qquad = H^p(\nabla v) - u^p\left[(1-p)H^p\left(\frac{\nabla v}{v}\right) + pH^{p-1}\left(\frac{\nabla v}{v}\right)\langle \nabla H(\nabla v), \frac{\nabla u}{u}\rangle\right] \\
&\qquad = H^p(\nabla v) - u^p\left[H^p(\nabla \ln v) + pH^{p-1}(\nabla \ln v)\langle \nabla H(\nabla \ln v), \nabla \ln u - \nabla \ln v\rangle\right].
\end{aligned} \tag{4.3.56}$$

(i) $p \geq 2$**.** We recall that when $p \geq 2$, it holds (1.2.13), i.e.

$$H^p(\eta) \geq H^p(\eta') + pH^{p-1}(\eta')\langle \nabla H(\eta'), \eta - \eta'\rangle + \frac{1}{2^{p-1}-1}H^p(\eta - \eta'), \qquad \forall \eta, \eta' \in \mathbb{R}^N. \tag{4.3.57}$$

Hence we can apply this inequality, in order to give an estimate from below for (4.3.55) and (4.3.56):

$$\begin{aligned}
&H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1\rangle \\
&= H^p(\nabla u) - v^p\left[H^p(\nabla \ln u) + pH^{p-1}(\nabla \ln u)\langle \nabla H(\nabla \ln u), \nabla \ln v - \nabla \ln u\rangle\right] \\
&\geq H^p(\nabla u) - v^p\left[H^p(\nabla \ln v) - C_pH^p(\nabla(\ln v - \ln u))\right] \\
&= H^p(\nabla u) - H^p(\nabla v) + C_pv^pH^p(\nabla(\ln v - \ln u)),
\end{aligned} \tag{4.3.58}$$

$$\begin{aligned}
&H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2\rangle \\
&= H^p(\nabla v) - u^p\left[H^p(\nabla \ln v) + pH^{p-1}(\nabla \ln v)\langle \nabla H(\nabla \ln v), \nabla \ln u - \nabla \ln v\rangle\right] \\
&\geq H^p(\nabla v) - u^p\left[H^p(\nabla \ln u) - C_pH^p(\nabla(\ln u - \ln v))\right] \\
&= H^p(\nabla v) - H^p(\nabla u) + C_pu^pH^p(\nabla(\ln u - \ln v)).
\end{aligned} \tag{4.3.59}$$

Adding both these two inequalities, we obtain

$$\begin{aligned}
&H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1\rangle + H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2\rangle \\
&\geq C_p(u^p + v^p)H^p(\nabla(\ln u - \ln v)).
\end{aligned} \tag{4.3.60}$$

(ii) $1 < p < 2$**.** We recall that when $1 < p < 2$, it holds (1.2.14), i.e.

$$\begin{aligned} H^p(\eta) \geq{} & H^p(\eta') + pH^{p-1}(\eta')\langle \nabla H(\eta'), \eta - \eta' \rangle \\ & + C_p[H(\eta) + H(\eta')]^{p-2} H^2(\eta - \eta'), \qquad \forall \eta, \eta' \in \mathbb{R}^N, \end{aligned} \tag{4.3.61}$$

where C_p is a positive constant depending only on p.

Now we proceed exactly as in the previous case to get an estimate from below for (4.3.55) and (4.3.56):

$$\begin{aligned} & H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1 \rangle \\ & = H^p(\nabla u) - v^p \left[H^p(\nabla \ln u) + pH^{p-1}(\nabla \ln u)\langle \nabla H(\nabla \ln u), \nabla \ln v - \nabla \ln u \rangle \right] \\ & \geq H^p(\nabla u) - v^p \left[H^p(\nabla \ln v) - C_p[H(\nabla \ln u) + H(\nabla \ln v)]^{p-2} H^2(\nabla(\ln v - \ln u)) \right] \\ & = H^p(\nabla u) - H^p(\nabla v) + C_p v^p \left[[H(\nabla \ln u) + H(\nabla \ln v)]^{p-2} H^2(\nabla(\ln v - \ln u) \right], \end{aligned} \tag{4.3.62}$$

$$\begin{aligned} & H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2 \rangle \\ & = H^p(\nabla v) - u^p \left[H^p(\nabla \ln v) + pH^{p-1}(\nabla \ln v)\langle \nabla H(\nabla \ln v), \nabla \ln u - \nabla \ln v \rangle \right] \\ & \geq H^p(\nabla v) - u^p \left[H^p(\nabla \ln u) - C_p[H(\nabla \ln u) + H(\nabla \ln v)]^{p-2} H^2(\nabla(\ln v - \ln u)) \right] \\ & = H^p(\nabla v) - H^p(\nabla u) + C_p u^p [H(\nabla \ln u) + H(\nabla \ln v)]^{p-2} H^2(\nabla(\ln v - \ln u)). \end{aligned} \tag{4.3.63}$$

Adding both these two inequalities, we obtain

$$\begin{aligned} & H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \psi_1 \rangle + H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \psi_2 \rangle \\ & \qquad \geq C_p(u^p + v^p)[H(\nabla \ln u) + H(\nabla \ln v)]^{p-2} H^2(\nabla(\ln v - \ln u)). \end{aligned} \tag{4.3.64}$$

□

Now, we are ready to prove the comparison principles in bounded and exteriors domains.

Proposition 4.3.6. *Let Ω be an open bounded smooth domain of $\mathbb{R}^N$ and $f \in L^{\frac{N}{p}}(\Omega)$. Let $u \in \mathcal{D}^{1,p}(\Omega)$ be a weak positive subsolution to (4.3.52) and $v \in \mathcal{D}^{1,p}(\Omega)$ be a weak positive supersolution of*

$$-\Delta_p^H v - \frac{\gamma}{H^\circ(x)^p} v^{p-1} = g(x) v^{p-1} \qquad \text{in } \Omega, \tag{4.3.65}$$

with $g \in L^{\frac{N}{p}}(\Omega)$. Assume that $\inf_\Omega v > 0$ and $f \leq g$ in Ω. If $u \leq v$ on $\partial\Omega$, then

$$u \leq v \quad \text{in } \Omega.$$

Proof. We will give the proof of this result in the case $p \geq 2$. The case $1 < p < 2$ is similar.

Let us define

$$\eta_1 := \frac{\min\{(u^p - v^p)^+, m\}}{u^{p-1}} \qquad \text{and} \qquad \eta_2 := \frac{\min\{(u^p - v^p)^+, m\}}{v^{p-1}}$$

where $m > 1$. It is quite standard to show that η_1 and η_2 are good test function that we can use in the weak formulations of (4.3.52) and (4.3.65). Taking both these test function and subtracting the two equations, we obtain

$$\begin{aligned}\int_\Omega H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \eta_1\rangle\,dx - \int_\Omega H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \eta_2\rangle\,dx&\\ \leq \int_\Omega \frac{\gamma}{H^\circ(x)^p}u^{p-1}\eta_1\,dx - \int_\Omega \frac{\gamma}{H^\circ(x)^p}v^{p-1}\eta_2\,dx&\\ + \int_\Omega f(x)u^{p-1}\eta_1\,dx - \int_\Omega g(x)v^{p-1}\eta_2\,dx&\\ = \int_\Omega \frac{\gamma}{H^\circ(x)^p}\left(u^{p-1}\eta_1 - v^{p-1}\eta_2\right)\,dx&\\ + \int_\Omega f(x)u^{p-1}\eta_1 - g(x)v^{p-1}\eta_2\,dx \leq 0,&\end{aligned} \tag{4.3.66}$$

since $f \leq g$ in Ω. Hence, setting $\Omega_1 := \{x \in \Omega \mid 0 \leq u^p - v^p \leq m\}$ and $\Omega_2 := \{x \in \Omega \mid u^p - v^p \geq m\}$, we deduce that

$$\begin{aligned}&\int_{\Omega_1} H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla\left(u - \frac{v^p}{u^p}u\right)\rangle\,dx + \int_{\Omega_1} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla\left(v - \frac{u^p}{v^p}v\right)\rangle\,dx\\ &+ m(1-p)\left[\int_{\Omega_2} H^{p-1}(\nabla u)\langle \nabla H(\nabla u), u^{-p}\nabla u\rangle\,dx - \int_{\Omega_2} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), v^{-p}\nabla v\rangle\,dx\right]\\ &\leq 0.\end{aligned} \tag{4.3.67}$$

Applying (4.3.53) in (4.3.67) and making some computations we obtain

$$\begin{aligned}C_p\int_{\Omega_1}(u^p+v^p)H^p(\nabla(\ln u - \ln v))\,dx&\\ + m(p-1)\left[\int_{\Omega_2} H^p(\nabla \ln v)\,dx - \int_{\Omega_2} H^p(\nabla \ln u)\,dx\right] \leq 0,&\end{aligned} \tag{4.3.68}$$

but this implies that

$$C_p\int_{\Omega_1}(u^p+v^p)H^p(\nabla(\ln u - \ln v))\,dx \leq m(p-1)\int_{\Omega_2} H^p(\nabla \ln u)\,dx. \tag{4.3.69}$$

For the right hand side of (4.3.69), we have

$$m(p-1)\int_{\Omega_2} H^p(\nabla \ln u)\,dx = (p-1)\int_{\Omega_2} mu^{-p}H^p(\nabla u)\,dx \leq (p-1)\int_{\Omega_2'} H^p(\nabla u)\,dx, \tag{4.3.70}$$

where $\Omega_2' := \{x \in \Omega \mid u^p \geq m\}$ and it holds that

$$\int_{\Omega_2'} H^p(\nabla u)\,dx \to 0 \qquad \text{for } m \to +\infty.$$

Hence, passing to the limit for $m \to +\infty$ in (4.3.69) we obtain that

$$\int_{\{u^p - v^p \geq 0\}}(u^p+v^p)H^p(\nabla(\ln u - \ln v))\,dx = 0, \tag{4.3.71}$$

which clearly implies that

$$u = Kv \qquad \text{in the set } \{x \in \Omega \mid u(x) \geq v(x)\},$$

for some positive constant K. By our assumptions $\inf_{x\in\Omega} v > 0$ and $u \leq v$ on $\partial\Omega$, hence it follows that $K = 1$. But this implies that

$$u \leq v \qquad \text{in } \Omega$$

and this complete the proof of this result in the case $p \geq 2$. The case $1 < p < 2$ follows repeating verbatim the proof of the case $p \geq 2$, but applying inequality (4.3.54) instead of (4.3.53). ☐

Now we want to prove the corresponding result of Proposition 4.3.6 in exterior domains.

Proposition 4.3.7. *Let Ω be an exterior domain such that $\mathbb{R}^N \setminus \Omega$ is bounded and $f \in L^{\frac{N}{p}}(\Omega)$. Let $u \in \mathcal{D}^{1,p}(\Omega)$ be a weak positive subsolutions to (4.3.52) and $v \in \mathcal{D}^{1,p}(\Omega)$ be a positive supersolution of*

$$-\Delta_p^H v - \frac{\gamma}{H^\circ(x)^p} v^{p-1} = g(x) v^{p-1} \qquad \text{in } \Omega, \tag{4.3.72}$$

with $g \in L^{\frac{N}{p}}(\Omega)$. Assume that $\inf_\Omega v > 0$ and $f \leq g$ in Ω. If $u \leq v$ on $\partial\Omega$ and

$$\limsup_{R\to+\infty} \frac{1}{R} \int_{B_{2R}\setminus B_R} u^p |\nabla \log v|^{p-1} = 0, \tag{4.3.73}$$

then

$$u \leq v \quad \text{in } \Omega.$$

Proof. In the same spirit of Proposition 4.3.6 we prove our result in the case $p \geq 2$. The other case is similar and it can be shown using the same arguments. To this aim, let $\varphi_R \in C_c^\infty(B_{2R})$ be a standard cut-off function such that

$$\begin{cases} 0 \leq \varphi_R \leq 1 \\ \varphi_R \equiv 1 & \text{on } B_R \\ |\nabla \varphi_R| \leq \frac{2}{R} & \text{on } B_{2R} \setminus B_R. \end{cases}$$

Let us define

$$\eta_1 := \varphi_R \frac{\min\{(u^p - v^p)^+, m\}}{u^{p-1}} \qquad \text{and} \qquad \eta_2 := \varphi_R \frac{\min\{(u^p - v^p)^+, m\}}{v^{p-1}}.$$

where $m > 1$. As pointed out in the proof of previous proposition, it is possible to show, by standard arguments, that η_1 and η_2 are good test functions for the weak formulations (4.3.52) and (4.3.72). Hence, we obtain

$$\begin{aligned} \int_\Omega H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \eta_1\rangle \, dx - \int_\Omega H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \eta_2\rangle \, dx& \\ \leq \int_\Omega \frac{\gamma}{H^\circ(x)^p} \left(u^{p-1}\eta_1 - v^{p-1}\eta_2\right) \, dx& \\ + \int_\Omega f(x) u^{p-1}\eta_1 - g(x) v^{p-1}\eta_2 \, dx \leq 0,& \end{aligned} \tag{4.3.74}$$

since $f \leq g$ in Ω. Now we explicitly compute the left hand side of (4.3.74).

$$
\begin{aligned}
0 \geq & \int_{\Omega} H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \eta_1\rangle \, dx - \int_{\Omega} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \eta_2\rangle \, dx \\
= & \int_{\Omega_1} H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \left(u - \frac{v^p}{u^p}u\right)\rangle \varphi_R \, dx \\
+ & \int_{\Omega_1} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \left(v - \frac{u^p}{v^p}v\right)\rangle \varphi_R \, dx \\
+ & \int_{\Omega_1} \left(u - \frac{v^p}{u^p}u\right) H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \varphi_R\rangle \, dx \\
+ & \int_{\Omega_1} \left(v - \frac{u^p}{v^p}v\right) H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \varphi_R\rangle \, dx \\
+ & \int_{\Omega_2} H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \left(u^{1-p}\right)\rangle m\varphi_R + H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla \varphi_R\rangle m u^{1-p} \, dx \\
- & \int_{\Omega_2} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \left(v^{1-p}\right)\rangle m\varphi_R + H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \varphi_R\rangle m v^{1-p} \, dx \\
=: & \, I_1 + I_2 + I_3 + I_4 + I_5 + I_6,
\end{aligned}
\tag{4.3.75}
$$

where $\Omega_1 := \{x \in \Omega \mid 0 \leq u^p - v^p \leq m\}$ and $\Omega_2 := \{x \in \Omega \mid u^p - v^p \geq m\}$. By Proposition 4.3.5 and using the definition of φ_R, it follows that there exits a positive constant depending only on p such that

$$
I_1 + I_2 \geq C_p \int_{\Omega_1 \cap B_R} (u^p + v^p) H^p(\nabla(\ln u - \ln v)) \, dx. \tag{4.3.76}
$$

Now we are going to give estimates for the other terms. We start with $I_3 + I_4$. Using (1.2.6) and the Cauchy-Schwarz inequality, setting $\tilde{\Omega}_1 := \{x \in \Omega : v^p \leq u^p\}$, we have

$$
\begin{aligned}
|I_3 + I_4| \leq & M \int_{\Omega_1} \left| u - \frac{v^p}{u^p}u \right| H^{p-1}(\nabla u)|\nabla \varphi_R| \, dx \\
& + M \int_{\Omega_1} \left| v - \frac{u^p}{v^p}v \right| H^{p-1}(\nabla v)|\nabla \varphi_R| \, dx \\
\leq & M \int_{\tilde{\Omega}_1} \left(2uH^{p-1}(\nabla u) + vH^{p-1}(\nabla v)\right)|\nabla \varphi_R| \, dx \\
& + M \int_{\tilde{\Omega}_1} u^p H^{p-1}(\nabla \ln v)|\nabla \varphi_R| \, dx \\
\leq & \frac{\mathcal{C}}{R} \int_{B_{2R} \setminus B_R} u \cdot H^{p-1}(\nabla u) \, dx + \frac{\mathcal{C}}{R} \int_{B_{2R} \setminus B_R} v \cdot H^{p-1}(\nabla v) \, dx \\
& + \frac{\mathcal{C}}{R} \int_{B_{2R} \setminus B_R} u^p H^{p-1}(\nabla \ln v) \, dx \\
\leq & \mathcal{C} \left(\int_{B_{2R} \setminus B_R} H^p(\nabla u) \, dx\right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} u^{p^*} \, dx\right)^{\frac{1}{p^*}} \\
& + \mathcal{C} \left(\int_{B_{2R} \setminus B_R} H^p(\nabla v) \, dx\right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} v^{p^*} \, dx\right)^{\frac{1}{p^*}} \\
& + \frac{\mathcal{C}}{R} \int_{B_{2R} \setminus B_R} u^p H^{p-1}(\nabla \ln v) \, dx,
\end{aligned}
\tag{4.3.77}
$$

where in the last line we applied the Hölder inequality with conjugate exponent $(N, p/(p-1), p^*)$ and $\mathcal{C} := 2M$. Passing to the limit for R that goes to $+\infty$ in the right hand side of (4.3.77), using also assumption (4.3.73) and (1.2.3), we deduce that $I_3 + I_4$ goes to zero.

Now we proceed with the estimate of the term I_5. By (1.2.4), we have

$$I_5 = m\int_{\Omega_2} H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla\varphi_R\rangle u^{1-p}\,dx + m(1-p)\int_{\Omega_2} H^p(\nabla u)\varphi_R u^{-p}\,dx$$

Therefore

$$|I_5| \leq mM\int_{\tilde{\Omega}_2} H^{p-1}(\nabla u)|\nabla\varphi_R|u^{1-p}\,dx + m(p-1)\int_{\tilde{\Omega}_2} H^p(\nabla u)\varphi_R u^{-p}\,dx,$$

where $\tilde{\Omega}_2 := \{x \in \Omega \mid u^p \geq m\}$. Using this definition and also the properties of φ_R we deduce that

$$\begin{aligned}
|I_5| &\leq \frac{2M}{R}\int_{\tilde{\Omega}_2} u\cdot H^{p-1}(\nabla u)\,dx + (p-1)\int_{\tilde{\Omega}_2} H^p(\nabla u)\,dx \\
&\leq \mathcal{C}\left(\int_{B_{2R}\setminus B_R} H^p(\nabla u)\,dx\right)^{\frac{p-1}{p}}\left(\int_{B_{2R}\setminus B_R} u^{p^*}\,dx\right)^{\frac{1}{p^*}} + (p-1)\int_{\tilde{\Omega}_2} H^p(\nabla u)\,dx,
\end{aligned} \tag{4.3.78}$$

where $\mathcal{C} := 2M$. Passing to the limit for m, R that go to $+\infty$, we deduce that

$$\lim_{m,R\to+\infty} I_5 = 0. \tag{4.3.79}$$

For the last term I_6, by (1.2.4), recalling $\tilde{\Omega}_2 := \{x \in \Omega : u^p \geq m\}$, we have

$$\begin{aligned}
I_6 &= m(p-1)\int_{\Omega_2} H^p(\nabla v)\varphi_R v^{-p}\,dx - \int_{\Omega_2} mH^{p-1}(\nabla v)\langle\nabla H(\nabla v), \nabla\varphi_R\rangle v^{1-p}\,dx \\
&\geq -M\int_{\Omega_2} mH^{p-1}(\nabla \ln v)|\nabla\varphi_R|\,dx \geq M\int_{\tilde{\Omega}_2} u^p H^{p-1}(\nabla\ln v)|\nabla\varphi_R|\,dx \\
&\geq -\frac{2M}{R}\int_{B_{2R}\setminus B_R} u^p H^{p-1}(\nabla\ln v)\,dx
\end{aligned} \tag{4.3.80}$$

Hence, passing to the limit the right hand side of (4.3.80), by (4.3.73) we have that I_6 goes to zero when R tends to $+\infty$.

Finally, if we combine all the estimates (4.3.76), (4.3.77), (4.3.79), (4.3.80) and we pass to the limit for $m, R \to +\infty$ we deduce that

$$\begin{aligned}
0 &\geq \int_\Omega H^{p-1}(\nabla u)\langle\nabla H(\nabla u), \nabla\eta_1\rangle\,dx - \int_\Omega H^{p-1}(\nabla v)\langle\nabla H(\nabla v), \nabla\eta_2\rangle\,dx \\
&= \limsup_{R\to+\infty}(I_1+I_2+I_3+I_4+I_5+I_6) \geq \liminf_{R\to+\infty}(I_1+I_2+I_3+I_4+I_5+I_6) \\
&\geq \int_{\{u\geq v\}} (u^p+v^p)H^p(\nabla(\ln u - \ln v))\,dx \geq 0,
\end{aligned}$$

which implies $u \leq v$ in Ω as we concluded in the proof of Proposition 4.3.6.

□

4.4 Proof of the main results

This section is dedicated to the proof of our main results: Theorem 4.1.1, Theorem 4.1.3 and Theorem 4.1.4.

Proof of Theorem 4.1.1. We start by proving (4.1.4). To this aim, let us consider u a solution of

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} u^{p-1} = u^{p^*-p} u^{p-1} \qquad \text{in } \mathbb{R}^N. \tag{4.4.1}$$

In particular, we have that u is a subsolution of (4.4.1) in any bounded domain $\Omega = \mathcal{B}_{R_1}^{H^\circ}$. We note that $f(x) := u^{p^*-p} \in L^{\frac{N}{p}}(\mathcal{B}_{R_1}^{H^\circ})$ and satisfies $|f(x)| \leq A[H^\circ(x)]^{-\alpha}$ with $\alpha = ((N-p)/p - \sigma_1)(p^* - p) < p$ for $x \in \mathcal{B}_{R_1}^{H^\circ}$ due to Theorem 4.3.3. By Proposition 4.3.4 we have that the function

$$v(H^\circ(x)) = \frac{1 - \delta[H^\circ(x)]^\varepsilon}{[H^\circ(x)]^{\mu_1}} \in \mathcal{D}^{1,p}(\mathcal{B}_{R_1}^{H^\circ})$$

is a positive supersolution of (4.3.65) in $\mathcal{B}_{R_1}^{H^\circ} \subset \Omega$, with $g \in L^{\frac{N}{p}}(\mathcal{B}_{R_1}^{H^\circ})$ satisfying $g(x) \geq A[H^\circ(x)]^{-\alpha}$ and where $0 < \delta, \varepsilon, R_1 < 1$ are positive constants depending only on N, p, γ, A and α.

Let us consider $\Gamma > 0$, $\mathcal{M} = \sup_{\partial \mathcal{B}_{R_1}^{H^\circ}} u$, $\mathcal{N} = \sup_{\partial \mathcal{B}_{R_1}^{H^\circ}} 1/v$ and define

$$w(x) := \mathcal{N}(\mathcal{M} + \Gamma) v(H^\circ(x)).$$

It is easy to chek that w is a positive supersolution of (4.3.65) in $\mathcal{B}_{R_1}^{H^\circ} \subset \Omega$ and $\inf_{\mathcal{B}_{R_1}^{H^\circ}} w = \mathcal{M} + \Gamma > 0$ and $u \leq w$ on $\partial \mathcal{B}_{R_1}^{H^\circ}$. Hence, by Proposition 4.3.6 we deduce that

$$u \leq w \qquad \text{in } \mathcal{B}_{R_1}^{H^\circ}.$$

Passing to the limit for $\Gamma \to 0$ we obtain that

$$u(x) \leq \frac{C}{[H^\circ(x)]^{\mu_1}} \qquad \text{in } \mathcal{B}_{R_1}^{H^\circ},$$

where $C = \mathcal{M} \cdot \mathcal{N}$.

Now we have to show the estimate from below. Let u be a weak solution of ($\mathcal{P}_H$), then u is a supersolution of

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} u^{p-1} = 0 \qquad \text{in } \mathcal{B}_{R_1}^{H^\circ}. \tag{4.4.2}$$

We set $c_1 := \inf_{\mathcal{B}_{R_1}^{H^\circ}} u > 0$.

Now, we define

$$\tilde{w}(x) := c \frac{c_1}{[H^\circ(x)]^{\mu_1}} \qquad \text{in } \mathcal{B}_{R_1}^{H^\circ},$$

where $c = \inf_{\partial \mathcal{B}_{R_1}^{H^\circ}} [H^\circ(x)]^{\mu_1} = R_1^{\mu_1}$. The function $\tilde{w}$ is a subsolution to (4.4.2) in $\mathcal{B}_{R_1}^{H^\circ}$. Since, it holds that $u \geq \tilde{w}$ in $\partial \mathcal{B}_{R_1}^{H^\circ}$, we conclude by using Proposition 4.3.6 to obtain

$$u \geq \tilde{w} \qquad \text{in } \mathcal{B}_{R_1}^{H^\circ},$$

and hence combining the estimates from above and below we deduce that (4.1.4) is proved.

Now, our aim is to prove (4.1.5). Let us consider u a subsolution of

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} u^{p-1} = u^{p^*-p} u^{p-1} \qquad \text{in } (\mathcal{B}_{R_2}^{H^\circ})^c. \tag{4.4.3}$$

We note that $f(x) := u^{p^*-p} \in L^{\frac{N}{p}}((\mathcal{B}_{R_2}^{H^\circ})^c)$ and by Theorem 4.3.3 we have $|f(x)| \leq A[H^\circ(x)]^{-\alpha}$ with $\alpha = ((N-p)/p + \sigma_2)(p^* - p) > p$ for $x \in (\mathcal{B}_{R_2}^{H^\circ})^c$. By Proposition 4.3.4 the function

$$v(H^\circ(x)) = \frac{1 - \delta[H^\circ(x)]^{-\varepsilon}}{[H^\circ(x)]^{\mu_2}} \in \mathcal{D}^{1,p}((\mathcal{B}_{R_2}^{H^\circ})^c)$$

is a positive supersolution of (4.3.72) in $(\mathcal{B}_{R_2}^{H^\circ})^c \subset \Omega$, with $g \in L^{\frac{N}{p}}((\mathcal{B}_{R_2}^{H^\circ})^c)$ satisfying $g(x) \geq A[H^\circ(x)]^{-\alpha}$ and where $0 < \delta, \varepsilon < 1$ and $R_2 > 1$ are positive constants depending only on N, p, γ, A and α.

Let us consider $\Gamma > 0$, $\mathcal{M} = \sup_{(\partial \mathcal{B}_{R_2}^{H^\circ})^c} u$, $\mathcal{N} = \sup_{(\partial \mathcal{B}_{R_2}^{H^\circ})^c} 1/v$ and define

$$w(x) := \mathcal{N}(\mathcal{M} + \Gamma) v(H^\circ(x)).$$

We note that w is a positive supersolution of (4.3.72) in $(\mathcal{B}_{R_2}^{H^\circ})^c \subset \Omega$ and $\inf_{(\mathcal{B}_{R_2}^{H^\circ})^c} w = \mathcal{M} + \Gamma > 0$ and $u \leq w$ on $(\partial \mathcal{B}_{R_2}^{H^\circ})^c$. We verify the condition (4.3.73). Since $|\nabla \log v(x)| \leq C|x|^{-1}$, by Hölder inequality we have

$$\limsup_{R \to +\infty} \frac{1}{R} \int_{B_{2R} \setminus B_R} u^p |\nabla \log v|^{p-1} \, dx \leq \limsup_{R \to +\infty} \frac{C}{R} \left(\int_{B_{2R} \setminus B_R} |u|^{p^*} \, dx \right)^{\frac{p}{p^*}} = 0, \tag{4.4.4}$$

where C is a constant independent of R.

Hence, by Proposition 4.3.7 we deduce that

$$u \leq w \qquad \text{in } (\mathcal{B}_{R_2}^{H^\circ})^c.$$

Passing to the limit for $\Gamma \to 0$ we obtain that

$$u(x) \leq \frac{C}{[H^\circ(x)]^{\mu_2}} \qquad \text{in } (\mathcal{B}_{R_2}^{H^\circ})^c,$$

where $C = \mathcal{M} \cdot \mathcal{N}$

We conclude with the estimate from below. Let u be a weak solution of ($\mathcal{P}_H$). Then u is a supersolution of

$$-\Delta_p^H u - \frac{\gamma}{[H^\circ(x)]^p} u^{p-1} = 0 \qquad \text{in } \mathbb{R}^N. \tag{4.4.5}$$

We claim that

$$\int_{B_{2R} \setminus B_R} |\nabla \log u|^p \leq C R^{N-p} \tag{4.4.6}$$

for R sufficiently large and constant C independent of R. Indeed, by (4.4.5), we have

$$-\Delta_p^H u \geq 0 \quad \text{in } \mathbb{R}^N. \tag{4.4.7}$$

Therefore, considering the test function $\eta = \zeta^p u^{1-p}$, with $\zeta \in C_c^1(\mathbb{R}^N)$ nonnegative function, and taking in (4.4.7) we have

$$-(p-1)\int_{\mathbb{R}^N} H^p(\nabla u)\zeta^p u^{-p}\,dx + p\int_{\mathbb{R}^N} H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla\zeta\rangle \zeta^{p-1}u^{1-p}\,dx \geq 0. \tag{4.4.8}$$

By Hölder inequality and by 0-homogeneity of ∇H we get

$$\int_{\mathbb{R}^N} H^p(\nabla \log u)\zeta^p\,dx \leq M\frac{p}{p-1}\left(\int_{\mathbb{R}^N} H^p(\nabla \log u)\zeta^p\,dx\right)^{\frac{p-1}{p}}\left(\int_{\mathbb{R}^N} |\nabla\zeta|^p\,dx\right)^{\frac{1}{p}}. \tag{4.4.9}$$

Taking a standard cutoff function ζ in (4.4.9) we get the claim (4.4.6).

Now we set $c_2 = \inf_{(\partial\mathcal{B}^{H^\circ}_{R_2})^c} u > 0$ and $c = \inf_{(\partial\mathcal{B}^{H^\circ}_{R_2})^c}[H^\circ(x)]^{\mu_2}$. We note that $v = c_2c[H^\circ(x)]^{-\mu_2}$ is a weak solution of (4.4.5). Moreover the condition (4.3.73) is verified. Indeed by Hölder inequality and (4.4.6) we have

$$\begin{aligned}\frac{1}{R}\int_{B_{2R}\setminus B_R} v^p|\nabla \log u|^{p-1}\,dx &\leq CR^{-1-p\mu_2+\frac{N}{p}}\left(\int_{B_{2R}\setminus B_R}|\nabla \log u|^p\,dx\right)^{\frac{p-1}{p}}\\ &\leq CR^{-1-p\mu_2+\frac{p-1}{p}(N-p)+\frac{N}{p}} \to 0\end{aligned} \tag{4.4.10}$$

since $\mu_2 > (N-p)/p$. Applying the Proposition 4.3.7 we conclude that

$$u(x) \geq v(x) = c\frac{c_2}{[H^\circ(x)]^{\mu_2}} \quad \text{in } (\mathcal{B}^{H^\circ}_{R_2})^c$$

and therefore the thesis. □

Now we prove Theorem 4.1.3 that will be essential to prove the asymptotic behavior of the gradient of solutions to ($\mathcal{P}_H$). For the reader convenience we state a more detailed statement contained in the following:

Theorem 4.4.1. *Let $v \in C^{1,\alpha}_{loc}(\mathbb{R}^N\setminus\{0\})$ be a positive weak solution of the equation*

$$-\Delta_p^H v - \frac{\gamma}{[H^\circ(x)]^p}v^{p-1} = 0 \qquad \text{in } \mathbb{R}^N\setminus\{0\}, \tag{4.4.11}$$

where $0 \leq \gamma < C_H$. Assume that there exist two positive constants C and c such that

$$\frac{c}{[H^\circ(x)]^{\mu_2}} \leq v(x) \leq \frac{C}{[H^\circ(x)]^{\mu_2}} \qquad \forall x \in \mathbb{R}^N\setminus\{0\}, \tag{4.4.12}$$

and suppose that there exists a positive constant $\hat{C}$ such that

$$|\nabla v(x)| \leq \frac{\hat{C}}{[H^\circ(x)]^{\mu_2+1}} \qquad \forall x \in \mathbb{R}^N\setminus\{0\}, \tag{4.4.13}$$

then

$$v(x) = \frac{\overline{c}_1}{[H^\circ(x)]^{\mu_2}}, \tag{4.4.14}$$

with

$$\overline{c_1} := \limsup_{|x|\to 0}[H^\circ(x)]^{\mu_2}v(x).$$

On the other hand, suppose that there exist two positive constants $\tilde{C}$ and $\tilde{c}$ such that

$$\frac{\tilde{c}}{[H^\circ(x)]^{\mu_1}} \leq v(x) \leq \frac{\tilde{C}}{[H^\circ(x)]^{\mu_1}} \qquad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{4.4.15}$$

and suppose that there exists a positive constant $\overline{C}$ such that

$$|\nabla v(x)| \leq \frac{\overline{C}}{[H^\circ(x)]^{\mu_1+1}} \qquad \forall x \in \mathbb{R}^N \setminus \{0\}, \tag{4.4.16}$$

then

$$v(x) = \frac{\overline{c}_2}{[H^\circ(x)]^{\mu_1}}, \tag{4.4.17}$$

with

$$\overline{c_2} := \limsup_{|x|\to+\infty} [H^\circ(x)]^{\mu_1} v(x).$$

Proof. We prove this result only for $p \geq 2$. The other case is similar. Suppose that (4.4.12) holds. From definition of $\overline{c}_1$, there exist r (sufficiently small) such that

$$v(x) \leq \frac{\overline{c}_1 + a_n}{[H^\circ(x)]^{\mu_2}} \quad \text{in } B_r(0) \setminus \{0\}, \tag{4.4.18}$$

given $a_n \to 0$.

On the other hand, there exist sequences of radii R_n and points x_n with R_n tending to zero and $H^\circ(x_n) = R_n$, such that

$$v(x_n) \geq \frac{\overline{c}_1 - a_n}{[H^\circ(x_n)]^{\mu_2}}. \tag{4.4.19}$$

Now we set

$$w_n(x) := R_n^{\mu_2} v(R_n x) \qquad \forall x \in \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}. \tag{4.4.20}$$

By (4.4.12), it follows that w_n is uniformly bounded in $L^\infty(\mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ})$ and, since w_n satisfies the equation (4.4.11), by [7, 113, 114], it is also uniformly bounded in $C^{1,\alpha}(K)$, for $0 < \alpha < 1$ and for any compact set $K \subset \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}$. Moreover, from Ascoli-Arzela's Theorem, we deduce that $w_n \to w_\infty$ in the norm $\|\cdot\|_{C^{1,\alpha}(K)}$, for any compact set $K \subset \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}$. By (4.4.18), we have

$$w_\infty \leq \frac{\overline{c}_1}{[H^\circ(x)]^{\mu_2}}$$

and by (4.4.19), there exist a point $\overline{x} \in \partial\mathcal{B}_1^{H^\circ}$ such that $w_\infty(\overline{x}) = \overline{c}_1$. By the strong comparison principle [53], that holds under our assumption on H (see Section 7.2), we have

$$w_\infty \equiv \frac{\overline{c}_1}{[H^\circ(x)]^{\mu_2}} \quad \text{in } \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}. \tag{4.4.21}$$

Now we set

$$\hat{v} := \frac{\overline{c}_1}{[H^\circ(x)]^{\mu_2}} \tag{4.4.22}$$

and we note that it solves (4.4.11). Fix $R > 0$ sufficiently large and let $\varphi_R \in C_c^\infty(B_{2R})$ be a standard cut-off function such that

$$\begin{cases} 0 \leq \varphi_R \leq 1 \\ \varphi_R \equiv 1 & \text{on } B_R \\ |\nabla\varphi_R| \leq \frac{2}{R} & \text{on } B_{2R} \setminus B_R. \end{cases} \tag{4.4.23}$$

Let us define

$$\varphi_1 := \varphi_R \frac{\min\{(v^p - \hat{v}^p)^+, m\}}{v^{p-1}} \qquad \text{and} \qquad \varphi_2 := \varphi_R \frac{\min\{(v^p - \hat{v}^p)^+, m\}}{\hat{v}^{p-1}}, \tag{4.4.24}$$

where $m > 1$.

We remark that φ_1 and φ_2 are good test function in any domain $0 \notin \overline{\Omega}$. Testing (4.4.24) in (4.4.11) on the domain $\mathbb{R}^N \setminus \mathcal{B}^{H^\circ}_{R_n}$, and, since the stress field $H(\nabla u)^{p-1}\nabla H(\nabla u) \in W^{1,2}_{loc}(\mathbb{R}^N \setminus \mathcal{B}^{H^\circ}_{R_n})$ (see [125]), exploiting the divergence theorem, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n}} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \varphi_1\rangle dx - \int_{\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n}} H^{p-1}(\nabla \hat{v})\langle \nabla H(\nabla \hat{v}), \nabla \varphi_2\rangle dx \\
&- \int_{\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n}} \frac{\gamma}{H^\circ(x)^p}\left(v^{p-1}\varphi_1 - v^{p-1}\varphi_2\right) dx \\
&= \int_{\partial \mathcal{B}^{H^\circ}_{R_n}} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \eta_n\rangle \varphi_1 - H^{p-1}(\nabla \hat{v})\langle \nabla H(\nabla \hat{v}), \eta_n\rangle \varphi_2 dx,
\end{aligned} \tag{4.4.25}$$

where η_n is the inner unite normal vector at $\partial \mathcal{B}^{H^\circ}_{R_n}$. Now we set $\mathcal{A} := \{0 \le v^p - \hat{v}^p \le m\}$ and $\mathcal{B} := \{v^p - \hat{v}^p \ge m\}$. Then (4.4.25) becomes:

$$\begin{aligned}
&\int_{\partial \mathcal{B}^{H^\circ}_{R_n}} -H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \eta_n\rangle \varphi_1 + H^{p-1}(\nabla \hat{v})\langle \nabla H(\nabla \hat{v}), \eta_n\rangle \varphi_2 \, dx \\
&= \int_{(\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n})\cap \mathcal{A}} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla\left(v - \frac{\hat{v}^p}{v^p} v\right)\rangle \varphi_R \, dx \\
&+ \int_{(\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n})\cap \mathcal{A}} H^{p-1}(\nabla \hat{v})\langle \nabla H(\nabla \hat{v}), \nabla\left(\hat{v} - \frac{v^p}{\hat{v}^p} \hat{v}\right)\rangle \varphi_R \, dx \\
&+ \int_{(\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n})\cap \mathcal{A}} \left(v - \frac{\hat{v}^p}{v^p} v\right) H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \varphi_R\rangle \, dx \\
&+ \int_{(\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n})\cap \mathcal{A}} \left(\hat{v} - \frac{v^p}{\hat{v}^p} \hat{v}\right) H^{p-1}(\nabla \hat{v})\langle \nabla H(\nabla \hat{v}), \nabla \varphi_R\rangle \, dx \\
&+ \int_{(\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n})\cap \mathcal{B}} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla\left(v^{1-p}\right)\rangle m \varphi_R \, dx \\
&+ \int_{(\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n})\cap \mathcal{B}} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \varphi_R\rangle m v^{1-p} \, dx \\
&- \int_{(\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n})\cap \mathcal{B}} H^{p-1}(\nabla \hat{v})\langle \nabla H(\nabla \hat{v}), \nabla\left(\hat{v}^{1-p}\right)\rangle m \varphi_R \, dx \\
&- \int_{(\mathbb{R}^N\setminus \mathcal{B}^{H^\circ}_{R_n})\cap \mathcal{B}} H^{p-1}(\nabla \hat{v})\langle \nabla H(\nabla \hat{v}), \nabla \varphi_R\rangle m \hat{v}^{1-p} \, dx \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7 + I_8.
\end{aligned} \tag{4.4.26}$$

Now we set $\Omega_1 := (\mathbb{R}^N \setminus \mathcal{B}^{H^\circ}_{R_n}) \cap \mathcal{A}$ and $\Omega_2 := (\mathbb{R}^N \setminus \mathcal{B}^{H^\circ}_{R_n}) \cap \mathcal{B}$. By Proposition 4.3.5, it follows that there exits a positive constant depending only on p such that

$$I_1 + I_2 \ge C_p \int_{\Omega_1 \cap B_{2R}} (v^p + \hat{v}^p) H^p(\nabla(\ln v - \ln \hat{v}))\varphi_R \, dx. \tag{4.4.27}$$

Now we estimate I_3+I_4. Using (1.2.6) and the Hölder inequality with conjugate exponent $(N, p/(p-1), p^*)$ we have

$$\begin{aligned}
|I_3+I_4| \leq & M \int_{\Omega_1} \left| v - \frac{\hat{v}^p}{v^p} v \right| H^{p-1}(\nabla v) |\nabla \varphi_R| \, dx + M \int_{\Omega_1} \left| \hat{v} - \frac{v^p}{\hat{v}^p} \hat{v} \right| H^{p-1}(\nabla \hat{v}) |\nabla \varphi_R| \, dx \\
\leq & M \int_{\Omega_1} \left(2v H^{p-1}(\nabla v) + \hat{v} H^{p-1}(\nabla \hat{v}) \right) |\nabla \varphi_R| \, dx + M \int_{\Omega_1} v^p H^{p-1}(\nabla \ln \hat{v}) |\nabla \varphi_R| \, dx \\
\leq & \frac{2M}{R} \int_{B_{2R} \setminus B_R} v H^{p-1}(\nabla v) \, dx + \frac{2M}{R} \int_{B_{2R} \setminus B_R} \hat{v} H^{p-1}(\nabla \hat{v}) \, dx \\
& + \frac{2M}{R} \int_{B_{2R} \setminus B_R} v^p H^{p-1}(\nabla \ln \hat{v}) \, dx \\
\leq & 2M \left(\int_{B_{2R} \setminus B_R} H^p(\nabla v) \, dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} v^{p^*} \, dx \right)^{\frac{1}{p^*}} \\
& + 2M \left(\int_{B_{2R} \setminus B_R} H^p(\nabla \hat{v}) \, dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} \hat{v}^{p^*} \, dx \right)^{\frac{1}{p^*}} \\
& + \frac{2M}{R} \int_{B_{2R} \setminus B_R} v^p H^{p-1}(\nabla \ln \hat{v}) \, dx.
\end{aligned} \tag{4.4.28}$$

By (4.4.22) we have $H(\nabla \ln \hat{v}(x)) \leq C[H^\circ(x)]^{-1}$, where $C = \mu_2$, we have that

$$\begin{aligned}
\frac{1}{R} \int_{B_{2R} \setminus B_R} v^p H^{p-1}(\nabla \ln \hat{v}) \leq & \frac{C}{R^p} \int_{B_{2R} \setminus B_R} |v|^p \\
\leq & C \left(\int_{B_{2R} \setminus B_R} |v|^{p^*} \right)^{\frac{p}{p^*}},
\end{aligned} \tag{4.4.29}$$

where we used the Hölder inequality. Since v has the right summability at the infinity, for R that goes to infinity in the right hand side of (4.4.28), we obtain that I_3+I_4 goes to zero.

Now we proceed with the estimate of the term I_5+I_6. Recalling that $v^p \geq m$ in Ω_2, we get

$$|I_5+I_6| \leq mM \int_{\Omega_2} H^{p-1}(\nabla v) |\nabla \varphi_R| v^{1-p} \, dx + m(p-1) \int_{\Omega_2} H^p(\nabla v) \varphi_R v^{-p} \, dx,$$

Using the properties of φ_R we obtain that

$$\begin{aligned}
|I_5+I_6| \leq & \frac{M}{R} \int_{B_{2R} \setminus B_R} v H^{p-1}(\nabla v) \, dx + (p-1) \int_{\Omega_2} H^p(\nabla v) \, dx \\
\leq & M \left(\int_{B_{2R} \setminus B_R} H^p(\nabla v) \, dx \right)^{\frac{p-1}{p}} \left(\int_{B_{2R} \setminus B_R} v^{p^*} \, dx \right)^{\frac{1}{p^*}} + (p-1) \int_{\Omega_2} H^p(\nabla v) \, dx,
\end{aligned} \tag{4.4.30}$$

Passing to the limit for m, R that go to $+\infty$, we deduce that I_5+I_6 goes to zero since the set Ω_2 vanishes as $m \to +\infty$.

For the last term $I_7 + I_8$ we obtain

$$\begin{aligned} I_7 + I_8 &= m(p-1)\int_{\Omega_2} H^p(\nabla\hat{v})\varphi_R\hat{v}^{-p}\,dx - \int_{\Omega_2} mH^{p-1}(\nabla\hat{v})\langle\nabla H(\nabla\hat{v}),\nabla\varphi_R\rangle\hat{v}^{1-p}\,dx \\ &\geq -M\int_{\Omega_2} mH^{p-1}(\nabla\ln\hat{v})|\nabla\varphi_R|\,dx \\ &\geq -\frac{2M}{R}\int_{B_{2R}\setminus B_R} v^pH^{p-1}(\nabla\ln\hat{v})\,dx \end{aligned} \tag{4.4.31}$$

Hence, passing to the limit in the right hand side of (4.4.31) and using (4.4.29) we have that the right hand of (4.4.31) tends to zero when R tends to the infinity.

Now we estimate the left hand of (4.4.26). By (4.4.24) and Proposition 1.2.3 (in particular see (1.2.14)), we get

$$\begin{aligned} &\int_{\partial\mathcal{B}^{H^\circ}_{R_n}} -H^{p-1}(\nabla\hat{v})\langle\nabla H(\nabla\hat{v}),\eta_n\rangle\varphi_2 + H^{p-1}(\nabla v)\langle\nabla H(\nabla v),\eta_n\rangle\varphi_1\,dx \\ &= -\int_{\partial\mathcal{B}^{H^\circ}_{R_n}} \varphi_1\left(H^{p-1}(\nabla\hat{v})\langle\nabla H(\nabla\hat{v}),\eta_n\rangle - H^{p-1}(\nabla v)\langle\nabla H(\nabla v),\eta_n\rangle\right)\,dx \\ &-\int_{\partial\mathcal{B}^{H^\circ}_{R_n}} (\varphi_2-\varphi_1)H^{p-1}(\nabla\hat{v})\langle\nabla H(\nabla\hat{v}),\eta_n\rangle\,dx \\ &\leq \int_{\partial\mathcal{B}^{H^\circ}_{R_n}} \frac{m}{v^{p-1}}\tilde{C}_p(|\nabla\hat{v}|^{p-2}+|\nabla v|^{p-2})|\nabla\hat{v}-\nabla v| + \frac{2m}{\hat{v}^{p-1}}M\alpha_2|\nabla\hat{v}|^{p-1}\,dx, \\ &:= J_1 + J_2. \end{aligned} \tag{4.4.32}$$

where, in the last line, we used (1.2.3) and (1.2.6).

We set $x = R_n y$, with $y \in \partial\mathcal{B}_1^{H^\circ}$. Using this change of variables and recalling (4.4.12), (4.4.20) and (4.4.21), J_1 can be estimated as

$$\begin{aligned} J_1 &\leq \tilde{C}_1\int_{\partial\mathcal{B}_1^{H^\circ}} mR_n^{N-1+\mu_2(p-1)-(\mu_2+1)(p-1)}|-\mu_2\bar{c}_1\nabla H^\circ(y) - \nabla w_n(y)|\,dy \\ &= \tilde{C}_1 R_n^{N-p-\epsilon} \end{aligned} \tag{4.4.33}$$

where we choose $m = R_n^{-\epsilon}$, for $\epsilon > 0$ fixed sufficiently small, and $\tilde{C}_1$ is a positive constant.

In a similar way J_2 is estimated as

$$J_2 \leq \tilde{C}_2 R_n^{N-p-\epsilon}, \tag{4.4.34}$$

where $\tilde{C}_2$ is a positive constant.

Finally, if we combine all the estimates (4.4.27), (4.4.28), (4.4.30), (4.4.31), (4.4.33), (4.4.34) and passing to the limit for $R_n \to 0$, and then, exploiting the Fatou's lemma, for $R \to +\infty$, we deduce that

$$\int_{\{v\geq\hat{v}\}} (v^p + \hat{v}^p)H^p(\nabla(\ln v - \ln\hat{v}))\,dx = 0,$$

which implies $v \leq \hat{v}$ as we concluded in the proof of Proposition 4.3.6.

To prove that $\hat{v} \leq v$, let us consider

$$\varphi_1 := \varphi_R\frac{\min\{(\hat{v}^p - v^p)^+, m\}}{\hat{v}^{p-1}} \quad \text{and} \quad \varphi_2 := \varphi_R\frac{\min\{(\hat{v}^p - v^p)^+, m\}}{v^{p-1}}, \tag{4.4.35}$$

where $m > 1$ and φ_R is the standard cutoff function defined in (4.4.23). Using (4.4.35) in the weak formulation of (4.4.11), proceeding in a similar way as above we obtain (4.4.14). The only thing to check is that

$$\frac{1}{R}\int_{B_{2R}\setminus B_R} \hat{v}^p H^{p-1}(\nabla \ln v)\, dx \to 0. \tag{4.4.36}$$

Assumption (4.4.13) ensures (4.4.36), and therefore we are done.

Now, assume that (4.4.15) holds and suppose that $p \geq 2$, the other case is similar. The proof is similar to the previous one and, for this reason, we omit some details. From definition of $\overline{c}_2$, there exist r (sufficiently large) such that

$$v(x) \leq \frac{\overline{c}_2 + a_n}{[H^\circ(x)]^{\mu_1}} \quad \text{in } \mathbb{R}^N \setminus B_r(0), \tag{4.4.37}$$

given $a_n \to 0$.

On the other hand, there exist sequences of radii R_n and points x_n with R_n tending to infinity and $H^\circ(x_n) = R_n$, such that

$$v(x_n) \geq \frac{\overline{c}_2 - a_n}{[H^\circ(x_n)]^{\mu_1}}. \tag{4.4.38}$$

Now we set

$$w_n(x) := R_n^{-\mu_1} v\left(\frac{x}{R_n}\right) \qquad \forall x \in \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}. \tag{4.4.39}$$

As in the previous case we obtain

$$w_\infty \equiv \frac{\overline{c}_2}{[H^\circ(x)]^{\mu_1}} \quad \text{in } \mathcal{B}_2^{H^\circ} \setminus \mathcal{B}_{1/2}^{H^\circ}. \tag{4.4.40}$$

Now we set

$$\hat{v} := \frac{\overline{c}_2}{[H^\circ(x)]^{\mu_1}}$$

and we remark that it solves (4.4.11). We show that $v = \hat{v}$ in $\mathbb{R}^N$. To this aim, fix $\varepsilon > 0$ sufficiently small and let $\varphi_\varepsilon \in C^\infty(\mathbb{R}^N)$ be a function such that

$$\begin{cases} 0 \leq \varphi_\varepsilon \leq 1 \\ \varphi_\varepsilon \equiv 0 & \text{on } B_\varepsilon \\ \varphi_\varepsilon \equiv 1 & \text{on } (B_{2\varepsilon})^c \\ |\nabla \varphi_\varepsilon| \leq \frac{2}{\varepsilon} & \text{on } B_{2\varepsilon} \setminus B_\varepsilon. \end{cases} \tag{4.4.41}$$

Let us define

$$\varphi_1 := \varphi_\varepsilon \frac{\min\{(v^p - \hat{v}^p)^+, m\}}{v^{p-1}} \qquad \text{and} \qquad \varphi_2 := \varphi_\varepsilon \frac{\min\{(v^p - \hat{v}^p)^+, m\}}{\hat{v}^{p-1}}, \tag{4.4.42}$$

where $m > 1$.

We note that φ_1 and φ_2 are good test function in any bounded domain $0 \in \Omega$. Testing (4.4.42) in (4.4.11) on the domain $\mathcal{B}_{R_n}^{H^\circ}$ we get

$$\begin{aligned} &\int_{\mathcal{B}_{R_n}^{H^\circ}} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \nabla \varphi_1\rangle dx - \int_{\mathcal{B}_{R_n}^{H^\circ}} H^{p-1}(\nabla \hat{v})\langle \nabla H(\nabla \hat{v}), \nabla \varphi_2\rangle dx \\ &- \int_{\mathcal{B}_{R_n}^{H^\circ}} \frac{\gamma}{H^\circ(x)^p}\left(v^{p-1}\varphi_1 - v^{p-1}\varphi_2\right)\, dx \\ &= \int_{\partial \mathcal{B}_{R_n}^{H^\circ}} H^{p-1}(\nabla v)\langle \nabla H(\nabla v), \eta_n\rangle \varphi_1 - H^{p-1}(\nabla \hat{v})\langle \nabla H(\nabla \hat{v}), \eta_n\rangle \varphi_2\, dx, \end{aligned} \tag{4.4.43}$$

where η_n is the outward unite normal vector at $\partial\mathcal{B}_{R_n}^{H^\circ}$. By (4.4.42) and Proposition 1.2.3, the right hand of (4.4.43) becomes

$$\begin{aligned}
&\int_{\partial\mathcal{B}_{R_n}^{H^\circ}} -H^{p-1}(\nabla\hat{v})\langle\nabla H(\nabla\hat{v}),\eta_n\rangle\varphi_2 + H^{p-1}(\nabla v)\langle\nabla H(\nabla v),\eta_n\rangle\varphi_1\,dx\\
&= \int_{\partial\mathcal{B}_{R_n}^{H^\circ}} -\varphi_1\left(H^{p-1}(\nabla\hat{v})\langle\nabla H(\nabla\hat{v}),\eta_n\rangle - H^{p-1}(\nabla v)\langle\nabla H(\nabla v),\eta_n\rangle\right)\,dx\\
&- \int_{\partial\mathcal{B}_{R_n}^{H^\circ}} (\varphi_2-\varphi_1)H^{p-1}(\nabla\hat{v})\langle\nabla H(\nabla\hat{v}),\eta_n\rangle\,dx\\
&\leq \int_{\partial\mathcal{B}_{R_n}^{H^\circ}} \tilde{C}_p v(|\nabla\hat{v}|^{p-2}+|\nabla v|^{p-2})(\nabla\hat{v}-\nabla v)+\frac{v^p}{\hat{v}^{p-1}}M\alpha_2|\nabla\hat{v}|^{p-1}\,dx,\\
&:= \tilde{J}_1+\tilde{J}_2.
\end{aligned} \tag{4.4.44}$$

where, in the last line, we used (1.2.3) and (1.2.6).

We consider the following change of variables $x = y/R_n$, with $y \in \partial\mathcal{B}_1^{H^\circ}$. Using this fact, by (4.4.39) and (4.4.40), we have

$$\begin{aligned}
\tilde{J}_1 &\leq \overline{C}_1\int_{\partial\mathcal{B}_1^{H^\circ}} R_n^{1-N-\mu_1+(\mu_1+1)(p-1)}|-\mu_1\overline{c}_2 y-\nabla w_n(y)|\,dy\\
&= \overline{C}_1 R_n^{-N+\mu_1 p+p}
\end{aligned} \tag{4.4.45}$$

where $\overline{C}_1$ is a positive constant.

In a similar way $\tilde{J}_2$ is estimated as

$$\tilde{J}_2 \leq \overline{C}_2 R_n^{-N+\mu_1 p+p}, \tag{4.4.46}$$

where $\overline{C}_2$ is a positive constant. We note that $-N+\mu_1 p+p<0$ since $0\leq\mu_1<(N-p)/p$. Proceeding as in the case (4.4.12), we prove that

$$\int_{\{v\geq\hat{v}\}}(v^p+\hat{v}^p)H^p(\nabla(\ln v-\ln\hat{v}))\,dx=0, \tag{4.4.47}$$

which implies $v\leq\hat{v}$ in Ω as we concluded in the proof of Proposition 4.3.6.

To prove that $\hat{v}\leq v$, let us consider

$$\varphi_1 := \varphi_\varepsilon\frac{\min\{(\hat{v}^p-v^p)^+,m\}}{\hat{v}^{p-1}} \qquad\text{and}\qquad \varphi_2 := \varphi_\varepsilon\frac{\min\{(\hat{v}^p-v^p)^+,m\}}{v^{p-1}}, \tag{4.4.48}$$

where $m>1$ and φ_ε is defined as (4.4.41). Testing (4.4.48) in (4.4.11), using the assumption (4.4.16) and proceeding in a similar way as above we get (4.4.17). □

At this point we are ready to prove

Proof of Theorem 4.1.4. We prove (4.1.12), the other case is similar and it can be proved in a similar way. We start by showing the estimate from above. For R_n tending to infinity, let us consider

$$w_n(x) := R_n^{\mu_2}u(R_n x) \qquad \forall x\in\mathcal{B}_2^{H^\circ}\setminus\mathcal{B}_{1/2}^{H^\circ}. \tag{4.4.49}$$

We remark that w_n is uniformly bounded in $L^\infty(\mathcal{B}_2^{H^\circ}\setminus\mathcal{B}_{1/2}^{H^\circ})$ and it weakly solves

$$-\Delta_p^H w_n-\frac{\gamma}{H^\circ(x)^p}w_n^{p-1}=R_n^{\mu_2(p-1)+p-\mu_2(p^*-1)}w_n^{p^*-1} \qquad \text{in } \mathbb{R}^N. \tag{4.4.50}$$

Since $\mu_2(p-1)+p-\mu_2(p^*-1)<0$, by [7, 113, 114], w_n is also uniformly bounded in $C^{1,\alpha}(K)$, for $0<\alpha<1$ and for any compact set $K\subset\mathcal{B}_2^{H^\circ}\setminus\mathcal{B}_{1/2}^{H^\circ}$. For R_n sufficiently large we get the estimate from above in (4.1.12).

Now we prove the estimate from below. Suppose by contradiction that there exist sequences of points x_n such that

$$[H^\circ(x_n)]^{\mu_2+1}|\nabla u(x_n)|\to 0 \qquad \text{for } |x_n|\to\infty. \tag{4.4.51}$$

For $0<a<A$ fixed, let us consider

$$w_n(x):=R_n^{\mu_2}u(R_nx).$$

For n sufficiently large, from Theorem 4.1.1, relabeling the constants, we have

$$\frac{c}{A^{\mu_2}}\le w_n(x)\le\frac{C}{a^{\mu_2}} \qquad \text{in } \overline{\mathcal{B}_A^{H^\circ}\setminus\mathcal{B}_a^{H^\circ}}.$$

Furthermore, recalling the estimate from above of the gradients of the weak solution u, proved previously, we get

$$|\nabla w_n(x)|\le\frac{\tilde{C}}{a^{\mu_2+1}} \quad \text{in } \overline{\mathcal{B}_A^{H^\circ}\setminus\mathcal{B}_a^{H^\circ}}. \tag{4.4.52}$$

For a, A fixed, by [7, 113, 114], w_n is also uniformly bounded in $C^{1,\alpha}(K)$, for $0<\alpha<1$ and for any compact set $K\subset\mathcal{B}_A^{H^\circ}\setminus\mathcal{B}_a^{H^\circ}$. Moreover

$$w_n(x)\to w_{a,A} \qquad \text{in } B_A\setminus B_a,$$

in the norm $C^{1,\alpha'}$, for $0<\alpha'<\alpha$. Moreover, since w_n weakly solves

$$-\Delta_p^H w_n-\frac{\gamma}{H^\circ(x)^p}w_n^{p-1}=R_n^{\mu_2(p-1)+p-\mu_2(p^*-1)}w_n^{p^*-1} \qquad \text{in } \mathcal{B}_A^{H^\circ}\setminus\mathcal{B}_a^{H^\circ}, \tag{4.4.53}$$

we deduce that

$$-\Delta_p^H w_{a,A}-\frac{\gamma}{H^\circ(x)^p}w_{a,A}^{p-1}=0 \qquad \text{in } \mathcal{B}_A^{H^\circ}\setminus\mathcal{B}_a^{H^\circ}. \tag{4.4.54}$$

Now we take $a_j=1/j$ and $A_j=j$, for $j\in\mathbb{N}$ and we construct w_{a_j,A_j} as above. For j goes to infinity, using a standard diagonal process, we construct a limiting profile w_∞ so that

$$-\Delta_p^H w_\infty-\frac{\gamma}{H^\circ(x)^p}w_\infty^{p-1}=0 \qquad \text{in } \mathbb{R}^N\setminus\{0\}, \tag{4.4.55}$$

with $w_\infty\equiv w_{a_j,A_j}$ in $\mathcal{B}_{A_j}^{H^\circ}\setminus\mathcal{B}_{a_j}^{H^\circ}$.

Since w_∞ satisfies the assumptions (4.4.12), (4.4.13) of the Theorem 4.4.1, we get

$$w_\infty(x)=\frac{\overline{c_2}}{[H^\circ(x)]^{\mu_2}}, \tag{4.4.56}$$

where we set $\overline{c_2}:=\limsup_{|x|\to 0}[H^\circ(x)]^{\mu_2}w_\infty(x)$.

Now we set $y_n=x_n/R_n$, and by (4.4.51), we deduce that $|\nabla w_n(y_n)|$ tends to zero as R_n tends to infinity. This fact and the uniform convergence of the gradients imply that there exist $\overline{y}\in\partial\mathcal{B}_1^{H^\circ}$ such that

$$|\nabla w_\infty(\overline{y})|=0.$$

This is an absurd since the solution w_∞ has no critical points. $\square$

4.5 Existence of solutions

In this section we prove the existence of a weak solution to problem ($\mathcal{P}_H$) by means of a minimization problem. For this reason we define the following minimization problem: Let $S(\gamma)$ defined as

$$S(\gamma) := \inf_{u\in\mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}} \frac{\mathcal{L}(u)}{\left(\displaystyle\int_{\mathbb{R}^N} |u|^{p^*}\, dx\right)^{\frac{p}{p^*}}}, \tag{4.5.1}$$

where

$$\mathcal{L}(u) = \int_{\mathbb{R}^N} H^p(\nabla u) - \gamma \frac{|u|^p}{H^\circ(x)^p}\, dx. \tag{4.5.2}$$

Thanks to the Hardy inequality (4.2.1) we deduce that $S(\gamma) \geq 0$.

Theorem 4.5.1. *Let $0 \leq \gamma < C_H$. The problem ($\mathcal{P}_H$) has a positive weak solution that it minimizes the quotient* (4.5.1).

Proof. Let $\{u_n\}$ be a minimizing sequence to (4.5.1). Without loss of generality, because of the homogeneity of the quotient in (4.5.1), we assume that

$$\mathcal{L}(u_n) = 1. \tag{4.5.3}$$

By Hardy inequality (observe that H is a norm equivalent to the euclidean one), $\mathcal{L}(\cdot)^{1/p}$ is an equivalent norm to the standard one $\|\cdot\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}$, we have that

$$\|u_n\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)} \leq C,$$

with C that does not depend on n. Hence up to a subsequence $u_n \rightharpoonup u_0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Moreover let us also assume that (we will prove it later)

$$u_0 \not\equiv 0. \tag{4.5.4}$$

Recalling (4.5.1), using the weak lower semicontinuity of the norm and (4.5.3), we deduce

$$\begin{aligned} S(\gamma) \leq \frac{\mathcal{L}(u_0)}{\left(\displaystyle\int_{\mathbb{R}^N} u_0^{p^*}\, dx\right)^{\frac{p}{p^*}}} &\leq \liminf_n \frac{\mathcal{L}(u_n)}{\left(\displaystyle\int_{\mathbb{R}^N} u_0^{p^*}\, dx\right)^{\frac{p}{p^*}}} \\ &= \frac{1}{\left(\displaystyle\int_{\mathbb{R}^N} u_0^{p^*}\, dx\right)^{\frac{p}{p^*}}} \end{aligned} \tag{4.5.5}$$

Using Sobolev inequality we have that $u_n \to u_0$ a.e. in $\mathbb{R}^N$. Therefore by the Breis-Lieb result [31], it follows that

$$\int_{\mathbb{R}^N} u_0^{p^*}\, dx = \int_{\mathbb{R}^N} u_n^{p^*}\, dx - \int_{\mathbb{R}^N} (u_n - u_0)^{p^*}\, dx + o(1).$$

Then from (4.5.5) we obtain

$$S(\gamma) \leq \frac{1}{\left(\displaystyle\int_{\mathbb{R}^N} u_n^{p^*}\, dx - \int_{\mathbb{R}^N} (u_n - u_0)^{p^*}\, dx + o(1)\right)^{\frac{p}{p^*}}} \tag{4.5.6}$$

From (4.5.1), we deduce

$$\mathcal{L}(u_n) = S(\gamma) \left(\int_{\mathbb{R}^N} u_n^{p^*} \, dx \right)^{\frac{p}{p^*}} + o(1)$$

and

$$\mathcal{L}(u_0 - u_n) \geq S(\gamma) \left(\int_{\mathbb{R}^N} |u_0 - u_n|^{p^*} \, dx \right)^{\frac{p}{p^*}}.$$

Using these last inequalities in (4.5.6) we get

$$\begin{aligned} S(\gamma) \leq S(\gamma) & \frac{1}{\left(\mathcal{L}^{\frac{p^*}{p}}(u_n) - \mathcal{L}^{\frac{p^*}{p}}(u_0 - u_n) + o(1) \right)^{\frac{p}{p^*}}} \\ \leq S(\gamma) & \frac{1}{\left(1 - \mathcal{L}^{\frac{p^*}{p}}(u_0 - u_n) + o(1) \right)^{\frac{p}{p^*}}}, \end{aligned} \tag{4.5.7}$$

where in the last line we used (4.5.3). Taking the limit superior of (4.5.7) we get a contradicition, i.e. $S(\gamma) < S(\gamma)$ unless the sequence $\mathcal{L}(u_0 - u_n) \to 0$. Hence, since $\mathcal{L}(\cdot)^{1/p}$ is an equivalent norm to $\| \cdot \|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}$, we finally get that $u_n \to u_0$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Therefore passing to the limit in (4.5.1), we obtain

$$S(\gamma) = \frac{\mathcal{L}(u_0)}{\left(\int_{\mathbb{R}^N} |u_0|^{p^*} \, dx \right)^{\frac{p}{p^*}}},$$

namely u_0 (eventually redefining it as Cu_0, for some positive constant) is a weak solution to ($\mathcal{P}_H$).

Now we prove that actually (4.5.4) holds, concluding indeed the proof. Let $\{u_n\}$ the minimizing sequence such that (4.5.3) holds. For every n let us take a sequence of radii R_n such that

$$\int_{B_{R_n}} H^p(\nabla u_n) - \gamma \frac{|u_n|^p}{H^\circ(x)^p} \, dx = \int_{\mathbb{R}^N \setminus B_{R_n}} H^p(\nabla u_n) - \gamma \frac{|u_n|^p}{H^\circ(x)^p} \, dx = \frac{\mathcal{L}(u_n)}{2} \tag{4.5.8}$$

and let us define the rescaled sequence

$$w_n = R_n^{\frac{p-N}{p}} u_n \left(\frac{x}{R_n} \right). \tag{4.5.9}$$

Using assumptions (h_H), (4.5.8) and (4.5.9) we deduce that

$$\int_{B_1} H^p(\nabla w_n) - \gamma \frac{|w_n|^p}{H^\circ(x)^p} \, dx = \int_{\mathbb{R}^N \setminus B_1} H^p(\nabla w_n) - \gamma \frac{|w_n|^p}{H^\circ(x)^p} \, dx = \frac{\mathcal{L}(u_n)}{2} \tag{4.5.10}$$

Now we prove the following

Lemma 4.5.2. *Let $\{u_n\}$ be a minimizing sequence, weakly converging to zero. Then, for every ball B_r and for every $\varepsilon \in (-r, r)$ there exists $\rho \in (0, \varepsilon) \cup (\varepsilon, 0)$ such that for a subsequence*

$$\textit{either} \int_{B_{r+\rho}} H^p(\nabla u_n) \, dx \to 0, \qquad \textit{or} \int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n) \, dx \to 0. \tag{4.5.11}$$

Proof. By the homogeneity of the quotient in (4.5.1) we can assume that the minimizing sequence $\{u_n\}$ is such that $\|u_n\|_{L^{p^*}(\mathbb{R}^N)} = 1$, so that $\mathcal{L}(u_n) \to S(\gamma)$. By Ekeland's $\varepsilon-$principle, we can suppose that the minimizing sequence has the Palais-Smale property, that is

$$\begin{aligned}&\int_{\mathbb{R}^N} H^{p-1}(\nabla u_n)\langle \nabla H(\nabla u_n), \nabla\varphi\rangle - \frac{\gamma}{H^\circ(x)^p} u_n^{p-1}\varphi\, dx \\ &= S(\gamma)\int_{\mathbb{R}^N} u_n^{p^*-1}\varphi\, dx + o(1)\|\varphi\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)},\end{aligned} \tag{4.5.12}$$

for all $\varphi \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ We have that

$$\int_r^{r+\varepsilon} d\rho \int_{\rho S^{N-1}} H^p(\nabla u_n) = \int_{B_{r+\varepsilon}\setminus B_r} H^p(\nabla u_n)$$

is bounded. Then we can find $\rho \in (0, \varepsilon)$ such that for infinitely many n's it holds

$$\int_{(r+\rho)S^{N-1}} H^p(\nabla u_n) \le C \int_{B_{r+\varepsilon}\setminus B_r} H^p(\nabla u_n),$$

for some positive constant C and hence up to redefining the constant

$$\int_{(r+\rho)S^{N-1}} |\nabla u_n|^p \le C \int_{B_{r+\varepsilon}\setminus B_r} H^p(\nabla u_n).$$

Therefore (see [152, Theorem A.8]) since

$$W^{1,p}((r+\rho)S^{N-1}) \hookrightarrow W^{1-\frac{1}{p},p}((r+\rho)S^{N-1}) \hookrightarrow L^p((r+\rho)S^{N-1}) \tag{4.5.13}$$

with both embedding compact, we can assume that a subsequence converges strongly to some limit, say u in the trace space $W^{1-\frac{1}{p},p}((r+\rho)S^{N-1})$. Using the fact that the trace operator has a continuous embedding from $W^{1,p}(B_{r+\rho})$ into $W^{1-\frac{1}{p},p}((r+\rho)S^{N-1})$, by the weak convergence to zero of $\{u_n\}$, we deduce that indeed $u \equiv 0$.

Now we show the following

CLAIM: Let $\Omega \subset \mathbb{R}^N$ a generic smooth bounded domain. The inverse operator

$$(-\Delta_p^H)^{-1} : W^{1-\frac{1}{p},p}(\partial\Omega) \to W^{1,p}(\Omega),$$

is continuous. Indeed we consider a succession $g_n \to g$ in $W^{1-\frac{1}{p},p}(\Omega)$, and let $u_n, u \in W^{1,p}(\Omega)$ be the solutions to

$$\begin{cases} -\Delta_p^H u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p^H u_n = 0 & \text{in } \Omega \\ u_n = g_n & \text{on } \partial\Omega. \end{cases} \tag{4.5.14}$$

The solution to (4.5.14) can be obtained minimizing the functional

$$J(u) = \frac{1}{p}\int_\Omega H^p(\nabla u)\, dx$$

on the set $\left\{\{g\}+W_0^{1,p}(\Omega)\right\}$, $\left\{\{g_n\}+W_0^{1,p}(\Omega)\right\}$ respectively. Since $(u-g),(u_n-g_n)\in W_0^{1,p}(\Omega)$, integrating by parts (4.5.14) and subtracting the equations, we obtain

$$\begin{aligned} 0 &= \int_\Omega H^{p-1}(\nabla u)\langle \nabla H(\nabla u), \nabla(u-g)\rangle\, dx \\ &- \int_\Omega H^{p-1}(\nabla u_n)\langle \nabla H(\nabla u_n), \nabla(u_n-g_n)\rangle\, dx \\ &= \int_\Omega \langle H^{p-1}(\nabla u)\nabla H(\nabla u) - H^{p-1}(\nabla u_n)\nabla H(\nabla u_n), \nabla u - \nabla u_n\rangle\, dx \\ &- \int_\Omega \langle H^{p-1}(\nabla u)\nabla H(\nabla u) - H^{p-1}(\nabla u_n)\nabla H(\nabla u_n), \nabla g - \nabla g_n\rangle\, dx. \end{aligned}$$

We recall (see [35, Lemma 4.1]) that for $x\in\mathbb{R}^N\setminus\{0\}$, $y\in\mathbb{R}^N$ there exist a constant $C>0$ such that

$$\langle H^{p-1}(x)\nabla H(x) - H^{p-1}(y)\nabla H(y), x-y\rangle \geq C(|x|+|y|)^{p-2}|x-y|^2, \tag{4.5.15}$$

Therefore, by (4.5.15) and (1.2.12) we get

$$\begin{aligned} &C\int_\Omega \left((|\nabla u|+|\nabla u_n|)^{p-2}\,(|\nabla u-\nabla u_n|)^2\right)\, dx \\ &\leq \tilde{C}_p \int_\Omega (|\nabla u|+|\nabla u_n|)^{p-2}|\nabla u-\nabla u_n||\nabla g-\nabla g_n|\, dx \\ &\leq \tilde{C}_p \left(\int_\Omega (|\nabla u|+|\nabla u_n|)^p\, dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |\nabla g-\nabla g_n|^p\, dx\right)^{\frac{1}{p}}, \end{aligned} \tag{4.5.16}$$

where in the last inequality we have used the Hölder inequality.

We recall that by $W^{1-\frac{1}{p},p}(\partial\Omega)$ we denote the space of traces $u_{|\partial\Omega}$, namely the set (of equivalence classes) $\left\{\{u\}+W_0^{1,p}(\Omega), u\in W^{1,p}(\Omega)\right\}$, endowed with the trace norm

$$\|u_{|\partial\Omega}\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} = \inf\{\|v\|_{W^{1,p}(\Omega)} \;:\; u-v\in W_0^{1,p}(\Omega)\}. \tag{4.5.17}$$

Hence using (4.5.16) letting the boundary data $g_n\to g$ in the sense of (4.5.17) we obtain the claim.

Let us define the two auxiliary sequences of $\mathcal{D}^{1,p}(\mathbb{R}^N)$ as follows:

$$u_{1,n}(x) = \begin{cases} u_n(x) & \text{if } x\in B_{r+\rho} \\ w_{1,n}(x) & \text{if } x\in B_{r+\varepsilon}\setminus B_{r+\rho} \\ 0 & \text{elsewhere;} \end{cases}$$

and

$$u_{2,n}(x) = \begin{cases} 0 & \text{if } x\in B_{r-\varepsilon} \\ w_{2,n}(x) & \text{if } x\in B_{r+\rho}\setminus B_{r-\varepsilon} \\ u_n(x) & \text{elsewhere,} \end{cases} \tag{4.5.18}$$

where $w_{1,n}$ respectively $w_{2,n}$ denote the solutions to

$$\begin{cases} -\Delta_p^H w_{1,n} = 0 & \text{in } B_{r+\varepsilon} \setminus B_{r+\rho} \\ w_{1,n} = 0 & \text{on } \partial B_{r+\varepsilon} \\ w_{1,n} = u_n & \text{on } \partial B_{r+\rho} \end{cases} \tag{4.5.19}$$

respectively

$$\begin{cases} -\Delta_p^H w_{2,n} = 0 & \text{in } B_{r+\rho} \setminus B_{r-\varepsilon} \\ w_{2,n} = u_n & \text{on } \partial B_{r+\rho} \\ w_{2,n} = 0 & \text{on } \partial B_{r-\varepsilon}. \end{cases} \tag{4.5.20}$$

Since $u_n \to 0$ on $\partial B_{r+\rho}$ in the $W^{1-\frac{1}{p},p}$ norm, see (4.5.13), by the above claim we immediately get that both

$$w_{1,n} \overset{W^{1,p}}{\longrightarrow} 0 \quad \text{in } B_{r+\varepsilon} \setminus B_{r+\rho} \qquad \text{and} \qquad w_{2,n} \overset{W^{1,p}}{\longrightarrow} 0 \quad \text{in } B_{r+\rho} \setminus B_{r-\varepsilon}. \tag{4.5.21}$$

Using $u_{1,n}$ as test function in (4.5.12) we obtain

$$\int_{\mathbb{R}^N} H^{p-1}(\nabla u_n)\langle \nabla H(\nabla u_n), \nabla u_{1,n}\rangle - \frac{\gamma}{H^\circ(x)^p} u_n^{p-1} u_{1,n}\, dx$$
$$= S(\gamma) \int_{\mathbb{R}^N} u_n^{p^*-1} u_{1,n}\, dx + o(1)\|u_{1,n}\|_{\mathcal{D}^{1,p}(\mathbb{R}^N)}$$

and recalling the definition of $u_{1,n}$ and by (4.5.21), we obtain

$$\int_{B_{r+\rho}} H^p(\nabla u_n) - \frac{\gamma}{H^\circ(x)^p} u_n^p\, dx = S(\gamma) \int_{B_{r+\rho}} u_n^{p^*}\, dx + o(1) = S(\gamma) \int_{B_{r+\rho}} u_{1,n}^{p^*}\, dx + o(1)$$

In the same way, using $u_{2,n}$ as test function in (4.5.12) we obtain

$$\int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n) - \frac{\gamma}{H^\circ(x)^p} u_n^p\, dx$$
$$= S(\gamma) \int_{\mathbb{R}^N \setminus B_{r+\rho}} u_n^{p^*}\, dx + o(1) = S(\gamma) \int_{\mathbb{R}^N \setminus B_{r+\rho}} u_{2,n}^{p^*}\, dx + o(1).$$

Moreover, by definition (4.5.2), using the two sequences $\{u_{1,n}\}$ and $\{u_{2,n}\}$ we infer that actually

$$\mathcal{L}(u_{1,n}) = \int_{B_{r+\rho}} H^p(\nabla u_n) - \frac{\gamma}{H^\circ(x)^p} u_n^p\, dx + o(1)$$

and

$$\mathcal{L}(u_{2,n}) = \int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n) - \frac{\gamma}{H^\circ(x)^p} u_n^p\, dx + o(1)$$

so that

$$\mathcal{L}(u_n) = \mathcal{L}(u_{1,n}) + \mathcal{L}(u_{2,n}) + o(1) \tag{4.5.22}$$

and $\|u_n\|_{p^*}^{p^*} = \|u_{1,n}\|_{p^*}^{p^*} + \|u_{2,n}\|_{p^*}^{p^*} + o(1)$. Let us assume for example, that $\{u_{1,n}\}$ does not converges to zero. Since $\{u_n\}$ is a minimizing sequence we have that (see (4.5.1))

$$\mathcal{L}(u_n) = S(\gamma)\|u_n\|_{p^*}^p + o(1)$$

and also that $\mathcal{L}(u_{2,n}) \geq S(\gamma)\|u_{2,n}\|_{p^*}^p$ and

$$\begin{aligned}
\frac{\mathcal{L}(u_{1,n})}{\|u_{1,n}\|_{p^*}^p} &= \frac{\mathcal{L}(u_n) - \mathcal{L}(u_{2,n}) + o(1)}{(\|u_n\|_{p^*}^{p^*} - \|u_{2,n}\|_{p^*}^{p^*} + o(1))^{\frac{p}{p^*}}} \\
&\leq S(\gamma) \frac{\mathcal{L}(u_n) - \mathcal{L}(u_{2,n}) + o(1)}{(\mathcal{L}(u_n)^{\frac{p^*}{p}} - \mathcal{L}(u_{2,n})^{\frac{p^*}{p}} + o(1))^{\frac{p}{p^*}}}.
\end{aligned}$$

By (4.5.22) we deduce that

$$\limsup_n \mathcal{L}(u_{2,n}) = \limsup_n (\mathcal{L}(u_n) - \mathcal{L}(u_{1,n}) + o(1)) < \limsup_n \mathcal{L}(u_n),$$

by some computations we deduce that actually

$$\limsup_n \frac{\mathcal{L}(u_{1,n})}{\|u_{1,n}\|_{p^*}^p} < S(\gamma),$$

a contradiction with (4.5.1) unless $\mathcal{L}(u_{2,n})$ tends to zero. Using Hardy inequality in (4.5.2), recalling (4.5.18), we obtain

$$\begin{aligned}
0 \leftarrow \int_{\mathbb{R}^N} H^p(\nabla u_{2,n})\, dx &= \int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n)\, dx + \int_{B_{r+\rho}\setminus B_{r-\varepsilon}} H^p(\nabla w_{2,n})\, dx \\
&= \int_{\mathbb{R}^N \setminus B_{r+\rho}} H^p(\nabla u_n)\, dx + o(1).
\end{aligned}$$

The other case of (4.5.11) can be proved arguing in the same as we have done above, assuming that $\{u_{2,n}\}$ does not converge to zero. This concludes the proof of the lemma. $\square$

Using the invariance of the problem under the scaling $R_n^{(p-N)/p} u(x/R_n)$, the sequence $\{w_n\}$ (see (4.5.9) and (4.5.10)) is still a minimizing sequence bounded in $\mathcal{D}^{1,p}(\mathbb{R}^N)$: hence it admits (up to a subsequence) a weakly convergence sequence. We want to show that the weak limit cannot be zero. We argue by contradiction and we apply Lemma 4.5.2 twice choosing $r = 1$ and $\varepsilon = \pm 1/4$ respectively. We find the existence of $\rho^+ \in (0, 1/4)$ and $\rho^- \in (-1/4, 0)$ such that (4.5.11) holds. Using the alternative (4.5.11) together with (4.5.10), we obtain that

$$\int_{B_{1+\rho^-}} H^p(\nabla w_n)\, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N \setminus B_{1+\rho^+}} H^p(\nabla w_n)\, dx \to 0. \tag{4.5.23}$$

Since $w_n \rightharpoonup 0$ using the strong convergence on compacts K of $w_n \to 0$ in $L^p(K)$ we deduce that

$$\int_{B(0)_{\frac{3}{2}} \setminus B(0)_{\frac{1}{2}}} \frac{|w_n|^p}{H^\circ(x)^p}\, dx \to 0. \tag{4.5.24}$$

Let us take a smooth cut-off function η, with $0 \leq \eta \leq 1$, such that $\eta \equiv 1$ in $B(0)_{5/4} \setminus B(0)_{3/4}$ and $\eta \equiv 0$ in $\mathbb{R}^N \setminus (B(0)_{3/2} \setminus B(0)_{1/2})$. Take in to account Hardy inequality, (4.5.23) and (4.5.24) we deduce that

$$\int_{\mathbb{R}^N} |H^p(\eta \nabla w_n) - H^p(\nabla w_n)|\, dx \to 0,$$

for $n \to +\infty$ and therefore we infer that

$$\int_{\mathbb{R}^N} H^p(\eta \nabla w_n)\, dx = \int_{\mathbb{R}^N} H^p(\nabla w_n)\, dx + o(1).$$

Then, recalling also (4.5.1) we get

$$S(0) \leq \frac{\displaystyle\int_{\mathbb{R}^N} H^p(\eta \nabla w_n)\, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |\eta w_n|^{p^*}\, dx\right)^{\frac{p}{p^*}}} \leq \frac{\mathcal{L}(w_n) + o(1)}{\left(\displaystyle\int_{\mathbb{R}^N} |w_n|^{p^*}\, dx\right)^{\frac{p}{p^*}} + o(1)}.$$

Passing to the limit we obtain

$$S(0) \leq S(\gamma). \tag{4.5.25}$$

We claim that (4.5.25) is not possible. Indeed let u_0 be a minimizer of (4.5.1) for $\gamma = 0$ ([49]). Therefore we clearly deduce the following

$$S(\gamma) \leq \frac{\displaystyle\int_{\mathbb{R}^N} H^p(\nabla u_0) - \gamma \frac{|u_0|^p}{H^\circ(x)^p}\, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_0|^{p^*}\, dx\right)^{\frac{p}{p^*}}} < \frac{\displaystyle\int_{\mathbb{R}^N} H^p(\nabla u_0)\, dx}{\left(\displaystyle\int_{\mathbb{R}^N} |u_0|^{p^*}\, dx\right)^{\frac{p}{p^*}}} = S(0). \tag{4.5.26}$$

Inequality (4.5.26) gives the desired contradiction, concluding the proof. □

Chapter 5

Second order regularity for solutions to anisotropic degenerate elliptic equations

The goal of this chapter is to investigate the second order regularity properties of solutions of degenarate elliptic problems based on the Uhlenbeck structure.

Let Ω be a domain in $\mathbb{R}^N$ and H be a Finsler norm (see Section 1.2 for more details), we are interested in the second order regularity of solutions of

$$-\operatorname{div}(A(H(\nabla u))H(\nabla u)\nabla H(\nabla u)) = f(x), \qquad \text{in } \Omega. \tag{5.0.1}$$

If $A(t) = t^{p-2}$, we immediately recognized that problem (5.0.1) reduces to Finsler p-Laplacian problem

$$-\Delta_p^H u := -\operatorname{div}(H(|\nabla u|)^{p-1}\nabla H(\nabla u)) = f(x), \qquad \text{in } \Omega. \tag{5.0.2}$$

Throughout the chapter we assume the following assumptions on A and f

(H_A) The function $A : (0, +\infty) \to (0, +\infty)$ is of class $C^{1,\alpha}_{loc}((0, +\infty))$ and fulfills the following

$$-1 < \inf_{t>0} \frac{tA'(t)}{A(t)} := m_A \leq M_A := \sup_{t>0} \frac{tA'(t)}{A(t)} < +\infty.$$

(H_f) the source term $f \in W^{1,\frac{N}{N-\gamma-s}}_{loc}(\Omega) \cap C^{0,\alpha}_{loc}(\Omega)$ where $s \in (0, N-\gamma)$ and $\gamma < N-2$ if $N > 2$ and $\gamma = 0$ if $N = 2$. If $\gamma = 0$, we consider $f \in W^{1,1}_{loc}(\Omega) \cap C^{0,\alpha}_{loc}(\Omega)$.

Recalling that for the p-Laplace equation, which is obtained when $A(t) = t^{p-2}$ and H is the Euclidean norm, solutions are understood in a weak sense, it is not surprising that a similar interpretation must be applied to solutions of (5.0.1). In particular, we choose as a natural space for solutions of (5.0.1) an Orlicz-Sobolev space that depend on the function $A(\cdot)$. This setting has been strongly used in [41, 42, 45]. For a precise definition of a weak solution to (5.0.1), we refer the reader to Section 5.1.

By standard regularity results (see [51, 67, 101, 109, 113, 114, 157]) weak solutions to equation (5.0.1) belong to $C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$ where Z_u denotes the set where the gradient vanishes.

As mentioned in the introduction, our first result concernes the regularity of the families of stress fields $A(H(\nabla u))^{\beta}\, H(\nabla u)\nabla H(\nabla u)$-like and involves again only m_A and M_A.

In [7], the authors proved that, if $u \in W^{1,p}_{loc}(\Omega)$ is a weak solution of (5.0.2) and for $f \in L^q_{loc}(\Omega)$ for a suitable choice of q, the stress field $H(|\nabla u|)^{p-1}\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega)$. Global second order estimates for anisotropic problems, in the Uhlenbeck structure has been obtained in very recent result due to Antonini, Cianchi, Ciraolo, Farina and Maz'ya [8]. This paper is focused on an L^2-second order regularity theory for solution to (5.0.1) and it can be stated as

$$f \in L^2 \iff A(H(\nabla u))H(\nabla u)\nabla H(\nabla u) \in W^{1,2}(\Omega).$$

under minimal assumptions on the regularity of $\partial\Omega$ and on H.

We have the following

Theorem 5.0.1. *Let $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$ be a weak solution to (5.0.1), where f satisfies (H_f).*
Then

- *If $0 < m_A < M_A$, then*
$$A(H(\nabla u))^{\frac{k-1}{m_A}}\, H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$
for any $k > 1 + \dfrac{m_A}{2} - \dfrac{m_A}{2M_A}$.

- *If $m_A < M_A < 0$, then*
$$A(H(\nabla u))^{\frac{k-1}{m_A}}\, H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$
for any $k > \dfrac{1}{2} + \dfrac{m_A}{2}$.

- *If $m_A < 0 < M_A$, then*
$$A(H(\nabla u))^{\frac{k-1}{M_A}}\, H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$
for any $k \in \left(\dfrac{1}{2} + \dfrac{M_A}{2}, 1 + \dfrac{M_A}{2} - \dfrac{M_A}{2m_A}\right)$.

- *If $m_A = 0$ and $M_A > 0$, then*
$$A(H(\nabla u))^{k-1}\, H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$
for any $k > \dfrac{3}{2} - \dfrac{1}{2M_A}$.

- *If $m_A < 0$ and $M_A = 0$, then*
$$A(H(\nabla u))^{k-1}\, H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N),$$
for any $k < \dfrac{3}{2} - \dfrac{1}{2m_A}$.

We remark that for a suitable choices of k (for $k = m_A + 1, k = M_A + 1$ and $k = 1$) we get

$$A(H(\nabla u))H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega).$$

In the second part of this chapter, we derive second-order estimates for the solution u of equation (5.0.1). In the isotropic case, as shown in [45], the authors proved that if $f \in L^{n,1}$ and $\inf_t A(t) > 0$ then $u \in W^{2,2}_{loc}(\Omega)$. Similar results are obtained in [40, 79], under stronger assumption on the functions f and A.

In [35], for the anisotropic case with A exhibiting p-type growth, and under stronger assumptions on f, the authors prove that if $u \in C^1(\overline{\Omega})$ is a weak solution of (5.0.1) then

- if $p \in (1, 3)$, then $u \in W^{2,2}(\Omega)$;
- if $p \geq 3$ and f is positive then $u \in W^{2,q}(\Omega)$ where $q < \frac{p-1}{p-2}$.

The hypothesis $u \in C^1(\overline{\Omega})$ is giustified by the fact the Ω is supposed to be smooth and hence Lieberman results [113] applied.

Our result does not require any growth conditions on the function A. In particular we have

Theorem 5.0.2. *Let $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$ be a weak solution to (5.0.1) where f satisfies (H_f). Suppose $\inf_{t\in[0,M]} A(t) = 0$. Then*

- *if $M_A < 1$, it holds*
$$u \in W^{2,2}_{loc}(\Omega). \tag{5.0.3}$$
- *If $M_A \geq 1$ and if we assume that*
$$f \geq \tau > 0 \quad \text{a.e. in } \Omega. \tag{5.0.4}$$
it holds
$$u \in W^{2,q}_{loc}(\Omega), \tag{5.0.5}$$
with $q < \dfrac{M_A + 1}{M_A}$.

If $\inf_{t\in[0,M]} A(t) > 0$, then (5.0.3) holds without further assumptions on M_A.

The chapter is organized as follows: In the next section, we introduce preliminaries and briefly define Orlicz-Sobolev spaces to clarify our notion of a weak solution for (5.0.1). In Section 5.2 we establish summability properties for the second derivative, which are crucial for our results. In the same section, we prove Theorem 5.0.1 by means of Propositions. In Section 5.3 we get a summability properties for the weight $A(H(\nabla u))^{-1}$, which is a key tool for proving the second part of Theorem 5.0.2, thus concluding the chapter.

5.1 Preliminaries

In what follows, we will assume standard hypotheses on the functions involved in our problem (see [41, 42, 45]).
We recall that the function $A \in C^{1,\alpha}_{loc}((0, +\infty))$ and fulfills the following:

$$-1 < \inf_{t>0} \frac{tA'(t)}{A(t)} := m_A \leq M_A := \sup_{t>0} \frac{tA'(t)}{A(t)} < +\infty. \tag{5.1.1}$$

We recall that above assumption gives us that (see [42, Proposition 4.1])

$$A(1)\min\{t^{m_A},t^{M_A}\}\le A(t)\le A(1)\max\{t^{m_A},t^{M_A}\}\qquad \text{for } t>0, \tag{5.1.2}$$

holds.

Moreover, since $m_A>-1$ there exists $\sigma\in[0,1)$ such that $m_A+\sigma>0$ and

$$t^\sigma A(t)\to 0,\quad \text{as } t\to 0. \tag{5.1.3}$$

Next Lemma will be useful later.

Lemma 5.1.1. *For any $v,w\in\mathbb{R}^N$ and for any $\xi\in\mathbb{R}^N\setminus\{0\}$, there exist two constants $\tilde C,\bar C>0$ such that:*

$$\left[A(H(\xi))+H(\xi)A'(H(\xi))\right]\langle\nabla H(\xi),v\rangle^2+H(\xi)A(H(\xi))\langle D^2H(\xi)v,v\rangle\ge \tilde C A(H(\xi))|v|^2 \tag{5.1.4}$$

$$\begin{aligned}\left[A(H(\xi))+H(\xi)A'(H(\xi))\right]\langle\nabla H(\xi),v\rangle\langle\nabla H(\xi),w\rangle+H(\xi)A(H(\xi))\langle D^2H(\xi)v,w\rangle\\ \le \bar C A(H(\xi))|v||w|\end{aligned} \tag{5.1.5}$$

where $\tilde C=\tilde C(\alpha_1,m_A,\Lambda)$ and $\bar C=\bar C(\alpha_2,M,\overline{M},M_A)$.

Proof. Let $\xi\in\mathbb{R}^N\setminus\{0\}$. By (1.2.4), we deduce that $\mathbb{R}^N=Span\{\xi,\nabla H(\xi)^\perp\}$. Now we consider

$$v=\alpha\xi+\eta,$$

with $\alpha\in\mathbb{R}$ and $\eta\in\nabla H(\xi)^\perp$. By (5.1.1), (1.2.4), (1.2.9), (1.2.1), since D^2H is a symmetric matrix we get:

$$\begin{aligned}
&\left[A(H(\xi))+H(\xi)A'(H(\xi))\right]\langle\nabla H(\xi),v\rangle^2+H(\xi)A(H(\xi))\langle D^2H(\xi)v,v\rangle\\
=&\left[A(H(\xi))+H(\xi)A'(H(\xi))\right](\alpha\langle\nabla H(\xi),\xi\rangle)^2+\alpha H(\xi)A(H(\xi))\langle D^2H(\xi)v,\xi\rangle\\
&+H(\xi)A(H(\xi))\langle D^2H(\xi)v,\eta\rangle \qquad (5.1.6)\\
=&\left[A(H(\xi))+H(\xi)A'(H(\xi))\right]\alpha^2H^2(\xi)+\alpha H(\xi)A(H(\xi))\langle D^2H(\xi)\eta,\xi\rangle\\
&+H(\xi)A(H(\xi))\langle D^2H(\xi)\eta,\eta\rangle\\
=&\left[A(H(\xi))+H(\xi)A'(H(\xi))\right]\alpha^2H^2(\xi)+H(\xi)A(H(\xi))\langle D^2H(\xi)\eta,\eta\rangle\\
\ge&\left[A(H(\xi))+H(\xi)A'(H(\xi))\right]\alpha^2H^2(\xi)+\Lambda H(\xi)A(H(\xi))|\xi|^{-1}|\eta|^2=\\
=&\ \alpha^2A(H(\xi))\left[1+\frac{H(\xi)A'(H(\xi))}{A(H(\xi))}\right]H^2(\xi)+\Lambda H(\xi)A(H(\xi))|\xi|^{-1}|\eta|^2\\
\ge&\ \alpha^2(m_A+1)A(H(\xi))H^2(\xi)+\Lambda H(\xi)A(H(\xi))|\xi|^{-1}|\eta|^2
\end{aligned}$$

We now consider two cases: if $2|\alpha\xi|\le|v|$,

$$|\eta|^2=|v-\alpha\xi|^2\ge(|v|-|\alpha\xi|)^2\ge\frac{|v|^2}{4}. \tag{5.1.7}$$

Therefore, by (5.1.7) and (1.2.3) we have that

$$\begin{aligned}
&\left[A(H(\xi))+H(\xi)A'(H(\xi))\right]\langle\nabla H(\xi),v\rangle^2+H(\xi)A(H(\xi))\langle D^2H(\xi)v,v\rangle\\
\ge&\ \Lambda H(\xi)A(H(\xi))|\xi|^{-1}|\eta|^2\ge\Lambda H(\xi)A(H(\xi))|\xi|^{-1}\frac{|v|^2}{4}\ge c_1\Lambda A(H(\xi))\frac{|v|^2}{4}\\
=&\ \tilde C_1A(H(\xi))|v|^2, \qquad (5.1.8)
\end{aligned}$$

where $\tilde C_1:=\dfrac{c_1\Lambda}{4}>0$.

On the contrary, if $2|\alpha\xi| > |v|$:

$$\begin{aligned} & \left[A(H(\xi)) + H(\xi)A'(H(\xi))\right] \langle \nabla H(\xi), v\rangle^2 + H(\xi)A(H(\xi))\langle D^2 H(\xi)v, v\rangle \\ \geq \; & \alpha^2 \alpha_1 (m_A + 1) A(H(\xi))|\xi|^2 \geq \tilde{C}_2 A(H(\xi))|v|^2, \end{aligned} \tag{5.1.9}$$

where $\tilde{C}_2 := \dfrac{\alpha_1(m_A+1)}{4} > 0$.

Choosing $\tilde{C} = \min\{\tilde{C}_1, \tilde{C}_2\}$, we have (5.1.4).

Let us now prove (5.1.5). By (5.1.1), (1.2.3), (1.2.6) and (1.2.8) we get

$$\begin{aligned} & \left[A(H(\xi)) + H(\xi)A'(H(\xi))\right] \langle \nabla H(\xi), v\rangle\langle \nabla H(\xi), w\rangle + H(\xi)A(H(\xi))\langle D^2 H(\xi)v, w\rangle \\ & \leq \left[A(H(\xi)) + H(\xi)A'(H(\xi))\right] |\nabla H(\xi)|^2 |v||w| + H(\xi)A(H(\xi))|D^2 H(\xi)||v||w| \\ & \leq M^2 \left(1 + |M_A|\right) A(H(\xi))|v||w| + \overline{M}\alpha_2 A(H(\xi))|v||w| \\ & =: \bar{C}(\alpha_2, M, \overline{M}, M_A) A(H(\xi))|v||w|, \end{aligned} \tag{5.1.10}$$

where $\bar{C}(\alpha_2, M, \overline{M}, M_A) > 0$. This completes the proof. □

5.1.1 Weak solution of our problem.

We conclude this section defining the meaning of weak solution for our main problem (5.0.1). Following [41, Section 2.2], in our hypotheses on A, one can verify that the function

$$\mathcal{A}(t) := \int_0^t A(s)s\,ds \tag{5.1.11}$$

is a Young function (i.e. it is convex and $\mathcal{A}(0) = 0$). By [41, Proposition 2.9], we get $\mathcal{A} \in \Delta_2$ (i.e. $\mathcal{A}(2t) \leq C\mathcal{A}(t)$ whenever $t > 0$) and $\mathcal{A}$ is a $N-$function (i.e. $\mathcal{A}(t)/t$ is a null function as $t \to 0$ and it goes to $+\infty$, as $t \to +\infty$). For a such function $\mathcal{A}$ define

$$W^{1,\mathcal{A}}(\Omega) = \{u \in L^{\mathcal{A}}(\Omega) : u \text{ is weakly-differentiable and } |\nabla u| \in L^{\mathcal{A}}(\Omega)\} \tag{5.1.12}$$

where $L^{\mathcal{A}}(\Omega)$ is the Banach space of the real-valued measurable functions on Ω such that the Luxemburg norm

$$\|u\|_{\mathcal{A}} = \inf \left\{ \mu > 0 : \int_\Omega \mathcal{A}\left(\frac{|u(x)|}{\mu}\right) dx \leq 1 \right\} < \infty.$$

The space $W_0^{1,\mathcal{A}}(\Omega)$ is the subset of $W^{1,\mathcal{A}}(\Omega)$ of those functions such that the continuation of u by 0 outside Ω is weakly differentiable in $\mathbb{R}^N$. The space $W_{loc}^{1,\mathcal{A}}(\Omega)$ is defined accordingly.

The next is an immediate corollary of classical results due to Donaldson and Trudinger [70]

Corollary 5.1.2. *Let Ω' be an open bounded set in $\mathbb{R}^N$ and $\mathcal{A}$ defined in (5.1.11), the space $C_c^\infty(\Omega')$ is dense in $W_0^{1,\mathcal{A}}(\Omega')$, $C^\infty(\bar{\Omega}')$ is dense in $W^{1,\mathcal{A}}(\Omega')$, for every $\Omega' \subset\subset \Omega$, where Ω' has a Lipschitz boundary. Moreover $W_0^{1,\mathcal{A}}(\Omega')$ and $W^{1,\mathcal{A}}(\Omega')$ are reflexive.*

Definition 5.1.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and f satisfying (H_f). A weak solution of (5.0.1) is a function $u \in W_{loc}^{1,\mathcal{A}}(\Omega)$ such that it holds*

$$\int_\Omega A(H(\nabla u))H(\nabla u)\langle \nabla H(\nabla u), \nabla \psi\rangle\,dx = \int_\Omega f\psi\,dx, \tag{5.1.13}$$

for every $\psi \in C_c^\infty(\Omega)$.

By Corollary 5.1.2, by density arguments, we can test our problem against $\psi \in W_0^{1,\mathcal{A}}(\Omega')$, *for any subset* $\Omega' \subset\subset \Omega$.

5.2 Local regularity

In this section we prove a result about the integrability of the second derivative of our solution and thus we exploit this last to prove the Theorem 5.0.1. The next summabilty properties is crucial in our method to investigate on the regularity of weak solutions to (5.0.1).

In what follows we will suppose that $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$. By Lemma 5.1.1 (see (5.1.4)) and [8, see (3.29)] we have

$$\begin{aligned}&\left[A(H(\xi)) + H(\xi)A'(H(\xi))\right] \langle \nabla H(\xi), v\rangle^2 + H(\xi)A(H(\xi))\langle D^2H(\xi)v, v\rangle \geq \tilde{C}A(H(\xi))|v|^2\\ &\geq \tilde{C}\frac{c(m_a, \lambda, \Lambda_2)}{\alpha_2}A(|\xi|)|v|^2\end{aligned}$$

therefore hypotheses (1.10) of Lieberman [114] are satisfied. Moreover, by [106, Theorem 5.1] we have that $u \in L^\infty_{loc}(\Omega)$. These permits to exploit [114, Theorem 1.7] to get $u \in C^{1,\alpha}_{loc}(\Omega)$.
Regularity outside the set of critical point Z_u is supposed for our solutions mainly taking into accout the results obtained in [101, 109].

Since we are supposing $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$ let us denote by

$$\tilde{u}_{ij}(x) := \begin{cases} u_{x_i x_j}(x) & x \in \Omega \setminus Z_u,\\ 0 & x \in Z_u.\end{cases} \tag{5.2.1}$$

Our first result give us the summability of the second derivative in the above sense. In order to make the proof readable, we usually omit the tilde-symbol on the derivatives.

Moreover, in what follows we will denote by $u_i := \partial_{x_i} u$.

Let us start proving the following:

Theorem 5.2.1. *Let* $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$ *be a weak solution of* (5.0.1),*where* f *satisfies* (H_f).
Fix $x_0 \in \Omega$ *and* $R > 0$ *such that* $B_{2R}(x_0) \subset\subset \Omega$, *and consider* $y \in \Omega$. *Then, for* $0 \leq \beta < 1$ *and* $\gamma < N - 2$ *for* $N \geq 3$ ($\gamma = 0$ *for* $N = 2$), *we have:*

$$\int_{B_R(x_0)\setminus Z_u} \frac{A(H(\nabla u))|\nabla u_i|^2}{|x-y|^\gamma |u_i|^\beta}\, dx \leq C \quad \forall\, i = 1, ..., N, \tag{5.2.2}$$

where $C = C(\alpha_1, \alpha_2, M, \overline{M}, m_A, M_A, x_0, R, \beta, \gamma, \Lambda, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, \|f\|_{1,\frac{N}{(N-\gamma)^-}})$ *is a positive constant.*

Proof. For $\varphi \in C_c^\infty(\Omega \setminus Z_u)$ and denote by $\varphi_i := \partial_{x_i}\varphi$. Using φ_i as a test function in (5.1.13), since $f(x) \in W^{1,\frac{N}{(N-\gamma)^-}}(\Omega)$, integrating by part we have

$$\begin{aligned} L_u(u_i, \varphi) \;:=\; &\int_\Omega A(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla\varphi\rangle\, dx\\ &+ \int_\Omega A'(H(\nabla u))H(\nabla u)\langle \nabla H(\nabla u), \nabla u_i\rangle\langle \nabla H(\nabla u), \nabla\varphi\rangle\, dx\\ &+ \int_\Omega A(H(\nabla u))H(\nabla u)\langle D^2H(\nabla u)\nabla u_i, \nabla\varphi\rangle\, dx - \int_\Omega f_i\varphi\, dx = 0.\end{aligned} \tag{5.2.3}$$

Let us now take an arbitrary $\epsilon > 0$ and define for $t \geq 0$:

$$G_\epsilon(t) := \begin{cases} 0 & \text{if} \quad t \in [0, \epsilon] \\ (2t - 2\epsilon) & \text{if} \quad t \in [\epsilon, 2\epsilon] \\ t & \text{if} \quad t \in [2\epsilon, \infty), \end{cases} \tag{5.2.4}$$

while $G_\epsilon(t) := -G_\epsilon(-t)$ if $t \leq 0$. Moreover we consider a cut-off function $\varphi_R := \varphi \in C_c^\infty(B_{2R}(x_0))$ such that:

$$\begin{cases} \varphi = 1 & \text{in} \quad B_R(x_0) \\ |\nabla \varphi| \leq \frac{2}{R} & \text{in} \quad B_{2R}(x_0) \setminus B_R(x_0) \\ \varphi = 0 & \text{in} \quad B_{2R}(x_0)^c. \end{cases} \tag{5.2.5}$$

For $0 \leq \beta < 1$, $\gamma < N - 2$ if $N \geq 3$ [1] and for every $\epsilon, \delta > 0$ we set:

$$T_\epsilon(t) := \frac{G_\epsilon(t)}{|t|^\beta}, \qquad H_\delta(t) := \frac{G_\delta(t)}{|t|^{\gamma+1}}. \tag{5.2.6}$$

Next, we test (5.2.3) by the function

$$\psi(x) := T_\epsilon(u_i) H_\delta(|x - y|) \varphi^2(x) \tag{5.2.7}$$

obtaining:

$$\begin{aligned}
&\int_\Omega A(H(\nabla u)) \langle \nabla H(\nabla u), \nabla u_i \rangle \langle \nabla H(\nabla u), \nabla u_i \rangle T'_\epsilon(u_i) H_\delta \varphi^2 \, dx \\
&+ \int_\Omega A(H(\nabla u)) \langle \nabla H(\nabla u), \nabla u_i \rangle \langle \nabla H(\nabla u), \nabla H_\delta \rangle T_\epsilon(u_i) \varphi^2 \, dx \\
&+ 2 \int_\Omega A(H(\nabla u)) \langle \nabla H(\nabla u), \nabla u_i \rangle \langle \nabla H(\nabla u), \nabla \varphi \rangle T_\epsilon(u_i) H_\delta \varphi \, dx \\
&+ \int_\Omega H(\nabla u) A'(H(\nabla u)) \langle \nabla H(\nabla u), \nabla u_i \rangle \langle \nabla H(\nabla u), \nabla u_i \rangle T'_\epsilon(u_i) H_\delta \varphi^2 \, dx \\
&+ \int_\Omega H(\nabla u) A'(H(\nabla u)) \langle \nabla H(\nabla u), \nabla u_i \rangle \langle \nabla H(\nabla u), \nabla H_\delta \rangle T_\epsilon(u_i) \varphi^2 \, dx \\
&+ 2 \int_\Omega H(\nabla u) A'(H(\nabla u)) \langle \nabla H(\nabla u), \nabla u_i \rangle \langle \nabla H(\nabla u), \nabla \varphi \rangle T_\epsilon(u_i) H_\delta \varphi \, dx \\
&+ \int_\Omega H(\nabla u) A(H(\nabla u)) \langle D^2 H(\nabla u) \nabla u_i, \nabla u_i \rangle T'_\epsilon(u_i) H_\delta \varphi^2 \, dx \\
&+ \int_\Omega H(\nabla u) A(H(\nabla u)) \langle D^2 H(\nabla u) \nabla u_i, \nabla H_\delta \rangle T_\epsilon(u_i) \varphi^2 \, dx \\
&+ 2 \int_\Omega H(\nabla u) A(H(\nabla u)) \langle D^2 H(\nabla u) \nabla u_i, \nabla \varphi \rangle T_\epsilon(u_i) H_\delta \varphi \, dx = \int_\Omega f_i T_\epsilon(u_i) H_\delta \varphi^2 \, dx.
\end{aligned} \tag{5.2.8}$$

[1] $\gamma = 0$ if $N = 2$.

Next we have to estimate every integral in above formula, hence let us rename:

$$\begin{aligned}
I_1 &:= \int_\Omega A(H(\nabla u))|\langle \nabla H(\nabla u), \nabla u_i\rangle|^2 T'_\epsilon(u_i) H_\delta \varphi^2 \, dx \\
I_2 &:= \int_\Omega H(\nabla u)A'(H(\nabla u))|\langle \nabla H(\nabla u), \nabla u_i\rangle|^2 T'_\epsilon(u_i) H_\delta \varphi^2 \, dx \\
I_3 &:= \int_\Omega H(\nabla u)A(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_i, \nabla u_i\rangle T'_\epsilon(u_i) H_\delta \varphi^2 \, dx \\
I_4 &:= \int_\Omega A(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle \langle \nabla H(\nabla u), \nabla H_\delta\rangle T_\epsilon(u_i)\varphi^2 \, dx \\
I_5 &:= 2\int_\Omega A(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle \langle \nabla H(\nabla u), \nabla \varphi\rangle T_\epsilon(u_i) H_\delta \varphi \, dx \\
I_6 &:= \int_\Omega H(\nabla u)A'(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle \langle \nabla H(\nabla u), \nabla H_\delta\rangle T_\epsilon(u_i)\varphi^2 \, dx \\
I_7 &:= 2\int_\Omega H(\nabla u)A'(H(\nabla u))\langle \nabla H(\nabla u), \nabla u_i\rangle \langle \nabla H(\nabla u), \nabla \varphi\rangle T_\epsilon(u_i) H_\delta \varphi \, dx \\
I_8 &:= \int_\Omega H(\nabla u)A(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_i, \nabla H_\delta\rangle T_\epsilon(u_i)\varphi^2 \, dx \\
I_9 &:= 2\int_\Omega H(\nabla u)A(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_i, \nabla \varphi\rangle T_\epsilon(u_i) H_\delta \varphi \, dx \\
I_{10} &:= \int_\Omega f_i T_\epsilon(u_i) H_\delta \varphi^2 \, dx,
\end{aligned} \tag{5.2.9}$$

in such a way that (5.2.8) is rewritten as

$$I_1 + I_2 + I_3 = I_{10} - \sum_{k=4}^{9} I_k.$$

Note that, by Lemma 5.1.1 (see in particular (5.1.4)), and since $T'_\varepsilon(t) \geq 0$, for $\beta < 1$, we have:

$$\tilde{C}(\alpha_1, m_A, \Lambda) \int_\Omega A(H(\nabla u))|\nabla u_i|^2 T'_\epsilon(u_i) H_\delta \varphi^2 \, dx \leq I_1 + I_2 + I_3. \tag{5.2.10}$$

By (5.1.5) in Lemma (5.1.1), by definition of T_ε and using a weighted Young inequality we get:

$$\begin{aligned}
&-(I_4 + I_6 + I_8) \\
&\quad = -\int_\Omega \big[(A(H(\nabla u)) + H(\nabla u)A'(H(\nabla u)))\langle \nabla H(\nabla u), \nabla u_i\rangle \langle \nabla H(\nabla u), \nabla H_\delta\rangle \\
&\qquad\quad + H(\nabla u)A(H(\nabla u))\langle D^2 H(\nabla u)\nabla u_i, \nabla H_\delta\rangle\big] T_\epsilon(u_i)\varphi^2 \, dx \\
&\leq \bar{C}(\alpha_2, M, \overline{M}, M_A) \int_\Omega A(H(\nabla u))|\nabla u_i||\nabla H_\delta||T_\epsilon(u_i)|\varphi^2 \, dx \\
&\leq \tilde{C} \int_\Omega A(H(\nabla u))|\nabla u_i| \frac{|T_\epsilon(u_i)|}{|x-y|^{\gamma+1}}\varphi^2 \, dx \\
&\leq \tilde{C} \int_\Omega \frac{A^{\frac{1}{2}}(H(\nabla u))|\nabla u_i|(|G_\varepsilon(u_i)|)^{\frac{1}{2}}\varphi}{|x-y|^{\frac{\gamma}{2}}|u_i|^{\frac{\beta+1}{2}}} \, \frac{A^{\frac{1}{2}}(H(\nabla u))|u_i|(|G_\varepsilon(u_i)|)^{\frac{1}{2}}\varphi}{|x-y|^{\frac{\gamma+2}{2}}|u_i|^{\frac{\beta+1}{2}}} \, dx \\
&\leq \theta \int_\Omega \frac{A(H(\nabla u))|\nabla u_i|^2|G_\varepsilon(u_i)|}{|x-y|^\gamma |u_i|^\beta |u_i|}\varphi^2 + C\int_\Omega \frac{A(H(\nabla u))|u_i|^{2-\beta}}{|x-y|^{\gamma+2}}\varphi^2 \, dx \\
&\leq \theta \int_\Omega \frac{A(H(\nabla u))|\nabla u_i|^2 G_\varepsilon(u_i)}{|x-y|^\gamma |u_i|^\beta u_i}\varphi^2 + C\int_\Omega \frac{A(H(\nabla u))|\nabla u|^{2-\beta}}{|x-y|^{\gamma+2}}\varphi^2 \, dx,
\end{aligned} \tag{5.2.11}$$

where in the last inequality we use that $\dfrac{|G_\varepsilon(u_i)|}{|u_i|}=\dfrac{G_\varepsilon(u_i)}{u_i}$ and denoting by $C(\alpha_2, M, \overline{M}, M_A, \gamma, \theta)$ a positive constant. Now we set $B := B_R(x_0)$, and

$$M_\gamma := \max\left\{\sup_{y\in\Omega}\int_B \frac{1}{|x-y|^\gamma}\,dx,\ \sup_{y\in\Omega}\int_B \frac{1}{|x-y|^{\gamma+2}}\,dx\right\}. \tag{5.2.12}$$

We note that M_γ does not depend on y. By (5.1.2)-(5.1.3), we have that the function $t\mapsto tA(t)$ is locally bounded, so:

$$\begin{aligned}&\int_\Omega \frac{A(H(\nabla u))|\nabla u|^{2-\beta}}{|x-y|^{\gamma+2}}\varphi^2\,dx\\ &\leq C(\|\nabla u\|_{L^\infty_{loc}(\Omega)})\int_B \frac{1}{|x-y|^{\gamma+2}}\,dx \leq C(m_A, M_A, M_\gamma, \|\nabla u\|_{L^\infty_{loc}(\Omega)}),\end{aligned} \tag{5.2.13}$$

where $C(m_A, M_A, M_\gamma, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant.

So by (5.2.11) and (5.2.13) we get:

$$-(I_4+I_6+I_8)\leq \theta\int_\Omega \frac{A(H(\nabla u))|\nabla u_i|^2G_\varepsilon(u_i)}{|x-y|^\gamma|u_i|^\beta u_i}\varphi^2\,dx + C, \tag{5.2.14}$$

where $C(\alpha_2, M, \overline{M}, m_a, M_A, M_\gamma, \gamma, \theta, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant.

In a similar way we can estimate $I_5+I_7+I_9$.

$$\begin{aligned}&-(I_5+I_7+I_9)\\ &= -2\int_\Omega \big[(A(H(\nabla u))+H(\nabla u)A'(H(\nabla u)))\langle\nabla H(\nabla u),\nabla u_i\rangle\langle\nabla H(\nabla u),\nabla\varphi\rangle\\ &+H(\nabla u)A(H(\nabla u))\langle D^2H(\nabla u)\nabla u_i,\nabla\varphi\rangle\big]T_\epsilon(u_i)H_\delta\varphi\,dx\\ &\leq \tilde{C}(\alpha_2, M, \overline{M}, M_A)\int_\Omega A(H(\nabla u))|\nabla u_i||\nabla\varphi||T_\epsilon(u_i)|H_\delta\varphi\,dx\\ &\leq \theta\int_\Omega \frac{A(H(\nabla u))|\nabla u_i|^2G_\varepsilon(u_i)}{|x-y|^\gamma|u_i|^\beta u_i}\varphi^2+\bar{C},\end{aligned} \tag{5.2.15}$$

where $\bar{C}(\alpha, M, \overline{M}, m_A, M_A, M_\gamma, \gamma, \theta, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant.

Finally we estimate the term I_{10}. By definition of T_ϵ and H_δ (see (5.2.6)), we have

$$\begin{aligned}I_{10}&\leq \int_\Omega |f_i||T_\epsilon(u_i)|H_\delta\varphi^2\,dx \leq \int_\Omega \frac{|f_i||T_\epsilon(u_i)|}{|x-y|^\gamma}\varphi^2\,dx\\ &\leq \int_\Omega \frac{|f_i||u_i|^{1-\beta}}{|x-y|^\gamma}\varphi^2\,dx\leq \bar{C},\end{aligned} \tag{5.2.16}$$

where $\bar{C}(M_\gamma, \|f\|_{1,\frac{N}{(N-\gamma)^-}}\, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant.

By (5.2.10), (5.2.14), (5.2.15), (5.2.16) we obtain:

$$\tilde{C}(\alpha_1, m_A, \Lambda)\int_\Omega A(H(\nabla u))|\nabla u_i|^2T'_\epsilon(u_i)H_\delta\varphi^2\,dx\leq \theta\int_\Omega \frac{A(H(\nabla u))|\nabla u_i|^2G_\varepsilon(u_i)}{|x-y|^\gamma|u_i|^\beta u_i}\varphi^2+C.$$

Passing $\delta\to 0$, by Fatou Lemma, we obtain:

$$\tilde{C}(m_A, \alpha_1, \Lambda)\int_\Omega \frac{A(H(\nabla u))|\nabla u_i|^2}{|x-y|^\gamma|u_i|^\beta}\left(G'_\epsilon(u_i)-(\beta+\theta)\frac{G_\epsilon(u_i)}{u_i}\right)\varphi^2\,dx\leq C. \tag{5.2.17}$$

Now choosing θ sufficiently small such that $1-\beta-\theta>0$, since for $\epsilon\to 0$,

$$\left(G'_\epsilon(u_i)-(\beta+\theta)\frac{G_\epsilon(u_i)}{u_i}\right)\to 1-\beta-\theta,$$

by Fatou Lemma, we get

$$\int_{B_R(x_0)\setminus Z_u}\frac{A(H(\nabla u))|\nabla u_i|^2}{|x-y|^\gamma|u_i|^\beta}\,dx\le C, \tag{5.2.18}$$

where $C:=C(\alpha_1,\alpha_2,M,\overline{M},m_A,M_A,x_0,R,\theta,\gamma,\beta,\Lambda,\|\nabla u\|_{L^\infty_{loc}(\Omega)},\|f\|_{1,\frac{N}{(N-\gamma)^-}})$. □

As a consequence of above result we get information about the regularity of the stress field of (5.0.1). For make readble the proof of Theorem 5.0.1, we consider each statementof it as an autonomous proposition that next we are going to prove.

Proposition 5.2.2. *Let $u\in C^{1,\alpha}_{loc}(\Omega)\cap C^2(\Omega\setminus Z_u)$ be a weak solution to (5.0.1), where f satisfies (H_f).*
If $0<m_A<M_A$, then

$$A(H(\nabla u))^{\frac{k-1}{m_A}}H(\nabla u)\nabla H(\nabla u)\in W^{1,2}_{loc}(\Omega,\mathbb{R}^N), \tag{5.2.19}$$

for any $k>1+\dfrac{m_A}{2}-\dfrac{m_A}{2M_A}$.

Proof. For $\epsilon>0$ define

$$V_{\epsilon,j}:=A(H(\nabla u))^{\frac{k-1}{m_A}}H(\nabla u)H_{\eta_j}(\nabla u)J_\epsilon(|\nabla u|), \tag{5.2.20}$$

where $J_\epsilon(t):=\dfrac{G_\epsilon(t)}{t}$, $H_{\eta_j}:=\dfrac{\partial H}{\partial\eta_j}$ and $H(\eta)=H(\eta_1,...,\eta_n)$. We have

$$\begin{aligned}\frac{\partial V_{\epsilon,j}}{\partial x_i}\;&=\;\left(\frac{k-1}{m_A}\right)A(H(\nabla u))^{\frac{k-1}{m_A}-1}A'(H(\nabla u))H(\nabla u)\langle\nabla H(\nabla u),\nabla u_i\rangle J_\epsilon(|\nabla u|)H_{\eta_j}(\nabla u)\\&+A(H(\nabla u))^{\frac{k-1}{m_A}}\langle\nabla H(\nabla u),\nabla u_i\rangle J_\epsilon(|\nabla u|)H_{\eta_j}(\nabla u)\\&+A(H(\nabla u))^{\frac{k-1}{m_A}}H(\nabla u)J'_\epsilon(|\nabla u|)\partial_{x_i}(|\nabla u|)\,H_{\eta_j}(\nabla u)\\&+A(H(\nabla u))^{\frac{k-1}{m_A}}H(\nabla u)J_\epsilon(|\nabla u|)\langle\nabla H_{\eta_j}(\nabla u),\nabla u_i\rangle.\end{aligned} \tag{5.2.21}$$

Using (5.1.1), (1.2.6), (1.2.8) and the definition of J_ϵ we get:

$$\begin{aligned}\left|\frac{\partial V_{\epsilon,j}}{\partial x_i}\right|&\le C(k,M,m_A,M_A)A(H(\nabla u))^{\frac{k-1}{m_A}}|\nabla u_i|\\&\quad+C(M)A(H(\nabla u))^{\frac{k-1}{m_A}}H(\nabla u)|J'_\epsilon(|\nabla u|)||\nabla u_i|\\&\qquad+C(\alpha_2,\overline{M})A(H(\nabla u))^{\frac{k-1}{m_A}}|\nabla u_i|\le CA(H(\nabla u))^{\frac{k-1}{m_A}}|\nabla u_i|,\end{aligned} \tag{5.2.22}$$

where $C=C(\alpha_2,k,M,\overline{M},m_A,M_A)$ is a positive constant.

Fixed $x_0 \in \Omega$ and $R > 0$ such that $B := B_R(x_0) \subset\subset \Omega$, using (5.2.22) and (5.1.2) we get

$$\begin{aligned}
\int_B |\nabla V_{\epsilon,j}|^2\, dx &\le C(\alpha_2, k, M, \overline{M}, m_A, M_A) \int_B A(H(\nabla u))^{\frac{2(k-1)}{m_A}} |D^2 u|^2\, dx \\
&\le C \int_B A(H(\nabla u))^{\frac{2(k-1)}{m_A}-1} \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} |\nabla u|^\beta\, dx \\
&\le C \int_B A(H(\nabla u))^{\frac{2(k-1)}{m_A}-1} \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} H(\nabla u)^\beta\, dx \\
&\le C \int_{B\cap H_>} A(H(\nabla u))^{\frac{2(k-1)}{m_A}-1} H(\nabla u)^\beta \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta}\, dx \\
&\quad + C \int_{B\cap H_<} A(H(\nabla u))^{\frac{2(k-1)}{m_A}-1} A(H(\nabla u))^{\frac{\beta}{M_A}} \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta}\, dx.
\end{aligned} \tag{5.2.23}$$

where $H_< := \{H(\nabla u) \le 1\}$, $H_> := \{H(\nabla u) \ge 1\}$ and $C = C(\alpha_1, \alpha_2, k, M, \overline{M}, m_A, M_A) > 0$.
Now, since $u \in C^2(\Omega \setminus Z_u)$ and by Theorem 5.2.1, choosing $\beta \sim 1^-$, for any $k > 1 + \frac{m_A}{2} - \frac{m_A}{2M_A}$ we get

$$\begin{aligned}
\int_B |\nabla V_{\epsilon,j}|^2 &\le C + C \int_{B\cap H_<} \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta}\, dx \\
&\le C(\alpha_1, \alpha_2, \hat{C}, k, M, \overline{M}, m_A, M_A, \|\nabla u\|_{L^\infty_{loc}(\Omega)}),
\end{aligned} \tag{5.2.24}$$

where $C(\alpha_1, \alpha_2, \hat{C}, k, M, \overline{M}, m_A, M_A, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant and $\hat{C}$ is given by Theorem 5.2.1.
Since $W^{1,2}_{loc}(\Omega)$ is a reflexive space, there exists $\tilde{V}_j \in W^{1,2}_{loc}(\Omega)$ such that

$$V_{\epsilon,j} \rightharpoonup \tilde{V}_j, \quad \text{for } \epsilon \to 0. \tag{5.2.25}$$

By the compact embedding, $V_{\epsilon,j} \to \tilde{V}_j$ in $L^q(\Omega)$ with $q < 2^*$ and up to subsequence $V_{\epsilon,j} \to \tilde{V}_j$ a.e. in Ω. Because of the choice of k, it is easy to verify that
$V_{\epsilon,j} \to A(H(\nabla u))^{\frac{k-1}{m_A}} H(\nabla u) H_{\eta_j}(\nabla u)$ a.e. in Ω, then

$$\tilde{V}_j = A(H(\nabla u))^{\frac{k-1}{m_A}} H(\nabla u) H_{\eta_j}(\nabla u) \in W^{1,2}_{loc}(\Omega).$$

$\square$

Remark 5.2.3. *The choice $k = m_A + 1$ permits us to cover the optimal result in [8] obtained for $f \in L^2$. Let us reveal that this choice will be possible in all next propositions, therefore, in all case, we cover the main result in [8].*

Proposition 5.2.4. *Let $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$ be a weak solution to (5.0.1) and f satisfying (H_f).*
If $m_A < M_A < 0$, then

$$A(H(\nabla u))^{\frac{k-1}{m_A}} H(\nabla u) \nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N) \tag{5.2.26}$$

for any $k > \dfrac{1}{2} + \dfrac{m_A}{2}$.

Proof. Consider $V_{\epsilon,j}$ in (5.2.20). We only prove that

$$\int_B |\nabla V_{\epsilon,j}|^2\,dx < +\infty$$

Indeed the thesis follows exploiting arguments in above Proposition 5.2.2 . Using (5.2.22) we already know that

$$\int_B |\nabla V_{\epsilon,j}|^2\,dx \le C\int_B A(H(\nabla u))^{\frac{2(k-1)-m_A}{m_A}}\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}|\nabla u|^\beta\,dx. \tag{5.2.27}$$

Now let us consider the two cases $2(k-1)-m_A \ge 0$ and $2(k-1)-m_A \le 0$.

In the first, i.e. $k \ge \frac{m_A+2}{2}$, using (5.1.2) and $u \in C^2(\Omega\setminus Z_u)$, from Theorem 5.2.1 we get

$$\begin{aligned}\int_B |\nabla V_{\epsilon,j}|^2\,dx &\le C\int_{B\cap H_<} A(H(\nabla u))^{\frac{2(k-1)-m_A}{m_A}}|\nabla u|^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\quad + C\int_{B\cap H_>} A(H(\nabla u))^{\frac{2(k-1)-m_A}{m_A}}|\nabla u|^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C\int_{B\cap H_<} H(\nabla u)^{(2(k-1)-m_A)\frac{M_A}{m_A}+\beta}\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx + C < C\end{aligned} \tag{5.2.28}$$

renaming $C = C(\alpha_1,\alpha_2,k,M,\overline{M},m_A,M_A,\hat{C},\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ a positive constant, $\hat{C}$ is given by Theorem 5.2.1 and

$$H_< := \{H(\nabla u)\le 1\},\qquad H_> := \{H(\nabla u)\ge 1\}.$$

Let us now cosider $2(k-1)-m_A \le 0$, that is $k \le \frac{m_A+2}{2}$. As in the previous case, using (5.2.27), (5.1.2) and since $u \in C^2(\Omega\setminus Z_u)$ we obtain

$$\begin{aligned}\int_B |\nabla V_{\epsilon,j}|^2\,dx &\le C\int_{B\cap H_<} A(H(\nabla u))^{\frac{2(k-1)-m_A}{m_A}}|\nabla u|^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\quad + C\int_{B\cap H_>} A(H(\nabla u))^{\frac{2(k-1)-m_A}{m_A}}|\nabla u|^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C\int_{B\cap H_<} H(\nabla u)^{2(k-1)-m_A}|\nabla u|^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx + C\\ &\le C\int_{B\cap H_>} |\nabla u|^{2(k-1)-m_A+\beta}\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx + C.\end{aligned} \tag{5.2.29}$$

From Theorem 5.2.1, for β close to 1, we get

$$\int_B |\nabla V_{\epsilon,j}|^2\,dx \le C(\alpha_2,k,M,\overline{M},m_A,M_A,\hat{C},\|\nabla u\|_{L^\infty_{loc}(\Omega)}) \tag{5.2.30}$$

for any $k \in \left(\frac{m_A+1}{2},\frac{m_A+2}{2}\right]$. Exploiting (5.2.28) and (5.2.30) the claim holds for any $k > \frac{m_A+1}{2}$. $\square$

Proposition 5.2.5. *Let $u \in C^{1,\alpha}_{loc}(\Omega)\cap C^2(\Omega\setminus Z_u)$ be a weak solution to (5.0.1), with f satisfying (H_f).*
If $m_A < 0 < M_A$, then

$$A(H(\nabla u))^{\frac{k-1}{M_A}}H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega,\mathbb{R}^N), \tag{5.2.31}$$

for any $k \in \left(\frac{1}{2}+\frac{M_A}{2}, 1+\frac{M_A}{2}-\frac{M_A}{2m_A}\right)$.

Proof. Here we set

$$V_{\epsilon,j} := A(H(\nabla u))^{\frac{k-1}{M_A}} H(\nabla u) J_\epsilon(|\nabla u|) H_{\eta_j}(\nabla u). \tag{5.2.32}$$

By the same computations of the previous two cases we get

$$\begin{aligned}\int_B |\nabla V_{\epsilon,j}|^2\,dx &\le C(\alpha_2,k,M,\overline{M},M_A)\int_{B\cap H_<} A(H(\nabla u))^{\frac{2(k-1)-M_A}{M_A}} A(H(\nabla u))|D^2u|^2\,dx\\ &\quad + C\int_{B\cap H_>} A(H(\nabla u))^{\frac{2(k-1)-M_A}{M_A}} A(H(\nabla u))|D^2u|^2\,dx\\ &=: I_1 + I_2.\end{aligned} \tag{5.2.33}$$

The integral I_2 is estimable exploiting that $u \in C^1(\Omega)\cap C^2(\Omega\setminus Z_u)$.
Hence we only explicitly estimate I_1. Suppose that $2(k-1) - M_A \ge 0$. By (5.1.2) and from Theorem 5.2.1, for β close to 1, we have

$$\begin{aligned} I_1 &\le C\int_{B\cap H_<} A(H(\nabla u))^{\frac{2(k-1)-M_A}{M_A}} |\nabla u|^\beta \frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C\int_{B\cap H_<} H(\nabla u)^{\beta+\frac{2(k-1)-M_A}{M_A}m_A}\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C(\alpha_1,\alpha_2,k,M,\overline{M},M_A,\hat{C},\|\nabla u\|_{L^\infty_{loc}(\Omega)}),\end{aligned} \tag{5.2.34}$$

for any $k \in \left[1+\dfrac{M_A}{2}, 1+\dfrac{M_A}{2}-\dfrac{M_A}{2m_A}\right)$, where $C(\alpha_1,\alpha_2,\hat{C},k,M,\overline{M},M_A,,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant and $\hat{C}$ is given by Theorem 5.2.1.

On the other hand, if $2(k-1)-M_A \le 0$, by (5.1.2) and from Theorem 5.2.1, with β close to 1, we obtain

$$\begin{aligned} I_1 &\le C\int_{B\cap H_<} A(H(\nabla u))^{\frac{2(k-1)-M_A}{M_A}} |\nabla u|^\beta \frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C\int_{B\cap H_<} H(\nabla u)^{\beta+2(k-1)-M_A}\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C(\alpha_1,\alpha_2,k,\hat{C},M,\overline{M},M_A,\|\nabla u\|_{L^\infty_{loc}(\Omega)}),\end{aligned} \tag{5.2.35}$$

for any $k \in \left(\dfrac{1}{2}+\dfrac{M_A}{2}, 1+\dfrac{M_A}{2}\right]$, where $C(\alpha_1,\alpha_2,k,\hat{C},M,\overline{M},M_A,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$, is a positive constant.

By (5.2.33), (5.2.34) and (5.2.35) we get

$$\int_B |\nabla V_{\epsilon,j}|^2\,dx \le C(\alpha_1,\alpha_2,k,\hat{C},M,\overline{M},M_A,\|\nabla u\|_{L^\infty_{loc}(\Omega)}). \tag{5.2.36}$$

The thesis (5.2.31) follows as in the previous cases. □

Proposition 5.2.6. *Let $u \in C^{1,\alpha}_{loc}(\Omega)\cap C^2(\Omega\setminus Z_u)$ be a weak solution to (5.0.1) and f satisfies (H_f).*
If $m_A = 0$ and $M_A > 0$, then

$$A(H(\nabla u))^{k-1} H(\nabla u)\nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega,\mathbb{R}^N) \tag{5.2.37}$$

for any $k > \dfrac{3}{2} - \dfrac{1}{2M_A}$.

Proof. Now we prove the case (5.2.37). Let us consider

$$V_{\epsilon,j} := A(H(\nabla u))^{k-1} H(\nabla u) J_\epsilon(|\nabla u|) H_{\eta_j}(\nabla u). \tag{5.2.38}$$

As we did before, we distinguish two cases. If $2(k-1)-1 \geq 0$, using (5.1.2) and Theorem 5.2.1 we can deduce

$$\begin{aligned}
&\int_B |\nabla V_{\epsilon,j}|^2 \, dx \leq C(\alpha_2, k, M, \overline{M}, M_A) \int_B A(H(\nabla u))^{2(k-1)} |D^2 u|^2 \, dx \\
&\leq C \int_B A(H(\nabla u))^{2(k-1)-1} H(\nabla u)^\beta \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} \, dx \\
&\leq C \int_{B\cap H_<} A(H(\nabla u))^{2(k-1)-1} H(\nabla u)^\beta \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} \, dx \\
&\quad + C \int_{B\cap H_>} A(H(\nabla u))^{2(k-1)-1} H(\nabla u)^\beta \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} \, dx \\
&\leq C \int_{B\cap H_<} \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} \, dx + C \\
&\leq C(\alpha_1, \alpha_2, \hat{C}, k, M, \overline{M}, M_A, \|\nabla u\|_{L^\infty_{loc}(\Omega)})
\end{aligned} \tag{5.2.39}$$

where $C(\alpha_1, \alpha_2, \hat{C}, k, M, \overline{M}, M_A, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant.

If $2(k-1)-1 \leq 0$, by (5.1.2) and Theorem 5.2.1, for β close to 1, and for any $k > \frac{3}{2} - \frac{1}{2M_A}$, we can deduce

$$\begin{aligned}
\int_B |\nabla V_{\epsilon,j}|^2 \, dx &\leq C(\alpha_1, \alpha_2, k, M, \overline{M}, M_A) \int_B A(H(\nabla u))^{2(k-1)} |D^2 u|^2 \, dx \\
&\leq C \int_B A(H(\nabla u))^{2(k-1)-1} H(\nabla u)^\beta \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} \, dx \\
&\leq C \int_{B\cap H_<} A(H(\nabla u))^{2(k-1)-1} H(\nabla u)^\beta \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} \, dx \\
&\quad + C \int_{B\cap H_>} A(H(\nabla u))^{2(k-1)-1} H(\nabla u)^\beta \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} \, dx \\
&\leq C \int_{B\cap H_<} H(\nabla u)^{\beta+(2(k-1)-1)M_A} \frac{A(H(\nabla u))|D^2 u|^2}{|u_i|^\beta} \, dx + C \\
&\leq C(\alpha_1, \alpha_2, \hat{C}, k, M, \overline{M}, M_A, \|\nabla u\|_{L^\infty_{loc}(\Omega)})
\end{aligned} \tag{5.2.40}$$

where $C(\alpha_1, \alpha_2, \hat{C}, k, M, \overline{M}, M_A, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant. □

Proposition 5.2.7. *Let $u \in C^{1,\alpha}_{loc}(\Omega) \cap C^2(\Omega \setminus Z_u)$ be a weak solution to (5.0.1), and f satisfies (H_f).*
If $m_A < 0$ and $M_A = 0$, then

$$A(H(\nabla u))^{k-1} H(\nabla u) \nabla H(\nabla u) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^N) \tag{5.2.41}$$

for any $k < \dfrac{3}{2} - \dfrac{1}{2m_A}$.

Proof. The proof of (5.2.41) is similar to the case (5.2.37). We take $V_{\epsilon,j}$ as in (5.2.38). Let us consider first the case $2(k-1)-1 \geq 0$. By Theorem 5.2.1, by (5.1.2) and choosing $\beta \sim 1^-$

we get

$$\begin{aligned}\int_B |\nabla V_{\epsilon,j}|^2\,dx &\le C(\alpha_2,k,M,\overline{M},M_A)\int_B A(H(\nabla u))^{2(k-1)}|D^2u|^2\,dx\\ &\le C\int_B A(H(\nabla u))^{2(k-1)-1}H(\nabla u)^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C\int_{B\cap H_<} A(H(\nabla u))^{2(k-1)-1}H(\nabla u)^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\quad+C\int_{B\cap H_>} A(H(\nabla u))^{2(k-1)-1}H(\nabla u)^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C\int_{B\cap H_<} H(\nabla u)^{\beta+(2(k-1)-1)m_A}\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C(\alpha_1,\alpha_2,\hat{C},k,M,\overline{M},M_A,\|\nabla u\|_{L^\infty_{loc}(\Omega)})\end{aligned}\tag{5.2.42}$$

for any $k\in\left[\frac{3}{2},\frac{3}{2}-\frac{1}{2m_A}\right)$, where $C(\alpha_1,\alpha_2,\hat{C},k,M,\overline{M},M_A,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is positive constant and $\hat{C}$ is given by Theorem 5.2.1.

If $2(k-1)-1<0$, by Theorem 5.2.1 and by (5.1.2) we get

$$\begin{aligned}\int_B |\nabla V_{\epsilon,j}|^2\,dx &\le C(\alpha_2,k,M,\overline{M},M_A)\int_B A(H(\nabla u))^{2(k-1)}|D^2u|^2\,dx\\ &\le C\int_B A(H(\nabla u))^{2(k-1)-1}H(\nabla u)^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C\int_{B\cap H_<} A(H(\nabla u))^{2(k-1)-1}H(\nabla u)^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\quad+C\int_{B\cap H_>} A(H(\nabla u))^{2(k-1)-1}H(\nabla u)^\beta\frac{A(H(\nabla u))|D^2u|^2}{|u_i|^\beta}\,dx\\ &\le C(\alpha_1,\alpha_2,\hat{C},k,M,\overline{M},M_A,\|\nabla u\|_{L^\infty_{loc}(\Omega)})\end{aligned}\tag{5.2.43}$$

for any $k<\frac{3}{2}$, where $C(\alpha_1,\alpha_2,\hat{C},k,M,\overline{M},M_A,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant. Therefore by (5.2.42) and (5.2.43), proceeding as in the previous cases, (5.2.41) holds for any $k<\frac{3}{2}-\frac{1}{2m_A}$. □

5.3 Second-oreder regularity of weak solutions.

We conclude our chapter proving our main result. In order to do this, we first prove an integrability property for of $A(H(\nabla u))^{-1}$.

Theorem 5.3.1. *Let $u\in C^{1,\alpha}_{loc}(\Omega)\cap C^2(\Omega\setminus Z_u)$ be a weak solution to (5.0.1), with f satisfying (H_f) and*

$$f(x)\ge c(x_0,R)>0\quad \text{in } B_{2R}(x_0),$$

where $B_{2R}(x_0)\subset\subset\Omega$, $x_0\in\Omega$ and $R>0$. Consider $y\in\Omega$.

Then, we have

$$\int_{B_R(x_0)}\frac{1}{A(H(\nabla u))^{\alpha r}}\frac{1}{|x-y|^\gamma}\,dx\le C\tag{5.3.1}$$

where $\alpha:=\frac{M_A+1}{M_A}$, $M_A>0$, $r<1$, $\gamma<N-2$ if $N\ge 3$ while $\gamma=0$ if $N=2$ and $C=C(M,m_A,M_A,r,R,x_0,\gamma,\sigma,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$.

Proof. For $n \in \mathbb{N}$ consider $G_{\frac{1}{n}}(t)$ following (5.2.4). Fix an arbitrary $\epsilon > 0$, let $n_\epsilon \in \mathbb{N}$ such that for any $n > n_\epsilon$ it holds

$$\sup_{t\geq 0} |G_{\frac{1}{n}}(t) - t| < \epsilon. \tag{5.3.2}$$

Choosing $N > n_\varepsilon$, consider the following test function:

$$\varphi = \frac{H_\delta(|x-y|)\psi^2}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}}, \tag{5.3.3}$$

where ψ and H_δ are defined in (5.2.5), (5.2.6). Then

$$\begin{aligned}\nabla\varphi = \frac{2\psi H_\delta \nabla\psi}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}} + \frac{\psi^2 \nabla H_\delta}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}} \\ - \alpha r \psi^2 H_\delta \frac{G'_{\frac{1}{N}}(A(H(\nabla u)))A'(H(\nabla u))D^2 u \nabla H(\nabla u)}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r+1}}.\end{aligned} \tag{5.3.4}$$

Since $f(x) \geq C(x_0, R) > 0$ in $B := B_{2R}(x_0)$, testing (5.3.3) in (5.1.13) we get

$$\begin{aligned} & C(R,x_0)\int_B \frac{H_\delta(|x-y|)\psi^2}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}}\,dx \qquad (5.3.5)\\ \leq \; & -\alpha r \int_B \frac{A(H(\nabla u))H(\nabla u)G'_{\frac{1}{N}}(A(H(\nabla u)))A'(H(\nabla u))\langle \nabla H(\nabla u), D^2 u \nabla H(\nabla u)\rangle}{\left(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon\right)^{\alpha r+1}} H_\delta \psi^2\,dx \\ & + \int_B \frac{A(H(\nabla u))H(\nabla u)\langle \nabla H(\nabla u), \nabla H_\delta\rangle}{\left(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon\right)^{\alpha r}}\psi^2\,dx \\ & + 2\int_\Omega \frac{A(H(\nabla u))H(\nabla u)\langle \nabla H(\nabla u), \nabla\psi\rangle}{\left(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon\right)^{\alpha r}} H_\delta \psi\,dx. \end{aligned}$$

Therefore, denoting by I_F the left-hand side of the previous inequality, we have

$$\begin{aligned} I_F \; := \; & \int_B \frac{H_\delta(|x-y|)\psi^2}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}}\,dx \\ \leq \; & C(r,R,x_0,\alpha)\int_B \frac{|H(\nabla u)A(H(\nabla u))A'(H(\nabla u))||D^2u||\nabla H(\nabla u)|^2}{\left(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon\right)^{\alpha r+1}} H_\delta\psi^2\,dx \\ & + C(R,x_0)\int_B \frac{H(\nabla u)A(H(\nabla u))|\nabla H(\nabla u)||\nabla H_\delta|}{\left(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon\right)^{\alpha r}}\psi^2\,dx \qquad (5.3.6) \\ & + C(R,x_0)\int_B \frac{H(\nabla u)A(H(\nabla u))|\nabla H(\nabla u)||\nabla\psi|}{\left(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon\right)^{\alpha r}} H_\delta\psi\,dx \\ =: \; & I_1 + I_2 + I_3, \end{aligned}$$

where $C(r,R,x_0,\alpha)$ and $C(R,x_0)$ are positive constants. First, we estimate the term I_3.

Using (5.1.2), (1.2.6), (5.3.2) and since $\alpha = (M_A + 1)/M_A$ we get:

$$
\begin{aligned}
I_3 &\leq C(M, R, x_0) \int_B \frac{A(H(\nabla u))H(\nabla u)}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}} H_\delta \psi \, dx \\
&\leq C \int_{B\cap\{A<1\}} \frac{A(H(\nabla u))H(\nabla u)}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}} H_\delta \psi \, dx \\
&\quad + C \int_{B\cap\{A\geq 1\}} \frac{A(H(\nabla u))H(\nabla u)}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}} H_\delta \psi \, dx \qquad (5.3.7)\\
&\leq C \int_{B\cap\{A<1\}} \frac{A(H(\nabla u))^r H(\nabla u)}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}} H_\delta \psi \, dx + \tilde{C}(M, r, R, x_0, \alpha, \gamma, \|\nabla u\|_{L^\infty_{loc}(\Omega)}) \\
&= C \int_{B\cap\{A<1\}} \frac{H(\nabla u)}{A(H(\nabla u))^{(\alpha-1)r}} H_\delta \psi \, dx + \tilde{C} \\
&= C \int_{B\cap\{A<1\}} \frac{H(\nabla u)}{A(H(\nabla u))^{\frac{r}{M_A}}} H_\delta \psi \, dx + \tilde{C} \\
&\leq C \int_{B\cap\{A<1\}} \frac{H(\nabla u)H_\delta \psi}{\min\{H(\nabla u)^r, H(\nabla u)^{\frac{m_A r}{M_A}}\}} \, dx + \tilde{C}, \qquad (5.3.8)
\end{aligned}
$$

(5.3.9)

Since $r < 1$ and $r\, m_A < M_A$, there exists a positive constant
$C = C(M, r, R, x_0, \alpha, \gamma, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ such that

$$
I_3 \leq C(M, r, R, x_0, \alpha, \gamma, \|\nabla u\|_{L^\infty_{loc}(\Omega)}). \qquad (5.3.10)
$$

By a similar computation, we get that there exists a positive constant C such that

$$
I_2 \leq C. \qquad (5.3.11)
$$

Now we estimate I_1; by (5.1.1), (1.2.6) and using a weighted Young inequality we get

$$
\begin{aligned}
I_1 &\leq C(M, r, R, x_0, \alpha) \int_B \frac{|A(H(\nabla u))H(\nabla u)A'(H(\nabla u))||D^2 u|\psi^2 H_\delta}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r + 1}} \, dx \\
&\leq C(M, m_A, M_A, r, R, x_0, \alpha) \int_B \frac{A^2(H(\nabla u))|D^2 u| H_\delta \psi^2}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r + 1}} \, dx \\
&\leq C \left[\theta \int_B \frac{\psi^2 H_\delta}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}} \, dx + \frac{1}{4\theta} \int_B \frac{A^4(H(\nabla u))|D^2 u|^2 \psi^2 H_\delta}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r + 2}} \, dx \right] \\
&\leq C \left[\theta I_F + \frac{1}{4\theta} \int_B \frac{A^2(H(\nabla u))|D^2 u|^2 \psi^2 H_\delta}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}} \, dx \right].
\end{aligned}
$$

(5.3.12)

Now, by the definition of $\alpha = (M_A + 1)/M_A$ and (5.3.2), we get

$$
\begin{aligned}
&\int_B \frac{A^2(H(\nabla u))|D^2u|^2\psi^2 H_\delta}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}}\, dx \\
&= \int_{B\cap\{A<1\}} A(H(\nabla u))|D^2u|^2\psi^2 H_\delta \frac{A(H(\nabla u))}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}}\, dx \\
&\quad + \int_{B\cap\{A\geq 1\}} A(H(\nabla u))|D^2u|^2\psi^2 H_\delta \frac{A(H(\nabla u))}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}}\, dx \\
&\leq \int_{B\cap\{A<1\}} A(H(\nabla u))|D^2u|^2\psi^2 H_\delta \frac{1}{A(H(\nabla u))^{\frac{r}{M_A}}}\, dx \\
&\quad + \int_{B\cap\{A\geq 1\}} A(H(\nabla u))|D^2u|^2\psi^2 H_\delta \frac{A(H(\nabla u))H(\nabla u)^\sigma}{H(\nabla u)^\sigma}\, dx \\
&\leq \frac{1}{A(1)^{\frac{r}{M_A}}} \int_{B\cap\{A<1\}} \frac{A(H(\nabla u))|D^2u|^2\psi^2 H_\delta}{\min\{H(\nabla u)^r, H(\nabla u)^{\frac{rm_A}{M_A}}\}}\, dx \\
&\quad + \sup_B |A(H(\nabla u))H(\nabla u)^\sigma| \int_{B\cap\{A\geq 1\}} \frac{A(H(\nabla u))|D^2u|^2\psi^2 H_\delta}{H(\nabla u)^\sigma}\, dx \\
&\leq \hat{C}\cdot C(m_A, M_A, r, R, x_0, \gamma, \sigma, \|\nabla u\|_{L^\infty_{loc}(\Omega)}),
\end{aligned}
\tag{5.3.13}
$$

where $\sigma < 1$ is given in (5.1.3), $C(m_A, M_A, r, R, x_0, \gamma, \sigma, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant and $\hat{C}$ is given by Theorem 5.2.1. By (5.3.12) and (5.3.13) we get

$$
I_1 \leq \theta C I_F + \frac{1}{4\theta} C(x_0, R, M, \alpha, r, \sigma, \gamma, m_A, M_A, \hat{C}, \|\nabla u\|_{L^\infty_{loc}(\Omega)}),
\tag{5.3.14}
$$

with $C(\hat{C}, M, m_A, M_A, r, R, x_0, \alpha, \gamma, \sigma, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ positive constant.

Choosing θ sufficiently small such that $1 - C\theta > 0$, letting $\delta \to 0$ in (5.3.6), by (5.3.10), (5.3.11), (5.3.14) and using Fatou's Lemma we get

$$
(1 - C\theta) \int_\Omega \frac{1}{(G_{\frac{1}{N}}(A(H(\nabla u))) + \epsilon)^{\alpha r}|x-y|^\gamma}\psi^2\, dx \quad \leq \quad \tilde{C},
\tag{5.3.15}
$$

where $\tilde{C} = \tilde{C}(\hat{C}, M, m_A, M_A, r, R, x_0, \gamma, \sigma, \theta, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant. Using (5.3.2) and letting $\epsilon \to 0$ by Fatou's Lemma we obtain

$$
\int_{B_R(x_0)} \frac{1}{A(H(\nabla u))^{\alpha r}|x-y|^\gamma}\, dx \leq C(\hat{C}, M, m_A, M_A, r, R, x_0, \gamma, \sigma, \theta, \|\nabla u\|_{L^\infty_{loc}(\Omega)}).
\tag{5.3.16}
$$

□

We conclude the section with the proof of our Theorem 5.0.2. Before to do this let us remark that

Remark 5.3.2. *Supposing* $\inf_t A(t) > 0$, *the following proof of* (5.0.3) *can easily simplified in* (5.3.18) *order to get* $u \in W^{2,2}_{loc}(\Omega)$, *without hypothesis on* M_A.

Proof of Theorem 5.0.2. First we prove (5.0.3). Let $x_0 \in \Omega$ and $R > 0$ such that $B := B_R(x_0) \subset\subset \Omega$. Fix $\epsilon > 0$ and consider G_ϵ defined in (5.2.4). Since $u \in C^2(\Omega \setminus Z_u)$, we have

$$
W_{\varepsilon,i} := G_\varepsilon(u_i) \in W^{1,2}_{loc}(\Omega).
$$

Since $\partial_{x_j}(W_{\varepsilon,i}) = G'_\varepsilon(u_i)u_{ij}$, we have

$$\int_B |\nabla W_{\varepsilon,i}|^2 \, dx \le 4 \int_{B\setminus Z_u} |D^2 u|^2 \, dx. \tag{5.3.17}$$

Taking $\beta < 1$ we get

$$\begin{aligned} &\int_{B\setminus Z_u} |D^2u|^2 \, dx \\ &= \int_{B\setminus Z_u} |D^2u|^2 \frac{\min\{H(\nabla u)^{m_A}, H(\nabla u)^{M_A}\}}{|\nabla u|^\beta} \frac{|\nabla u|^\beta}{\min\{H(\nabla u)^{m_A}, H(\nabla u)^{M_A}\}} \, dx. \end{aligned} \tag{5.3.18}$$

Using Theorem 5.2.1, we can choose $\beta \approx 1^-$. Since we are assuming $M_A < 1$, then (5.1.2) and (1.2.3) we get

$$\begin{aligned} \int_{B\setminus Z_u} |D^2u|^2 \, dx &\le C(\alpha_1, \|\nabla u\|_{L^\infty_{loc}(\Omega)}) \int_{B\setminus Z_u} \frac{\min\{H(\nabla u)^{m_A}, H(\nabla u)^{M_A}\}}{|\nabla u|^\beta} |D^2u|^2 \, dx \\ &\le C(\alpha_1, \|\nabla u\|_{L^\infty_{loc}(\Omega)}) \int_{B\setminus Z_u} \frac{A(H(\nabla u))|D^2u|^2}{|\nabla u|^\beta} \, dx \le \hat{C} \cdot C(\|\nabla u\|_{L^\infty_{loc}(\Omega)}), \end{aligned} \tag{5.3.19}$$

where $C(\alpha_1, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant and $\hat{C}$ is given by Theorem 5.2.1. Following the proof of the Proposition 5.2.2 we get the thesis.

Now we suppose $M_A \ge 1$. In this case,

$$\int_B |\nabla W_{\varepsilon,i}|^q \, dx \le 2^q \int_{B\setminus Z_u} |D^2u|^q \, dx. \tag{5.3.20}$$

If we set $B_1 := B \cap \{H(\nabla u) < 1\}$, using Holder inequality and (5.1.2) we can deduce

$$\begin{aligned} &\int_{B_1} |D^2u|^q \, dx \\ &= \int_{B_1} \left(|D^2u|^2\right)^{\frac{q}{2}} \frac{A(H(\nabla u))^{\frac{q}{2}}}{|\nabla u|^{\beta\frac{q}{2}}} \frac{|\nabla u|^{\beta\frac{q}{2}}}{A(H(\nabla u))^{\frac{q}{2}}} \, dx \\ &\le \left(\int_{B_1} \frac{A(H(\nabla u))|D^2u|^2}{|\nabla u|^\beta} \, dx \right)^{\frac{q}{2}} \left(\int_{B_1} \left(\frac{|\nabla u|^\beta}{A(H(\nabla u))} \right)^{\frac{q}{2-q}} dx \right)^{\frac{2-q}{2}} \\ &\le C(\alpha_1, \beta) \left(\int_B \frac{A(H(\nabla u))|D^2u|^2}{|\nabla u|^\beta} \, dx \right)^{\frac{q}{2}} \left(\int_{B_1} \left(\frac{A(H(\nabla u))^{\frac{\beta}{M_A}}}{A(H(\nabla u))} \right)^{\frac{q}{2-q}} dx \right)^{\frac{2-q}{2}} \\ &\le C(\alpha_1, \beta) \left(\int_B \frac{A(H(\nabla u))|D^2u|^2}{|\nabla u|^\beta} \, dx \right)^{\frac{q}{2}} \left(\int_{B_1} \frac{1}{A(H(\nabla u))^{\frac{q(M_A-\beta)}{M_A(2-q)}}} \, dx \right)^{\frac{2-q}{2}}. \end{aligned} \tag{5.3.21}$$

Applying Theorem 5.2.1 and Theorem 5.3.1 we have

$$\int_{B_1} |D^2u|^q \, dx < +\infty. \tag{5.3.22}$$

On the other hand, since $u \in C^2(\Omega \setminus Z_u)$ we note that on the set $B_2 := B \cap \{H(\nabla u) \geq 1\}$, we have

$$\int_{B_2} |D^2 u|^q \, dx < +\infty. \tag{5.3.23}$$

By (5.3.22) and (5.3.23) we can conclude that $u \in W^{2,q}_{loc}(\Omega)$. $\square$

Chapter 6

Regularity and symmetry results for the vectorial p-Laplacian

6.1 Main results

Let Ω be a bounded smooth domain in $\mathbb{R}^n$ with $n \geq 2$. In this chapter we will consider problems involving the vectorial operator $-\boldsymbol{\Delta}_p \boldsymbol{u}$ defined, for smooth functions, by

$$-\boldsymbol{\Delta}_p \boldsymbol{u} = -\operatorname{\mathbf{div}}(|D\boldsymbol{u}|^{p-2} D\boldsymbol{u}) \tag{6.1.1}$$

where $p > 1$ and $D\boldsymbol{u}$ is the Jacobian of the vector field $\boldsymbol{u} : \Omega \to \mathbb{R}^N$, $N \geq 2$. We shall use the notation $\boldsymbol{u} = (u^1, \ldots, u^N)$, for a weak $C^1(\overline{\Omega})$ solution to the p-Laplace system

$$\begin{cases} -\operatorname{\mathbf{div}}(|D\boldsymbol{u}|^{p-2} D\boldsymbol{u}) = \boldsymbol{f}(x, \boldsymbol{u}) & \text{in } \Omega \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega, \end{cases} \tag{p-S}$$

where $\boldsymbol{f} : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies a suitable set of assumptions (h_f) stated below.
The first step of our study concerns the regularity of the solutions, an issue still actively explored in the literature despite its significance. It is well known that, under suitable assumptions on the source term, solutions to (p-S) lie in $C^{1,\alpha}(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus Z_{\boldsymbol{u}})$, where $Z_{\boldsymbol{u}}$ is the set where the gradient vanishes, for every $p > 1$ under suitable assumptions on the source term.

We extend the regularity estimates for the second derivatives of solutions to (p-S) and, as a consequence, we also deduce integrability proprieties of $|D\boldsymbol{u}|^{-1}$.
Our first result is the following:

Theorem 6.1.1. *Let Ω be a bounded smooth domain and $p > 1$. Let $\boldsymbol{u} \in C^1(\Omega)$ be a weak solution of (p-S), with*

$$\boldsymbol{f}(x, \boldsymbol{u}) := \boldsymbol{f}(x) \in W^{1, \frac{n}{n-\gamma-s}}_{loc}(\Omega) \cap C^{0,\beta}_{loc}(\Omega),$$

where $0 < s < n - \gamma$ and $\gamma < n - 2$ ($\gamma = 0$ if $n = 2$).

If $1 < p \leq 2$ let us assume $0 \leq \alpha < p - 1$ and if $p > 2$ let $\alpha \in [0, 1)$. Then, for any $\tilde{\Omega} \subset\subset \Omega$, it follows that

$$\int_{\tilde{\Omega} \setminus Z_{\boldsymbol{u}}} \frac{|D\boldsymbol{u}|^{p-2-\alpha} \|D^2 \boldsymbol{u}\|^2}{|x - y|^\gamma} dx \leq C, \tag{6.1.2}$$

uniformly for any $y \in \mathbb{R}^n$, with $C = C(\boldsymbol{f}, n, p, \alpha, \gamma, \tilde{\Omega})$.

In the case $1 < p < 3$*, as a consequence, we have that*

$$\int_{\tilde{\Omega}\setminus Z_{\boldsymbol{u}}} \frac{\|D^2\boldsymbol{u}\|^2}{|x-y|^\gamma} dx \leq C\,. \tag{6.1.3}$$

Remark 6.1.2. *The estimate obtained in Theorem 6.1.1 holds true, as a consequence, also in the case* $\alpha \leq 0$*, which is actually an easiest case when the gradient is bounded, see the proof of* (6.1.3)*.*

We also point out that Theorem 6.1.1 holds without any sign assumption on the source term $\boldsymbol{f}$. If else we assume that $\boldsymbol{f}$ has a sign, as a consequence of the previous result and we obtain the integrability properties of the inverse of the weight $|D\boldsymbol{u}|^{p-2}$.

Theorem 6.1.3. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded smooth domain and let* $\boldsymbol{u} \in C^1(\Omega)$ *be a weak solution of* (p-S) *with* $\boldsymbol{f}(x,\boldsymbol{u}) := \boldsymbol{f}(x) \in W^{1,\frac{n}{n-\gamma-s}}_{loc}(\Omega) \cap C^{0,\beta}_{loc}(\Omega)$*, where* $0 < s < n-\gamma$ *and* $\gamma < n-2$ *if* $n \geq 3$ *(*$\gamma = 0$ *if* $n = 2$*). Suppose that for some* i*:*

$$f^i \geq \tau > 0 \quad in \quad \Omega\,.$$

Then, for any $\tilde{\Omega} \subset\subset \Omega$

$$\int_{\tilde{\Omega}} \frac{1}{|D\boldsymbol{u}|^\sigma} \frac{1}{|x-y|^\gamma} \leq C, \tag{6.1.4}$$

for any

$$\sigma < \begin{cases} 2p-3 & \text{if } 1 < p \leq 2 \\ p-1 & \text{if } p > 2, \end{cases} \tag{6.1.5}$$

and $C = C(\boldsymbol{f}, n, p, \gamma, \sigma, \tau, \tilde{\Omega})$ *is a positive constant. In particular the Lebesgue measure of* $Z_{\boldsymbol{u}}$ *is zero.*

Remark 6.1.4. *The reader will observe in the proof of Theorems 6.1.1-6.1.3 that the case* $\gamma = 0$ *follows assuming* $\boldsymbol{f}(x) \in W^{1,1}_{loc}(\Omega) \cap C^{0,\beta}_{loc}(\Omega)$*.*

Remark 6.1.5. *If* $\boldsymbol{u} \in C^1(\bar{\Omega})$ *and* $Z_{\boldsymbol{u}} \subset\subset \Omega$*, it follows that the estimate* (6.1.4) *holds up to the boundary, i.e.*

$$\int_{\Omega} \frac{1}{|D\boldsymbol{u}|^\sigma} \frac{1}{|x-y|^\gamma} \leq C. \tag{6.1.6}$$

Note that Theorem 6.1.3 will be crucial in our application since it will allow us to deduce a weak comparison principle in small domains.

As an important consequence, exploiting our Theorem 6.1.1 and Theorem 6.1.3, we recover that $\boldsymbol{u} \in W^{2,2}_{loc}(\Omega)$ if $p \in (1,3)$ and, under the assumption that $\boldsymbol{f}$ has a sign, we prove that for $p \geq 3$, $\boldsymbol{u} \in W^{2,q}_{loc}(\Omega)$ with $1 \leq q < \frac{p-1}{p-2}$, see Theorem 6.2.2 in Section 6.2.

The second part of this chapter focuses on establishing monotonicity and symmetry results using the moving plane method. This approach is closely linked to weak and strong comparison principles, which are generally known to fail in the vectorial case. Therefore, we first derive a comparison principle for our system under very general assumptions on $\boldsymbol{f}$. We begin with the following:

Theorem 6.1.6 (Weak Comparison Principle in small domains)**.** *Let* $p > 1$ *and* $\boldsymbol{u}, \boldsymbol{v} \in C^1(\overline{\Omega})$ *such that either* $\boldsymbol{u}$ *or* $\boldsymbol{v}$ *is solution to* (p-S) *and*

$$-\boldsymbol{\Delta}_p \boldsymbol{u} - \boldsymbol{f}(x,\boldsymbol{u}) \leq -\boldsymbol{\Delta}_p \boldsymbol{v} - \boldsymbol{f}(x,\boldsymbol{v}) \quad in\ \Omega, \tag{6.1.7}$$

where $\boldsymbol{f}$ satisfies (h_f). Let $\tilde{\Omega} \subset\subset \Omega$ be open and assume that

$$(u^i - v^i)(u^j - v^j) \geq 0 \quad \text{in } \tilde{\Omega}, \quad \text{for } i \neq j, \tag{6.1.8}$$

and $\boldsymbol{u} \leq \boldsymbol{v}$ on $\partial\tilde{\Omega}$.
Then there exist $\lambda = \lambda(\boldsymbol{f}, p, N, \|D\boldsymbol{u}\|_\infty) > 0$ such that, if $|\tilde{\Omega}| < \lambda$, it follows that

$$\boldsymbol{u} \leq \boldsymbol{v} \quad \text{in } \tilde{\Omega}.$$

If $\tilde{\Omega} \subseteq \Omega$, the same result holds assuming, for $p > 2$, that $Z_{\boldsymbol{u}} \subset\subset \Omega$.

Remark 6.1.7. *Here and in the following, for a vector valued function $\boldsymbol{w}$, we adopt the notation $\boldsymbol{w} := (w_1, \ldots, w_n) \geq 0$ if*

$$w_i \geq 0 \qquad \forall\, i = 1, \ldots, n\,.$$

Consequently we say that $\boldsymbol{u} \leq \boldsymbol{v}$ if $\boldsymbol{v} - \boldsymbol{u} \geq 0$.

We conclude the chapter deducing a symmetry and monotonicity result. To get this result we need further structural assumptions on the nonlinear term $\boldsymbol{f}$. For the convenience of the reader, we denote by (h_f) the following

(h_f) (i) $\boldsymbol{f}(x, t_1, \ldots, t_N)$ is locally Lipschitz continuous and $\boldsymbol{f}(x, \cdot)$ is a locally Lipschitz continuous vector field, uniformly with respect to x.
Consequently, each component $\ell \in \{1, \ldots, N\}$

$$f^\ell(x, t_1, \ldots, t_N) : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$$

is a Lipschitz function with respect variable to t_j namely, for every $\Omega' \subseteq \overline{\Omega}$ and for every $M > 0$, there is a positive constant $L_\ell = L_\ell(M, \Omega')$ such that for every $x \in \Omega'$ and every $t_j, s_j \in [0, M]$ it holds:

$$|f^\ell(x, t_1, \ldots, t_j, \ldots, t_N) - f^\ell(x, t_1, \ldots, s_j, \ldots, t_N)| \leq L_\ell |t_j - s_j|.$$

(ii) $\boldsymbol{f}$ is a nonnegative vector field and there exists $\ell \in \{1, ..., N\}$ such that

$$f^\ell(x, t_1, \ldots, t_N) > 0$$

for all $x \in \Omega$ and for every $t_j > 0$.

(iii) $\boldsymbol{f}(x, \cdot)$ satisfies

$$\frac{\partial f^\ell}{\partial t_k}(x, t_1, \ldots, t_j, \ldots, t_N) \geq 0, \quad \text{a.e. and for } k \neq l.$$

We succeed in adapting the moving plane procedure in the context of vectorial p-Laplace equations proving the following:

Theorem 6.1.8. *Let Ω be a bounded smooth domain of $\mathbb{R}^n$, convex with respect to the x_1-direction and symmetric with respect to T_0, where*

$$T_0 = \{x \in \Omega : x_1 = 0\}.$$

Let $\boldsymbol{u}$ be a positive $C^1(\overline{\Omega})$ weak solution to (p-S) such that

$$|D\boldsymbol{u}|^{p-2} D\boldsymbol{u} \in W^{1,2}_{loc}(\Omega), \qquad Z_{\boldsymbol{u}} \subset\subset \Omega, \tag{6.1.9}$$

with $\boldsymbol{f}$ satisfying (h_f) and

$$\boldsymbol{f}(x_1, x_2, \ldots, x_n, \boldsymbol{u}) < \boldsymbol{f}(y_1, x_2, \ldots, x_n, \boldsymbol{u}) \quad \text{where } x_1 < y_1 < 0, \tag{6.1.10}$$

$$\boldsymbol{f}(-x_1, \ldots, x_{n-1}, x_n, \boldsymbol{u}) = \boldsymbol{f}(x_1, \ldots, x_{n-1}, x_n, \boldsymbol{u}).$$

We set $a := \inf_{x\in\Omega} x_1$ and $b := \sup_{x\in\Omega} x_1$. Assume that

$$(u^i(x) - u^i(2\lambda - x_1, ..., x_n))(u^j(x) - u^j(2\lambda - x_1, ..., x_n)) \geq 0 \quad i, j = 1, ..., N, \tag{6.1.11}$$

either for any $a < \lambda < 0$ with $x \in \Omega$ such that $x_1 < \lambda$ or for any $0 < \lambda < b$ with $x \in \Omega$ such that $x_1 > \lambda$.
Then $\boldsymbol{u}$ is symmetric with respect to T_0 and non-decreasing with respect to the x_1-direction in $\Omega_0 = \{x \in \Omega \, : \, x_1 < 0\}$.

Remark 6.1.9. *Hypothesis (6.1.9) is quite natural. Indeed it follows e.g. from (6.2.29) of Theorem 6.2.2 or using [12, 46].*

This chapter is structured as follows. In Section 6.2 we prove the local regularity results given by Theorem 6.1.1 and by Theorem 6.1.3. In Section 6.3 we prove the Symmetry result, i.e. Theorem 6.1.8.

6.2 Regularity results for second derivative

Notation

We recall and we introduce some notations that will be useful in the following sections. We will use the bold style to stress the vectorial nature of different quantities. For instance, a N-vectorial function $\boldsymbol{w}$ defined in Ω will be written as

$$\boldsymbol{w}(x) = \left(w^1(x), \ldots, w^N(x)\right),$$

where w^ℓ is a scalar function defined in Ω for $\ell = 1, \ldots, N$.

We say that a vector field $\boldsymbol{u}$ weakly solves (p-S) if and only if $\boldsymbol{u} \in W_0^{1,p}(\Omega) := W_0^{1,p}(\Omega; \mathbb{R}^N)$ and

$$\int_\Omega |D\boldsymbol{u}|^{p-2} D\boldsymbol{u} : D\boldsymbol{\varphi} \, dx = \int_\Omega \langle \boldsymbol{f}, \boldsymbol{\varphi} \rangle \, dx, \quad \forall \boldsymbol{\varphi} \in W_0^{1,p}(\Omega). \tag{6.2.1}$$

Here

$$D\boldsymbol{u} = \begin{pmatrix} \nabla u^1 \\ \vdots \\ \nabla u^N \end{pmatrix},$$

where

$$\nabla u^\ell = \left(\frac{\partial u^\ell}{\partial x_1}, \ldots, \frac{\partial u^\ell}{\partial x_n}\right) \text{ for } \ell = 1, \ldots, N,$$

and

$$|D\boldsymbol{u}| = \sqrt{\sum_{\ell=1}^N \sum_{j=1}^n \left(\frac{\partial u^\ell}{\partial x_j}\right)^2}.$$

We also use the notation $u_j^\ell = \dfrac{\partial u^\ell}{\partial x_j}$.
We point out that along this chapter the symbol $:$ stands for the scalar product of the matrices rows, namely

$$\mathcal{M} : \mathcal{N} = \sum_{i=1}^{q} \mathcal{M}^i \cdot \mathcal{N}^i = \sum_{i=1}^{q}\sum_{j=1}^{r} \mathcal{M}^i_j \mathcal{N}^i_j,$$

where $\mathcal{M}, \mathcal{N}$ are $(q \times r)$ real matrices, whereas $\cdot$ denotes the scalar product of two real vectors.
In the following we will denote the space of the $q \times r$ matrices as $\mathbb{R}^{q\times r}$ with $q, r \in \mathbb{N}$.
Observe that (6.2.1) can be rewritten as

$$\sum_{\ell=1}^{N} \int_\Omega |D\boldsymbol{u}|^{p-2}\langle \nabla u^\ell, \nabla \varphi^\ell\rangle \, dx = \sum_{\ell=1}^{N} \int_\Omega f^\ell \varphi^\ell \, dx, \quad \forall \varphi \in W_0^{1,p}(\Omega).$$

In order to simplify the computations below, we define the vector

$$D^{1,2}\boldsymbol{u} := \begin{pmatrix} D\boldsymbol{u} : D\boldsymbol{u}_1 \\ \vdots \\ D\boldsymbol{u} : D\boldsymbol{u}_i \\ \vdots \\ D\boldsymbol{u} : D\boldsymbol{u}_n \end{pmatrix}. \tag{6.2.2}$$

Moreover we define

$$\|D^2\boldsymbol{u}\| = \sqrt{\sum_{i=1}^{N}\sum_{j,k} (u^i_{jk})^2}.$$

In what follows we often use the inequality

$$|D^{1,2}\boldsymbol{u}| \le |D\boldsymbol{u}| \|D^2\boldsymbol{u}\|. \tag{6.2.3}$$

We are now ready to prove Theorem 6.1.1. The proof will be mainly based on the idea of testing the linearized equation with

$$\boldsymbol{\varphi} = \frac{1}{|x-y|^\gamma} \frac{\boldsymbol{u}_i}{|D\boldsymbol{u}|^\alpha}.$$

This is clearly not allowed and, therefore, we pass to the regularized problem and we exploit a regularization and localization argument. More precisely let $x_0 \in \Omega$ and $B = B_{2\rho(x_0)} \subset\subset \Omega$. Since $\boldsymbol{u} \in C^1(\Omega)$, for any $\varepsilon > 0$, let $\boldsymbol{u}_\varepsilon \in \boldsymbol{u} + W_0^{1,p}(B)$ be the unique solution to

$$\begin{cases} -\,\mathbf{div}((\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}} D\boldsymbol{u}_\varepsilon) = \boldsymbol{f}(x) & \text{in } B_{2\rho(x_0)} \\ \boldsymbol{u}_\varepsilon = \boldsymbol{u} & \text{on } \partial B_{2\rho(x_0)}. \end{cases} \tag{p-S_ε}$$

Testing (p-S_ε) by $\boldsymbol{\varphi} \in C_c^\infty(B)$ we get

$$\int_B (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}} (D\boldsymbol{u}_\varepsilon : D\boldsymbol{\varphi}) dx = \int_B \langle \boldsymbol{f}, \boldsymbol{\varphi}\rangle dx. \tag{6.2.4}$$

The existence of such a weak solution follows by classical minimization procedure. Notice that by [139, Corollary 1.26] and [68, Corollary 5.5], it follows that $\boldsymbol{u}_\varepsilon \in C^{1,\alpha}_{loc}(B)$. Then, looking at the single equations and considering that the therm $(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}}$ is now holder continuous, we deduce from standard Schauder theory ([101]) that the solution $\boldsymbol{u}_\varepsilon \in C^{2,\alpha}_{loc}(B)$.

Lemma 6.2.1. *Using the same notation introduced here above, we have that*

$$\begin{aligned}
&\beta\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2\frac{\psi^2}{|x-y|^\gamma}\,dx \\
&\le \int_B\sum_{i=1}^n\sum_{j=1}^N f_i^j(x)\varphi^j\,dx+C(n,p,\alpha,\gamma,\delta,\rho)\le C.
\end{aligned} \tag{6.2.5}$$

Proof. Fixing $i=1,\dots,n$, using $\boldsymbol{\varphi}_i\in C_c^\infty(B)$ in (6.2.4), we obtain

$$\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}}(D\boldsymbol{u}_\varepsilon:D\boldsymbol{\varphi}_i)dx=\int_B\langle\boldsymbol{f},\boldsymbol{\varphi}_i\rangle dx$$

Integrating by parts, we get

$$\begin{aligned}
&\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}}(D\boldsymbol{u}_{\varepsilon,i}:D\boldsymbol{\varphi})\,dx \\
&+(p-2)\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4}{2}}(D\boldsymbol{u}_\varepsilon:D\boldsymbol{u}_{\varepsilon,i})(D\boldsymbol{u}_\varepsilon:D\boldsymbol{\varphi})\,dx \\
=\ &\int_B\langle\boldsymbol{f}_i,\boldsymbol{\varphi}\rangle dx.
\end{aligned}$$

For any $m\in\mathbb{R}^+$, let $G_{\frac{1}{m}}(t)$ be defined by

$$G_{\frac{1}{m}}(t):=\begin{cases}0 & \text{if } 0\le t\le\frac{1}{m},\\ 2t-\dfrac{2}{m} & \text{if } \frac{1}{m}<t<\frac{2}{m},\\ t & \text{if } t\ge\frac{2}{m}.\end{cases}$$

and for a given $\gamma>0$,

$$H_{\frac{1}{m}}(t):=\frac{G_{\frac{1}{m}}(t)}{t^{\gamma+1}}. \tag{6.2.6}$$

Let $\psi\in C_c^\infty(B_{2\rho}(x_0))$ be such that $\psi=1$ in $B_\rho(x_0)$, $\psi=0$ in $(B_{2\rho}(x_0))^c$ and $|\nabla\psi|\le\frac{2}{\rho}$ in the annulus $B_{2\rho}(x_0)\setminus B_\rho(x_0)$.

For $y\in\mathbb{R}^n$ let us define,

$$\boldsymbol{\varphi}=\frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha}{2}}}\boldsymbol{u}_{\varepsilon,i}.$$

Then using φ as a test function we deduce

$$\begin{aligned}
&\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha}{2}}}|D\boldsymbol{u}_{\varepsilon,i}|^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx\\
&-\alpha\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha+2}{2}}}(D\boldsymbol{u}_{\varepsilon,i}:\boldsymbol{u}_{\varepsilon,i}\left(D^{1,2}\boldsymbol{u}_\varepsilon\right)^T)H_{\frac{1}{m}}(|x-y|)\psi^2\,dx\\
&+(p-2)\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha}{2}}}(D\boldsymbol{u}_\varepsilon:D\boldsymbol{u}_{\varepsilon,i})^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx \qquad (6.2.7)\\
&-\alpha(p-2)\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha+2}{2}}}(D\boldsymbol{u}_\varepsilon:D\boldsymbol{u}_{\varepsilon,i})(D\boldsymbol{u}_\varepsilon:\boldsymbol{u}_{\varepsilon,i}\left(D^{1,2}\boldsymbol{u}_\varepsilon\right)^T)H_{\frac{1}{m}}(|x-y|)\psi^2\,dx\\
=\;&-2\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha}{2}}}(D\boldsymbol{u}_{\varepsilon,i}:\boldsymbol{u}_{\varepsilon,i}\nabla\psi^T)H_{\frac{1}{m}}(|x-y|)\psi\,dx\\
&-\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha}{2}}}\left(D\boldsymbol{u}_{\varepsilon,i}:\boldsymbol{u}_{\varepsilon,i}\left(\nabla H_{\frac{1}{m}}(|x-y|)\right)^T\right)\psi^2\,dx\\
&-2(p-2)\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha}{2}}}(D\boldsymbol{u}_\varepsilon:D\boldsymbol{u}_{\varepsilon,i})(D\boldsymbol{u}_\varepsilon:\boldsymbol{u}_{\varepsilon,i}\nabla\psi^T)H_{\frac{1}{m}}(|x-y|)\psi\,dx\\
&-(p-2)\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha}{2}}}(D\boldsymbol{u}_\varepsilon:D\boldsymbol{u}_{\varepsilon,i})\left(D\boldsymbol{u}_\varepsilon:\boldsymbol{u}_{\varepsilon,i}\left(\nabla H_{\frac{1}{m}}(|x-y|)\right)^T\right)\psi^2\,dx\\
&+\int_B\langle\boldsymbol{f}_i,\boldsymbol{\varphi}\rangle dx.
\end{aligned}$$

Exploiting also a nice computation found in [123], we deduce on (6.2.7)-left hand-side that

$$\begin{aligned}
&\sum_{i=1}^n\left\{\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha}{2}}}|D\boldsymbol{u}_{\varepsilon,i}|^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx\right. \qquad (6.2.8)\\
&-\alpha\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha+2}{2}}}(D\boldsymbol{u}_{\varepsilon,i}:\boldsymbol{u}_{\varepsilon,i}\left(D^{1,2}\boldsymbol{u}_\varepsilon\right)^T)H_{\frac{1}{m}}(|x-y|)\psi^2\,dx\\
&+(p-2)\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha}{2}}}(D\boldsymbol{u}_\varepsilon:D\boldsymbol{u}_{\varepsilon,i})^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx\\
&\left.-\alpha(p-2)\int_B \frac{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4}{2}}}{(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{\alpha+2}{2}}}(D\boldsymbol{u}_\varepsilon:D\boldsymbol{u}_{\varepsilon,i})(D\boldsymbol{u}_\varepsilon:\boldsymbol{u}_{\varepsilon,i}D^{1,2}\boldsymbol{u}_\varepsilon^T)H_{\frac{1}{m}}(|x-y|)\psi^2\,dx\right\}\\
&=\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx\\
&+(p-2-\alpha)\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4-\alpha}{2}}|D^{1,2}\boldsymbol{u}_\varepsilon|^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx\\
&-\alpha(p-2)\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-6-\alpha}{2}}\sum_{k=1}^N\langle D^{1,2}\boldsymbol{u}_\varepsilon,\nabla u^k\rangle^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx := \mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3.
\end{aligned}$$

We distinguish two case: if $1<p\leq 2$, note that $\mathcal{J}_2\leq 0$. Using (6.2.3) in $\mathcal{J}_2$ and noting that the $\mathcal{J}_3\geq 0$, we deduce

$$\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3\geq(p-1-\alpha)\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx. \qquad (6.2.9)$$

In the case $p > 2$, we observe that

$$\sum_{k=1}^{N} \langle D^{1,2}\boldsymbol{u}_\varepsilon, \nabla u^k\rangle^2 = \|D\boldsymbol{u}_\varepsilon D^{1,2}\boldsymbol{u}_\varepsilon\|^2 \leq |D\boldsymbol{u}_\varepsilon|^2 |D^{1,2}\boldsymbol{u}_\varepsilon|^2. \tag{6.2.10}$$

Therefore plugging again (6.2.10) in $\mathcal{J}_3$, we obtain

$$\begin{aligned}
&\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3 \geq \mathcal{J}_1 + (p-2+\alpha-\alpha p)\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4-\alpha}{2}} |D^{1,2}\boldsymbol{u}_\varepsilon|^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx \\
&\geq \min\{1,(p-1)(1-\alpha)\}\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} \|D^2\boldsymbol{u}_\varepsilon\|^2 H_{\frac{1}{m}}(|x-y|)\psi^2\,dx. \qquad (6.2.11)
\end{aligned}$$

where we also use (6.2.3), in $\mathcal{J}_2$ if $p-2+\alpha-\alpha p<0$.

Let us continue working on each term on (6.2.7)-right-hand-side, first considering their norm and then summing over i. Observe that

$$\begin{aligned}
|(D\boldsymbol{u}_{\varepsilon,i} : \boldsymbol{u}_{\varepsilon,i}\nabla\psi^T)| &\leq |D\boldsymbol{u}_{\varepsilon,i}||\boldsymbol{u}_{\varepsilon,i}\nabla\psi^T| = |D\boldsymbol{u}_{\varepsilon,i}|\sqrt{\sum_{j=1}^{N}\sum_{k=1}^{n}(u^j_{\varepsilon,i}\psi_k)^2} \\
&= |D\boldsymbol{u}_{\varepsilon,i}|\sqrt{|\nabla\psi|^2\sum_{j=1}^{N}(u^j_{\varepsilon,i})^2} \leq |D\boldsymbol{u}_{\varepsilon,i}||D\boldsymbol{u}_\varepsilon||\nabla\psi|. \qquad (6.2.12)
\end{aligned}$$

Using weighted Young inequality and (6.2.12), taking into account that $G_{\frac{1}{m}}(|x-y|) \leq |x-y|$, we have

$$\begin{aligned}
& 2\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} |(D\boldsymbol{u}_{\varepsilon,i} : \boldsymbol{u}_{\varepsilon,i}\nabla\psi^T)| H_{\frac{1}{m}}(|x-y|)\psi\,dx \\
\leq\ & 2\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} |D\boldsymbol{u}_{\varepsilon,i}||D\boldsymbol{u}_\varepsilon||\nabla\psi| H_{\frac{1}{m}}(|x-y|)\psi\,dx \\
\leq\ & 2\delta\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} |D\boldsymbol{u}_{\varepsilon,i}|^2 \frac{\psi^2}{|x-y|^\gamma}\,dx \\
& + \frac{1}{2\delta}\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} |D\boldsymbol{u}_\varepsilon|^2|\nabla\psi|^2 \frac{1}{|x-y|^\gamma}\,dx \\
\leq\ & 2\delta\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} |D\boldsymbol{u}_{\varepsilon,i}|^2 \frac{\psi^2}{|x-y|^\gamma}\,dx \qquad (6.2.13) \\
& + \frac{1}{2\delta}\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-\alpha}{2}} |\nabla\psi|^2 \frac{1}{|x-y|^\gamma}\,dx \\
\leq\ & 2\delta\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} |D\boldsymbol{u}_{\varepsilon,i}|^2 \frac{\psi^2}{|x-y|^\gamma}\,dx + C(p,\alpha,\gamma,\delta,\rho),
\end{aligned}$$

where in the last line we used the fact $p > \alpha$ and $|D\boldsymbol{u}_\varepsilon| \leq C$ for some constant uniformly on ε (see [68]).

Summing up on $i = 1, \ldots, n$

$$\begin{aligned}
& 2\sum_{i=1}^{n}\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} |(D\boldsymbol{u}_{\varepsilon,i} : \boldsymbol{u}_\varepsilon\nabla\psi^T)| H_{\frac{1}{m}}(|x-y|)\psi\,dx \qquad (6.2.14) \\
\leq\ & 2\delta\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} \|D^2\boldsymbol{u}_\varepsilon\|^2 \frac{\psi^2}{|x-y|^\gamma}\,dx + C(n,p,\alpha,\gamma,\delta,\rho),
\end{aligned}$$

As in (6.2.13), by weighted Young inequality, we get

$$\begin{aligned}
& 2|(p-2)|\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4-\alpha}{2}}|(D\boldsymbol{u}_\varepsilon : D\boldsymbol{u}_{\varepsilon,i})||(D\boldsymbol{u}_\varepsilon : \boldsymbol{u}_{\varepsilon,i}\nabla\psi^T)|H_{\frac{1}{m}}(|x-y|)\psi\,dx \\
\le\;& 2|p-2|\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_\varepsilon||D\boldsymbol{u}_{\varepsilon,i}||\nabla\psi|H_{\frac{1}{m}}(|x-y|)\psi\,dx \\
\le\;& 2\delta|p-2|\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_{\varepsilon,i}|^2\frac{\psi^2}{|x-y|^\gamma}\,dx \\
& +\frac{|p-2|}{2\delta}\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_\varepsilon|^2|\nabla\psi|^2\frac{1}{|x-y|^\gamma}\,dx \qquad (6.2.15)\\
\le\;& 2\delta|p-2|\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_{\varepsilon,i}|^2\frac{\psi^2}{|x-y|^\gamma}\,dx \\
& +\frac{|p-2|}{2\delta}\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-\alpha}{2}}|\nabla\psi|^2\frac{1}{|x-y|^\gamma}\,dx \\
\le\;& 2\delta|p-2|\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_{\varepsilon,i}|^2\frac{\psi^2}{|x-y|^\gamma}\,dx + C(p,\alpha,\gamma,\delta,\rho).
\end{aligned}$$

Then

$$\begin{aligned}
& 2|p-2|\sum_{i=1}^n\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4-\alpha}{2}}|(D\boldsymbol{u}_\varepsilon : D\boldsymbol{u}_{\varepsilon,i})||(D\boldsymbol{u}_\varepsilon : \boldsymbol{u}_{\varepsilon,i}\nabla\psi^T)|H_{\frac{1}{m}}(|x-y|)\psi\,dx \\
\le\;& 2\delta|p-2|\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2\frac{\psi^2}{|x-y|^\gamma}\,dx \qquad (6.2.16)\\
& +C(n,p,\alpha,\gamma,\delta,\rho).
\end{aligned}$$

Arguing as in (6.2.13), following (6.2.12), inasmuch

$$\left|\left(D\boldsymbol{u}_{\varepsilon,i} : \boldsymbol{u}_{\varepsilon,i}\left(\nabla H_{\frac{1}{m}}(|x-y|)\right)^T\right)\right| \le |D\boldsymbol{u}_{\varepsilon,i}||D\boldsymbol{u}_\varepsilon||\nabla H_{\frac{1}{m}}(|x-y|)|. \qquad (6.2.17)$$

By (6.2.6), (6.2.17) and weighted Young inequality, we get

$$\begin{aligned}
& \int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\left|\left(D\boldsymbol{u}_{\varepsilon,i} : \boldsymbol{u}_{\varepsilon,i}\left(\nabla H_{\frac{1}{m}}(|x-y|)\right)^T\right)\right|\psi^2\,dx \\
\le\;& \int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_{\varepsilon,i}||D\boldsymbol{u}_\varepsilon||\nabla H_{\frac{1}{m}}(|x-y|)|\psi^2\,dx \\
\le\;& \int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_{\varepsilon,i}||D\boldsymbol{u}_\varepsilon|\frac{G'_{\frac{1}{m}}(|x-y|)+(\gamma+1)\frac{G_m(|x-y|)}{|x-y|}}{|x-y|^{\gamma+1}}\psi^2\,dx \\
\le\;& C_\gamma\delta\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_{\varepsilon,i}|^2\frac{\psi^2}{|x-y|^\gamma}\,dx \\
& +\frac{C_\gamma}{4\delta}\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-\alpha}{2}}\frac{\psi^2}{|x-y|^{\gamma+2}}\,dx \\
\le\;& C_\gamma\delta\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_{\varepsilon,i}|^2\frac{\psi^2}{|x-y|^\gamma}\,dx + C(p,\alpha,\gamma,\delta,\rho),
\end{aligned}$$

where we have used the fact that $\gamma+2<n$, and C_γ and $C(p,\alpha,\gamma,\delta,\rho)$ are positive

constants. This implies that

$$\begin{aligned}&\sum_{i=1}^{n}\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\left|\left(D\boldsymbol{u}_{\varepsilon,i}:\boldsymbol{u}_{\varepsilon,i}\left(\nabla H_{\frac{1}{m}}(|x-y|)\right)^T\right)\right|\psi^2\,dx \qquad (6.2.18)\\ \leq\;& C_\gamma\delta\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2\frac{\psi^2}{|x-y|^\gamma}\,dx+C(n,p,\alpha,\gamma,\delta).\end{aligned}$$

Similarly, we see that

$$\begin{aligned}&|p-2|\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4-\alpha}{2}}|(D\boldsymbol{u}_\varepsilon:D\boldsymbol{u}_{\varepsilon,i})|\left|\left(D\boldsymbol{u}_\varepsilon:\boldsymbol{u}_{\varepsilon,i}\left(\nabla H_{\frac{1}{m}}(|x-y|)\right)^T\right)\right|\psi^2\,dx\\ \leq\;& |p-2|\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4-\alpha}{2}}(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)|D\boldsymbol{u}_{\varepsilon,i}||D\boldsymbol{u}_\varepsilon||\nabla H_{\frac{1}{m}}(|x-y|)|\psi^2\,dx\\ \leq\;& C_\gamma\delta|p-2|\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_{\varepsilon,i}|^2\frac{\psi^2}{|x-y|^\gamma}\,dx\\ &+C_\gamma\frac{|p-2|}{4\delta}\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-\alpha}{2}}\frac{\psi^2}{|x-y|^{\gamma+2}}\,dx \qquad (6.2.19)\\ \leq\;& C_\gamma\delta|p-2|\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}|D\boldsymbol{u}_{\varepsilon,i}|^2\frac{\psi^2}{|x-y|^\gamma}\,dx+C(p,\alpha,\gamma,\delta,\rho).\end{aligned}$$

So we conclude that

$$\begin{aligned}&|p-2|\sum_{i=1}^{n}\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-4-\alpha}{2}}|(D\boldsymbol{u}_\varepsilon:D\boldsymbol{u}_{\varepsilon,i})|\left|\left(D\boldsymbol{u}_\varepsilon:\boldsymbol{u}_{\varepsilon,i}\left(\nabla H_{\frac{1}{m}}(|x-y|)\right)^T\right)\right|\psi^2\,dx\\ \leq\;& C_\gamma\delta|p-2|\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2\frac{\psi^2}{|x-y|^\gamma}\,dx+C(n,p,\alpha,\gamma,\delta,\rho). \qquad (6.2.20)\end{aligned}$$

Next we use lower bounds on (6.2.7)-LHS obtained in (6.2.9) and (6.2.11), as well as upper bounds on (6.2.7)-RHS from (6.2.14), (6.2.16), (6.2.18) and (6.2.20). Exploiting the Fatou Lemma, for $m\to+\infty$ and redenoting by $C_\gamma:=2+C_\gamma$ we deduce:

if $1<p\leq 2$

$$\begin{aligned}&(p-1-\alpha)\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2\frac{\psi^2}{|x-y|^\gamma}\,dx \qquad (6.2.21)\\ \leq\;& C_\gamma\delta\max\{(p-1),(3-p)\}\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2\frac{\psi^2}{|x-y|^\gamma}\,dx\\ &+\sum_{i=1}^{n}\int_B\langle\boldsymbol{f}_i,\boldsymbol{\varphi}\rangle dx+C(n,p,\alpha,\gamma,\delta,\rho);\end{aligned}$$

if $p>2$

$$\begin{aligned}&\min\{1,(p-1)(1-\alpha)\}\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2\frac{\psi^2}{|x-y|^\gamma}\,dx \qquad (6.2.22)\\ \leq\;& C_\gamma\delta\max\{(p-1),(3-p)\}\int_B(\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2\frac{\psi^2}{|x-y|^\gamma}\,dx\\ &+\sum_{i=1}^{n}\int_B\langle\boldsymbol{f}_i,\boldsymbol{\varphi}\rangle dx+C(n,p,\alpha,\gamma,\delta,\rho).\end{aligned}$$

There exists $\delta=\delta(p,\alpha,\gamma)$ such that both

$$(p-1-\alpha)-C_\gamma\delta\max\{(p-1),(3-p)\}:=\beta>0,$$

and

$$\min\{1,(p-1)(1-\alpha)\} - C_\gamma\delta\max\{(p-1),(3-p)\} := \beta > 0,$$

hold. Therefore from (6.2.21) and (6.2.22) we obtain

$$\beta\int_B (\varepsilon+|D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}}\|D^2\boldsymbol{u}_\varepsilon\|^2\frac{\psi^2}{|x-y|^\gamma}\,dx$$
$$\le \int_B\sum_{i=1}^n\sum_{j=1}^N f_i^j(x)\varphi^j\,dx + C(n,p,\alpha,\gamma,\delta,\rho)\le C,$$

where in the last inequality we have used $\boldsymbol{f}\in W_{loc}^{1,\frac{n}{n-\gamma-s}}(\Omega)$ and where $C=C(\boldsymbol{f},n,p,\alpha,\gamma,\rho)$ is a positive constant. This concludes the proof of the lemma.

□

Proof of Theorem 6.1.1. Let us consider a compact set $K\subset\subset B\setminus Z_{\boldsymbol{u}}$. We note that, by [68], $\boldsymbol{u}_\varepsilon$ is uniformly bounded in $C^{1,\beta}(K)$, for some $0<\beta<1$. Moreover, by Dirichlet datum in (p-S_ε), it follows that

$$\boldsymbol{u}_\varepsilon\to\boldsymbol{u}\quad\text{in the norm }\|\cdot\|_{C^{1,\beta}(K)}. \tag{6.2.23}$$

By Schauder estimates (that it can be applied to each component) we remark that

$$\|\boldsymbol{u}_\varepsilon\|_{C^{2,\beta}(K)}\le C$$

where $\beta\in(0,1)$ and C is a positive constant not depending on ε. So, by last inequality, it follows that the second limit in (6.2.23) holds in $C^2(K)$. Therefore, exploiting Fatou's Lemma, by Lemma 6.2.1 we deduce that

$$\int_{B_\rho(x_0)\setminus Z_{\boldsymbol{u}}}\frac{|D\boldsymbol{u}|^{p-2-\alpha}\|D^2\boldsymbol{u}\|^2}{|x-y|^\gamma}\,dx\le C,$$

where $C=C(\boldsymbol{f},n,p,\alpha,\gamma,\rho)$ is a positive constant.
If $\tilde\Omega\subset\subset\Omega$, by finite covering argument,

$$\int_{\tilde\Omega\setminus Z_{\boldsymbol{u}}}\frac{|D\boldsymbol{u}|^{p-2-\alpha}\|D^2\boldsymbol{u}\|^2}{|x-y|^\gamma}\,dx\le C.$$

To conclude and deduce (6.1.3) by (6.1.2), note that

$$\|D^2\boldsymbol{u}\|^2=|D\boldsymbol{u}|^{-p+2+\alpha}|D\boldsymbol{u}|^{p-2-\alpha}\|D^2\boldsymbol{u}\|^2\le C|D\boldsymbol{u}|^{p-2-\alpha}\|D^2\boldsymbol{u}\|^2$$

since, for $1<p<3$, we can fix α so that $p-2-\alpha<0$ and $|D\boldsymbol{u}|$ is bounded. □

Now we are in the position to prove:

Proof of Theorem 6.1.3. Once we have at hand the summability properties of the second derivatives proved in Theorem 6.1.1, we proceed exploiting such information, testing the equation. The idea is mainly to plug into the equation the function $\boldsymbol{\varphi}=(0,\cdots,\varphi^i,\cdots,0)$, where

$$\varphi^i=\frac{1}{|D\boldsymbol{u}|^\sigma}\frac{1}{|x-y|^\gamma}.$$

To allow such a procedure, we exploit the regularized problem and a suitable truncation and regularization argument.

For any $m \in \mathbb{R}^+$ let $G_{\frac{1}{m}}(t)$ and

$$H_{\frac{1}{m}}(t) := \frac{G_{\frac{1}{m}}(t)}{t^{\gamma+1}},$$

as in Theorem 6.1.1. Moreover let $x_0 \in \Omega$ and $\rho > 0$ such that $B = B_{2\rho}(x_0) \subset \Omega$. Define $\varphi \in C_c^\infty(\Omega)$ such that $\varphi = 1$ on $B_\rho(x_0)$, $|\nabla\varphi| < \dfrac{2}{r}$ on $B_{2\rho}(x_0) \setminus B_\rho(x_0)$ and $\varphi = 0$ otherwise. Let $\boldsymbol{u}_\varepsilon$ be the solution to (p-S_ε) with $\boldsymbol{f}(x) := \boldsymbol{f}(x, \boldsymbol{u}(x))$ and $\boldsymbol{u}$ solution to (p-S).
Consider the test function $\boldsymbol{\varphi} = (0, \cdots, \varphi^i, \cdots, 0)$, where

$$\varphi^i = \frac{\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}} H_{\frac{1}{m}}(|x-y|).$$

By (p-S_ε) and (6.2.3) we obtain

$$\begin{aligned}
&\int_\Omega f^i \frac{\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}} H_{\frac{1}{m}}(|x-y|)\,dx = \int_\Omega (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u^i_\varepsilon, \nabla\varphi^i\rangle\,dx \\
= \;& -\sigma \int_\Omega (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u^i_\varepsilon, D^{1,2}\boldsymbol{u}_\varepsilon\rangle \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma+2}{2}}}\,dx \\
& +2\int_\Omega (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u^i_\varepsilon, \nabla\psi\rangle \frac{H_{\frac{1}{m}}(|x-y|)\psi}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\,dx \\
& + \int_\Omega (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2}{2}} \langle \nabla u^i_\varepsilon, \nabla H_{\frac{1}{m}}(|x-y|)\rangle \frac{\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\,dx \\
\le \;& \sigma \int_\Omega (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-1}{2}} |D\boldsymbol{u}_\varepsilon| \|D^2\boldsymbol{u}_\varepsilon\| \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma+2}{2}}}\,dx \\
& +2\int_{B_{2\rho}} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-1}{2}} |\nabla\psi| \frac{H_{\frac{1}{m}}(|x-y|)}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\,dx \\
& + \int_{B_{2\rho}} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-1}{2}} |\nabla H_{\frac{1}{m}}(|x-y|)| \frac{1}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\,dx.
\end{aligned} \tag{6.2.24}$$

We estimate the three terms in the right hand side of (6.2.24). By weighted Young inequality we have

$$\begin{aligned}
&\sigma \int_\Omega (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-1}{2}} |D\boldsymbol{u}_\varepsilon| \|D^2\boldsymbol{u}_\varepsilon\| \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma+2}{2}}}\,dx \\
&\le \theta\sigma \int_\Omega \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\,dx + \frac{\sigma}{4\theta}\int_\Omega \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma+2}{2}}} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{(p-1)} \|D^2\boldsymbol{u}_\varepsilon\|^2\,dx.
\end{aligned} \tag{6.2.25}$$

For the second term, since $|D\boldsymbol{u}_\varepsilon| \le C$ for some constant uniformly on ε (see [68]), we deduce

$$\begin{aligned}
&\int_{B_{2\rho}} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-1}{2}} |\nabla\psi| \frac{H_{\frac{1}{m}}(|x-y|)}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\,dx \\
\le \;& C(\rho)\int_{B_{2\rho}} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-1}{2} - \frac{\sigma}{2}} \frac{1}{|x-y|^\gamma}\,dx \le C(p, \rho, \sigma, \gamma),
\end{aligned} \tag{6.2.26}$$

where we use that $\sigma < p-1$ and $\gamma < n-2$. We observe that the constant C does not depend on the position of y in $\mathbb{R}^n$. Similarly to above we obtain

$$\begin{aligned} &\int_{B_{2\rho}} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-1}{2}} |\nabla H_{\frac{1}{m}}(|x-y|)| \frac{1}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\, dx \\ \leq\ & C_\gamma \int_{B_{2\rho}} \left(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2\right)^{\frac{p-1}{2} - \frac{\sigma}{2}} \frac{1}{|x-y|^{\gamma+1}}\, dx \leq C(p,\rho,\sigma,\gamma). \end{aligned} \tag{6.2.27}$$

Using estimates (6.2.25), (6.2.26) and (6.2.27) in (6.2.24) we deduce, recalling that, $f^i \geq \tau > 0$,

$$\begin{aligned} &\tau \int_\Omega \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\, dx \leq \theta\sigma \int_\Omega \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\, dx \\ &+ \frac{\sigma}{4\theta} \int_\Omega \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma+2}{2}}} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{(p-1)} \|D^2\boldsymbol{u}_\varepsilon\|^2\, dx + C(p,\rho,\sigma,\gamma). \end{aligned}$$

Choosing $\theta < \dfrac{\tau}{\sigma}$ we infer that

$$\begin{aligned} &\int_{B_\rho(x_0)} \frac{H_{\frac{1}{m}}(|x-y|)}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\, dx \leq \int_\Omega \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\, dx \\ \leq\ & C(\sigma,\tau) \int_\Omega \frac{H_{\frac{1}{m}}(|x-y|)\psi^2}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma+2}{2}}} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{(p-1)} \|D^2\boldsymbol{u}_\varepsilon\|^2\, dx + C(p,\rho,\sigma,\gamma). \end{aligned}$$

Let $\varepsilon = \frac{1}{m}$. We obtain

$$\begin{aligned} &\int_{B_\rho(x_0)} \frac{H_\varepsilon(|x-y|)}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\, dx \\ \leq\ & C(\sigma,\tau) \int_{B_{2\rho}} \left(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2\right)^{\frac{\alpha-\sigma+(p-2)}{2}} \frac{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} \|D^2\boldsymbol{u}_\varepsilon\|^2\psi^2}{|x-y|^\gamma}\, dx \\ &+ C(p,\rho,\sigma,\gamma). \end{aligned}$$

Since σ is as in (6.1.4), there exists $0 \leq \alpha < p-1$, if $1 < p \leq 2$ or $\alpha \geq 0$ if $p > 2$ such that $\alpha - \sigma + (p-2) \geq 0$ so that (6.2.5) of Lemma 6.2.1 holds. We obtain

$$\int_{B_\rho(x_0)} \frac{H_\varepsilon(|x-y|)}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\sigma}{2}}}\, dx \leq C(f,n,p,\gamma,\rho,\sigma,\tau). \tag{6.2.28}$$

Using Fatou Lemma in (6.2.28) and a classical finite covering argument, we get the thesis. $\square$

The estimates for $D^2\boldsymbol{u}$ and $|D\boldsymbol{u}|^{-1}$ obtained in Theorem 6.1.1 and Theorem 6.1.3 gives us tools for proving more regularity for our weak solution $\boldsymbol{u}$. We have

Theorem 6.2.2. *Let Ω be a bounded smooth domain and $p > 1$. Let $\boldsymbol{u} \in C^1(\Omega)$ be a weak solution of (p-S) and assume $\boldsymbol{f}(x) \in W^{1,1}_{loc}(\Omega) \cap C^{0,\beta}_{loc}(\Omega)$. Then*

(i) if $p > \frac{3}{2}$, recovering the result already proved in [46]

$$|D\boldsymbol{u}|^{p-2} D\boldsymbol{u} \in W^{1,2}_{loc}(\Omega); \tag{6.2.29}$$

(ii) *if* $1 < p < 3$, *then*

$$\boldsymbol{u} \in W^{2,2}_{loc}(\Omega);$$

(iii) *if* $p \geq 3$ *and for some* $i = 1, \dots, N$

$$f^i \geq \tau > 0 \quad in \quad \Omega,$$

then

$$\boldsymbol{u} \in W^{2,q}_{loc}(\Omega), \text{ with } 1 \leq q < \frac{p-1}{p-2}.$$

Proof of Theorem 6.2.2. In this proof we use the regularity results contained in Theorems 6.1.1-6.1.3 in the case $\gamma = 0$, see Remark 6.1.4.
We start proving (iii). Let $\boldsymbol{u}_\varepsilon$ be the solution to (p-S_ε) with $\boldsymbol{f}(x) := \boldsymbol{f}(x, \boldsymbol{u}(x))$ and $\boldsymbol{u}$ solution to (p-S). We split $|D^2\boldsymbol{u}_\varepsilon|^q$ as

$$|D^2\boldsymbol{u}_\varepsilon|^q = |D\boldsymbol{u}_\varepsilon|^{\frac{(p-2-\alpha)q}{2}} |D^2\boldsymbol{u}_\varepsilon|^q \frac{1}{|D\boldsymbol{u}_\varepsilon|^{\frac{(p-2-\alpha)q}{2}}}.$$

Let us exploit estimation (6.2.28) of Theorem 6.1.3 together with the fact that, in this case, if $p \geq 3$, $q < 2$. Then

$$\begin{aligned}
&\int_{B_\rho(x_0)} |D^2\boldsymbol{u}_\varepsilon|^q dx \\
&= \int_{B_\rho(x_0)} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{(p-2-\alpha)q}{4}} |D^2\boldsymbol{u}_\varepsilon|^q \frac{1}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{(p-2-\alpha)q}{4}}} dx \\
&\leq \left(\int_{B_\rho(x_0)} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{p-2-\alpha}{2}} |D^2\boldsymbol{u}_\varepsilon|^2 dx \right)^{\frac{q}{2}} \left(\int_{B_\rho(x_0)} \frac{1}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{(p-2-\alpha)q}{4-2q}}} dx \right)^{\frac{2-q}{2}} \\
&\leq C \left(\int_{B_\rho(x_0)} \frac{1}{(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{(p-2-\alpha)q}{4-2q}}} dx \right)^{\frac{2-q}{2}},
\end{aligned}$$

by (6.2.5) of Lemma 6.2.1. Whenever the right hand side is bounded (see Theorem 6.1.3 and (6.2.28)), namely for

$$\frac{(p-2-\alpha)q}{4-2q} < \frac{p-1}{2},$$

choosing $\alpha \simeq 1$ since $q < \frac{p-1}{p-2}$, we deduce that

$$\int_{B_\rho(x_0)} |D^2\boldsymbol{u}_\varepsilon|^q dx \leq C.$$

If $\tilde{\Omega} \subset\subset \Omega$, by finite covering argument we get

$$\int_{\tilde{\Omega}} |D^2\boldsymbol{u}_\varepsilon|^q dx \leq C,$$

where C does not depend on ε. It is now easy to see that, consequently, $\boldsymbol{u} \in W^{2,q}_{loc}(\Omega)$ in this case. Indeed $\boldsymbol{u}_\varepsilon \rightharpoonup \boldsymbol{w} \in W^{2,q}(\tilde{\Omega})$. Moreover because of the compact embedding in $L^q(\tilde{\Omega})$, up to a subsequence, $\boldsymbol{u}_\varepsilon \to \boldsymbol{w}$ a.e. in $\tilde{\Omega}$. On the other hand (see [68])

$$\boldsymbol{u}_\varepsilon \to \boldsymbol{u} \quad \text{a.e.}$$

Hence

$$\boldsymbol{u} \equiv \boldsymbol{w} \in W^{2,q}(\tilde{\Omega}).$$

We prove now the cases (i)-(ii). We set

$$V_{\varepsilon,\eta,i,j}(x) := (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\eta-1}{2}} u^j_{\varepsilon,i}. \tag{6.2.30}$$

Since

$$\nabla V_{\varepsilon,\eta,i,j} = (\eta-1)(\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\eta-3}{2}} |D\boldsymbol{u}_\varepsilon| \nabla(|D\boldsymbol{u}_\varepsilon|)(u^j_{\varepsilon,i}) + (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\frac{\eta-1}{2}} \nabla u^j_{\varepsilon,i}.$$

We have

$$\int_{B_\rho(x_0)} |DV_{\varepsilon,\eta}|^2 \, dx \le C \int_{B_\rho(x_0)} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{\eta-1} |D^2\boldsymbol{u}_\varepsilon|^2 \, dx,$$

for a suitable positive constant $C = C(\eta)$.
Using (6.2.5) of Lemma 6.2.1, if $p > \frac{3}{2}$ and $\eta = p-1$ we get

$$\int_{B_\rho(x_0)} |DV_{\varepsilon,\eta}|^2 \, dx \le C \int_{B_\rho(x_0)} (\varepsilon + |D\boldsymbol{u}_\varepsilon|^2)^{p-2} |D^2\boldsymbol{u}_\varepsilon|^2 \, dx \le C.$$

Using this estimate, arguing exactly as in the previous case, we deduce (i)

$$|D\boldsymbol{u}|^{p-2} D\boldsymbol{u} \in W^{1,2}_{loc}(\Omega).$$

Finally (ii) follows setting $\eta = 1$ in (6.2.30) and recalling (6.1.3). □

6.3 Comparison Principles and Symmetry

In this section we will first prove some comparison principles that are new for vectorial p-Laplace equation. First of all, let us state a useful inequality; for given $\eta, \eta' \in \mathbb{R}^{q\times r}$, there exists a positive constant $C_p = C_p(p)$ such that

$$(|\eta|^{p-2}\eta - |\eta'|^{p-2}\eta') : (\eta - \eta') \ge C_p(|\eta| + |\eta'|)^{p-2}|\eta - \eta'|^2. \tag{6.3.1}$$

As a consequence of Theorem 6.1.3, we deduce a weighted Poincaré type inequality and then we use it to get a weak comparison principle in small domains. The integrability property (6.1.4) is the key to get weighted Poincaré inequality. Indeed once (6.1.4) is in force, we deduce the Poincaré inequality from [87, Theorem 8]. Let us remark that the Poincaré inequality is relevant to the case $p > 2$ in the proof of the symmetry result. We have the following

Theorem 6.3.1 ([57, 87]). *Let $p > 2$, $\boldsymbol{u} \in C^1(\Omega)$ be a weak solution of (p-S) with $\Omega \subset \mathbb{R}^n$ a smooth bounded domain and assume that $\boldsymbol{f}(x,\cdot)$ satisfies (h_f) (i)-(ii).*
For any $\Omega' \subset\subset \Omega$, let $\boldsymbol{u}$ be such that

$$\int_{\Omega'} \frac{1}{|D\boldsymbol{u}|^{(p-2)t}|x-y|^\gamma} \, dx \le C,$$

with $1 < t < \frac{p-1}{p-2}$, $\gamma < n-2$ if $n \ge 3$ ($\gamma = 0$ if $n = 2$). Then for any $w \in C^\infty_c(\Omega')$ there exist a constant $C_S(|\Omega'|)$, with $C_S(|\Omega'|) \to 0$ if $|\Omega'| \to 0$, such that

$$\|w\|_{L^q(\Omega')} \le C_S(|\Omega'|) \left(\int_\Omega |D\boldsymbol{u}|^{p-2} |\nabla w|^2 \right)^{\frac{1}{2}}, \tag{6.3.2}$$

for any $1 \le q < 2^*(t)$, where

$$\frac{1}{2^*(t)} = \frac{1}{2} - \frac{1}{n} + \frac{1}{t}\left(\frac{1}{2} - \frac{\gamma}{2n}\right).$$

Furthermore (6.3.2) holds for any $w \in H^1_{0,\rho}(\Omega')$, [48].

Remark 6.3.2. *The proof of the weighted Sobolev embedding, although it goes back to [57], the reader should look at [87, Theorem 8] since the embedding is proved there for general weights, so that no matter if the weight $\rho = |D\boldsymbol{u}|^{p-2}$ arises from a scalar or from a vectorial equation. Note that [87, Theorem 8] is proved in a more general version that can be reduced to our case without splitting the domain.*

Remark 6.3.3. *As already noted in Remark 4 in [87], the largest value of $2^*(t)$ is obtained for $\gamma \simeq n-2$ ($\gamma = 0$, whenever $n = 2$) and $t \simeq \frac{p-1}{p-2}$. In this case*

$$\frac{1}{2^*(p)} \simeq \frac{1}{2} - \frac{1}{n} + \frac{p-2}{p-1}\frac{1}{n}$$

and therefore $2^*(p) > 2$.
Moreover we point out that if $\boldsymbol{u} \in C^1(\bar\Omega)$ and $Z_{\boldsymbol{u}} \subset\subset \Omega$, it follows that the estimate (6.3.2) holds for any $\Omega' \subseteq \Omega$.

Theorem 6.3.1 allows us to prove Theorem 6.1.6:

Proof of Theorem 6.1.6. Let us consider the vectorial function

$$(\boldsymbol{u}-\boldsymbol{v})^+ := \Big((u^1-v^1)^+, \dots, (u^\ell - v^\ell)^+, \dots, (u^N - v^N)^+\Big). \tag{6.3.3}$$

Since $(\boldsymbol{u}-\boldsymbol{v})^+$ is a good test function for equations (6.1.7), using (6.1.8) together with (6.3.1), we obtain

$$\begin{aligned}
&C_p \int_{\tilde\Omega} (|D\boldsymbol{u}| + |D\boldsymbol{v}|)^{p-2} |D(\boldsymbol{u}-\boldsymbol{v})^+|^2 dx \\
&\le \int_{\tilde\Omega} (|D\boldsymbol{u}|^{p-2} D\boldsymbol{u} - |D\boldsymbol{v}|^{p-2} D\boldsymbol{v} : D(\boldsymbol{u}-\boldsymbol{v})^+)\, dx \\
&\le \int_{\tilde\Omega} \langle (\boldsymbol{f}(x,\boldsymbol{u}) - \boldsymbol{f}(x,\boldsymbol{v})), (\boldsymbol{u}-\boldsymbol{v})^+ \rangle\, dx.
\end{aligned} \tag{6.3.4}$$

Now we can evaluate the right term of (6.3.4) as

$$\begin{aligned}
&\int_{\tilde\Omega} \langle (\boldsymbol{f}(x,\boldsymbol{u}) - \boldsymbol{f}(x,\boldsymbol{v})), (\boldsymbol{u}-\boldsymbol{v})^+ \rangle\, dx \\
&= \int_{\tilde\Omega} \sum_{\ell=1}^N \Big[f^\ell(x,u^1,\dots,u^N) - f^\ell(x,v^1,\dots,v^N) \Big] (u^\ell - v^\ell)^+\, dx \\
&= \int_{\tilde\Omega} \sum_{\ell=1}^N \Big[f^\ell(x,u^1,u^2,\dots,u^N) - f^\ell(x,v^1,u^2\dots,u^N) + f^\ell(x,v^1,u^2\dots,u^N) \\
&\qquad\qquad -f^\ell(x,v^1,\dots,v^N) \Big] (u^\ell - v^\ell)^+\, dx.
\end{aligned}$$

Iterating this procedure for the remaining variables we get

$$
\begin{aligned}
&\int_{\tilde\Omega} \langle (\boldsymbol{f}(x,\boldsymbol{u}) - \boldsymbol{f}(x,\boldsymbol{v})), (\boldsymbol{u}-\boldsymbol{v})^+\rangle \, dx \qquad (6.3.5)\\
= \ &\int_{\tilde\Omega} \sum_{\ell=1}^{N} \Big[f^\ell(x,u^1,u^2,\dots,u^N) - f^\ell(x,v^1,u^2\dots,u^N) + f^\ell(x,v^1,u^2,\dots,u^N)\\
&-f^\ell(x,v^1,v^2,\dots,u^N) + \dots + f^\ell(x,v^1,v^2,\dots,u^\ell,\dots,u^N)\\
&\qquad -f^\ell(x,v^1,v^2,\dots,v^\ell,\dots,u^N)\\
&\quad \vdots\\
&+\dots + f^\ell(x,v^1,v^2,\dots,u^N) - f^\ell(x,v^1,v^2,\dots,v^N)\Big] (u^\ell - v^\ell)^+ \, dx.
\end{aligned}
$$

Now, having assumend among the others the cooperative condition (h_f)-(iii), (6.3.5) becomes

$$
\begin{aligned}
&\int_{\tilde\Omega} \langle (\boldsymbol{f}(x,\boldsymbol{u}) - \boldsymbol{f}(x,\boldsymbol{v})), (\boldsymbol{u}-\boldsymbol{v})^+ \rangle dx\\
\leq \ &\int_{\tilde\Omega} \sum_{\ell=1}^{N} \Big[\frac{f^\ell(x,u^1,u^2,\dots,u^N) - f^\ell(x,v^1,u^2\dots,u^N)}{(u^1-v^1)^+}(u^1-v^1)^+(u^\ell-v^\ell)^+\\
&+\frac{f^\ell(x,v^1,u^2,\dots,u^N) - f^\ell(x,v^1,v^2,\dots,u^N)}{(u^2-v^2)^+}(u^2-v^2)^+(u^\ell-v^\ell)^+\\
&\vdots\\
&+\frac{f^\ell(x,v^1,v^2,\dots,u^\ell,\dots,u^N) - f^\ell(x,v^1,v^2,\dots,v^\ell,\dots,u^N)}{(u^\ell-v^\ell)^+}[(u^\ell-v^\ell)^+]^2\\
&\vdots\\
&+\frac{f^\ell(x,v^1,v^2,\dots,u^N) - f^\ell(x,v^1,v^2,\dots,v^N)}{(u^N-v^N)^+}(u^N-v^N)^+(u^\ell-v^\ell)^+\Big] \, dx\\
\leq \ & L_f \int_{\tilde\Omega} \sum_{\ell=1}^{N}\sum_{j=1}^{N} (u^j-v^j)^+(u^\ell-v^\ell)^+ \, dx \leq N L_f \int_{\tilde\Omega} \sum_{\ell=1}^{N} [(u^\ell-v^\ell)^+]^2 \, dx, \qquad (6.3.6)
\end{aligned}
$$

where in the last inequality we used the Young's inequality, and L_f is a positive constant depending on the constants L_ℓ in (h_f)-(i).
If $1 < p \leq 2$ we use classical Poincaré inequality in the right hand side of (6.3.6). In this case we obtain

$$
\begin{aligned}
\int_{\tilde\Omega} \sum_{\ell=1}^{N} [(u^\ell-v^\ell)^+]^2 \, dx \ &\leq \ C_P(|\tilde\Omega|) \int_{\tilde\Omega} \sum_{\ell=1}^{N} |\nabla (u^\ell-v^\ell)^+|^2 \, dx = C_P \int_{\tilde\Omega} |D(\boldsymbol{u}-\boldsymbol{v})^+|^2 \, dx\\
&\leq \ C \int_{\tilde\Omega} (|D\boldsymbol{u}| + |D\boldsymbol{v}|)^{p-2} |D(\boldsymbol{u}-\boldsymbol{v})^+|^2 \, dx, \qquad (6.3.7)
\end{aligned}
$$

where C is a positive constant depending on $\|D\boldsymbol{u}\|_\infty^{2-p}$ and $|\tilde\Omega|$, since $1 < p \leq 2$. As well known $C \to 0$, as $|\tilde\Omega| \to 0$.
In the case $p > 2$ since either $\boldsymbol{u}$ or $\boldsymbol{v}$ is a solution to (p-S), (6.1.4) and (6.3.2) hold, the right

hand side of (6.3.6) becomes

$$\begin{aligned}\int_{\tilde\Omega}\sum_{\ell=1}^{N}[(u^\ell-v^\ell)^+]^2\,dx &\le C_S(|\tilde\Omega|)\int_{\tilde\Omega}\sum_{\ell=1}^{N}(|D\boldsymbol{u}|+|D\boldsymbol{v}|)^{p-2}|\nabla(u^\ell-v^\ell)^+|^2\,dx\\ &\le C_S(|\tilde\Omega|)\int_{\tilde\Omega}(|D\boldsymbol{u}|+|D\boldsymbol{v}|)^{p-2}|D(\boldsymbol{u}-\boldsymbol{v})^+|^2\,dx \qquad (6.3.8)\end{aligned}$$

Collecting (6.3.4)-(6.3.8) we deduce

$$\int_{\tilde\Omega}(|D\boldsymbol{u}|+|D\boldsymbol{v}|)^{p-2}|D(\boldsymbol{u}-\boldsymbol{v})^+|^2dx\le C\int_{\tilde\Omega}(|D\boldsymbol{u}|+|D\boldsymbol{v}|)^{p-2}|D(\boldsymbol{u}-\boldsymbol{v})^+|^2\,dx, \qquad (6.3.9)$$

where $C=C(\boldsymbol{f},p,N,|\tilde\Omega|,\|D\boldsymbol{u}\|_\infty)$ goes to zero as $|\tilde\Omega|$ goes to zero.
There exists $\lambda:=\lambda(\boldsymbol{f},p,N,\|D\boldsymbol{u}\|_\infty)$ sufficiently small such that, if $|\tilde\Omega|<\lambda$, then $C<\frac{1}{2}$ in (6.3.9). This is a contradiction unless $\boldsymbol{u}\le\boldsymbol{v}$ in $\tilde\Omega$. □

The following result that will be useful in the proofs of Theorem 6.1.8.

Proposition 6.3.4. *Let $\Omega\subset\mathbb{R}^n$ be a bounded smooth domain and let $\boldsymbol{u},\boldsymbol{v}\in C^1(\Omega)$ such that*

$$|D\boldsymbol{u}|^{p-2}D\boldsymbol{u},|D\boldsymbol{v}|^{p-2}D\boldsymbol{v}\in W^{1,2}_{loc}(\Omega) \qquad (6.3.10)$$

and satisfying

$$-\boldsymbol{\Delta}_p\boldsymbol{u}-\boldsymbol{f}(x,\boldsymbol{u})=0 \quad -\boldsymbol{\Delta}_p\boldsymbol{v}-\boldsymbol{g}(x,\boldsymbol{v})=0 \quad \text{in } \Omega. \qquad (6.3.11)$$

We set $A:=\{x\in\Omega:\boldsymbol{u}(x)=\boldsymbol{v}(x)\}$ and suppose that

$$\boldsymbol{f}(x,\boldsymbol{u})<\boldsymbol{g}(x,\boldsymbol{v}) \quad \text{a.e. in } A. \qquad (6.3.12)$$

Then $|A|=0$.

Proof. Using Stampacchia's theorem (see [112, Theorem 6.19]) we have that

$$|D\boldsymbol{u}|^{p-2}D\boldsymbol{u}=|D\boldsymbol{v}|^{p-2}D\boldsymbol{v} \quad \text{a.e. on } A.$$

By (6.3.10) and Stampacchia's theorem we get

$$\mathbf{div}(|D\boldsymbol{u}|^{p-2}D\boldsymbol{u})=\mathbf{div}(|D\boldsymbol{v}|^{p-2}D\boldsymbol{v}) \quad \text{a.e. on } A.$$

By (6.3.11) and (6.3.12) we get the thesis. □

The last part of this section is devoted to proving the symmetry result Theorem 6.1.8. For a real number λ, we set

$$\Omega_\lambda=\Omega\cap\{x_1<\lambda\},$$
$$x_\lambda=R_\lambda(x):=(2\lambda-x_1,x_2,\dots,x_n),$$

which is the point reflected through the hyperplane $T_\lambda=\{x\in\mathbb{R}^n\,:\,x_1=\lambda\}$. Also define

$$\boldsymbol{u}_\lambda(x):=\boldsymbol{u}(x_\lambda)=\boldsymbol{u}(2\lambda-x_1,x_2,\dots,x_{n-1},x_n).$$

We recall the $\boldsymbol{u}_\lambda$ solves the following equation

$$\int_{\Omega_\lambda}|D\boldsymbol{u}_\lambda|^{p-2}(D\boldsymbol{u}_\lambda:D\boldsymbol{\varphi})\,dx=\int_{\Omega_\lambda}\langle\boldsymbol{f}(x_\lambda,\boldsymbol{u}_\lambda),\boldsymbol{\varphi}\rangle dx,$$

for any $\boldsymbol{\varphi}\in W^{1,p}_0(\Omega)$. Finally we define

$$\Lambda_0:=\{a<\lambda<0:\boldsymbol{u}\le\boldsymbol{u}_t \text{ in } \Omega_t \text{ for all } t\in(a,\lambda]\}.$$

Proof of Theorem 6.1.8. We start showing that

$$\Lambda_0 \neq \emptyset.$$

To prove this, let us consider $\lambda > a$ with $\lambda - a$ small enough, so that $|\Omega_\lambda|$ is small. As remarked above and by (6.1.10), we deduce that $\boldsymbol{u}_\lambda$ is such that

$$-\boldsymbol{\Delta}_p \boldsymbol{u}_\lambda = \boldsymbol{f}(x_\lambda, \boldsymbol{u}_\lambda) > \boldsymbol{f}(x, \boldsymbol{u}_\lambda) \quad \Omega_\lambda. \tag{6.3.13}$$

Since $\boldsymbol{u} \leq \boldsymbol{u}_\lambda$ on $\partial\Omega_\lambda$ and (6.1.11) holds, by Theorem 6.1.6 it follows that

$$\boldsymbol{u} \leq \boldsymbol{u}_\lambda \quad \text{in } \Omega_\lambda.$$

Since $\Lambda_0 \neq \emptyset$, let us now set

$$\overline{\lambda} = \sup \Lambda_0.$$

We shall show that $\overline{\lambda} = 0$.

To do this assume that $\overline{\lambda} < 0$ and we reach a contradiction by proving that $\boldsymbol{u} \leq \boldsymbol{u}_{\overline{\lambda}+\tau}$ in $\Omega_{\overline{\lambda}+\tau}$ for any $0 < \tau < \overline{\tau}$, for some small $\overline{\tau} > 0$.

We set $A := \{x \in \Omega_{\overline{\lambda}} : \boldsymbol{u}(x) = \boldsymbol{u}_{\overline{\lambda}}(x)\}$ and let $\mathcal{B} \subset \Omega_{\overline{\lambda}}$ be an open set such that $(A \cup Z_{\boldsymbol{u}} \cup Z_{\boldsymbol{u}_{\overline{\lambda}}}) \cap \Omega_{\overline{\lambda}} \subset \mathcal{B}$. Theorem 6.1.3 and Proposition 6.3.4 (using (6.1.10) with $\boldsymbol{g}(x, \boldsymbol{v}) = \boldsymbol{f}(x_{\overline{\lambda}}, \boldsymbol{u}_{\overline{\lambda}})$) guarantee that $|A| = |Z_{\boldsymbol{u}}| = 0$; therefore $\mathcal{B}$ will be of an arbitrarily small measure.

By continuity we have

$$\boldsymbol{u} \leq \boldsymbol{u}_{\overline{\lambda}} \quad \text{in } \Omega_{\overline{\lambda}}.$$

We choose a compact set $K \subset \Omega_{\overline{\lambda}} \setminus \mathcal{B}$ such that Theorem 6.1.6 applies, e.g. $|\Omega_{\overline{\lambda}} \setminus K| < \frac{\delta}{2}$, where $\delta = \delta(\boldsymbol{f}, p, N, \|D\boldsymbol{u}\|_\infty)$.

We set

$$S_1 := \left\{ x \in K : u^1(x) < u^1_{\overline{\lambda}}(x) \right\}, \quad \tilde{S}_1 := \left\{ x \in K : u^1(x) = u^1_{\overline{\lambda}}(x) \right\}.$$

Since $K \cap A = \emptyset$, without loss of generality we consider $S_1 \neq \emptyset$. We can select a compact set $K_1 \subset S_1$ such that $S_1 \setminus K_1$ has small measure. By (6.1.11) and by the uniform continuity of $\boldsymbol{u}$, we deduce that

$$\boldsymbol{u} \leq \boldsymbol{u}_{\overline{\lambda}+\tau} \text{ in } K_1, \tag{6.3.14}$$

for any $0 < \tau < \overline{\tau}_1$ for some small $\overline{\tau}_1 > 0$.

Since $\tilde{S}_1 \cap A = \emptyset$, for any $x \in \tilde{S}_1$ there exist at least an index $i = 2, ..., N$ such that $u^i < u^i_{\overline{\lambda}}$ in $\overline{B_r(x)} \subset\subset \Omega_{\overline{\lambda}}$, for some $r > 0$.

Since $\tilde{S}_1$ is closed (and obviously bounded), therefore there exist compact sets $\tilde{K}_2 \subset \left\{ u^2 < u^2_{\overline{\lambda}} \right\}, ..., \tilde{K}_N \subset \left\{ u^N < u^N_{\overline{\lambda}} \right\}$ in $\Omega_{\overline{\lambda}}$ (some of them eventually empty) such that

$$\tilde{S}_1 = (\tilde{S}_1 \cap \tilde{K}_2) \cup \cdots \cup (\tilde{S}_1 \cap \tilde{K}_N).$$

By (6.1.11) and by the uniform continuity of $\boldsymbol{u}$, we have that

$$\boldsymbol{u} \leq \boldsymbol{u}_{\overline{\lambda}+\tau} \text{ in } \tilde{S}_1, \tag{6.3.15}$$

for any $0 < \tau < \overline{\tau}_2$ for some small $\tau_2 > 0$. Let us choose $0 < \overline{\tau} < \min\{\tau_1, \tau_2\}$ sufficiently small such that $|\Omega_{\overline{\lambda}+\tau} \setminus (\tilde{S}_1 \cup K_1)| < \delta$, for any $0 < \tau < \overline{\tau}$.

Since $\boldsymbol{u} \leq \boldsymbol{u}_{\overline{\lambda}+\tau}$ on $\partial(\Omega_{\overline{\lambda}+\tau} \setminus (\tilde{S}_1 \cup K_1))$ and (6.1.11), Theorem 6.1.6 permits to have

$$\boldsymbol{u} \leq \boldsymbol{u}_{\overline{\lambda}+\tau} \quad \text{in } \Omega_{\overline{\lambda}+\tau} \setminus (\tilde{S}_1 \cup K_1)),$$

for any $0 < \tau < \overline{\tau}$. Exploiting also (6.3.15) and (6.3.14), we have that

$$\boldsymbol{u} \le \boldsymbol{u}_{\overline{\lambda}+\tau} \quad \text{in } \Omega_{\overline{\lambda}+\tau},$$

for any $0 < \tau < \overline{\tau}$.

This is a contradiction with the definition of $\overline{\lambda}$, hence we have $\overline{\lambda} = 0$ and $\boldsymbol{u} \le \boldsymbol{u}_0$ in Ω_0.

Performing the moving plane technique in the opposite direction, we deduce that $\boldsymbol{u}$ is symmetric with respect to the hyperplane $\{x_1 = 0\}$. We also remark that the solution is increasing in the x_1-direction in $\{x_1 < 0\}$ since it is implicit in the moving plane procedure.

$\square$

Chapter 7

Third order estimates and the regularity of the stress field for solutions to p-Laplace equations

7.1 Main results

We deal with the regularity of the third derivatives of weak solutions to:

$$-\Delta_p u = f(x) \quad \text{in } \Omega \tag{7.1.1}$$

where $\Delta_p u := -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplace operator and Ω is a domain of $\mathbb{R}^n$ with $n \geq 2$.

We consider possibly sign-changing solutions $u \in W^{1,p}_{loc}(\Omega)$ such that

$$\int_\Omega |\nabla u|^{p-2}(\nabla u, \nabla \psi)\, dx = \int_\Omega f\psi\, dx \quad \forall\, \psi \in C^\infty_c(\Omega). \tag{7.1.2}$$

Here we address the study of the integrability of the third derivatives of the solutions. The latter can be defined as distributional derivatives in a suitable meaning. As mentioned in the Introduction, to our knowledge, no other results in the literature address this topic. Consequently, our estimates will enable us to deduce regularity results for the stress field $|\nabla u|^{p-2}\nabla u$.

Our results are influenced by the classical theory of elliptic regularity, particularly by the constant $C(n,q)$ found in the Calderón-Zygmund estimate

$$\|D^2 w\|_{L^q(\Omega)} \leq C(n,q)\|\Delta w\|_{L^q(\Omega)}. \tag{7.1.3}$$

Under the influence of this factor, we obtain second order regularity results for the stress field $|\nabla u|^{p-2}\nabla u$:

Theorem 7.1.1. *Let $\Omega \subset \mathbb{R}^N$ be a domain and $u \in C^{1,\beta}_{loc}(\Omega)$ be a weak solution of (7.1.1). We set*

$$\begin{cases} q_N = 2(q_{N-1} - 1) \\ q_0 = 4 \end{cases} \tag{7.1.4}$$

Let $q > q_0$, with $q_N < q \leq q_{N+1}$ for some $N \in \mathbb{N}_0$.
Assume p such that

$$2 - \frac{1}{C(n,q)} < p < \min\left\{2 + \frac{1}{q-1}, 2 + \frac{1}{C(n,q)}\right\}. \tag{7.1.5}$$

Let $f(x) \in W^{2,q/(q-1)}_{loc}(\Omega) \cap C^{1,\beta'}_{loc}(\Omega)$ *and*

$$f \geq \tau > 0 \quad in \quad \Omega\,, \quad (or\ f \leq \tau < 0 \quad in \quad \Omega\,).$$

Then we have

$$|\nabla u|^{k-1}\nabla u \in W^{2,1}_{loc}(\Omega, \mathbb{R}^N) \tag{7.1.6}$$

for any $k > (\alpha+1)/2$*, where*

$$\alpha > \frac{(q-q_N)}{2^N q}(1-p) + \frac{3-p}{2^N} + 1. \tag{7.1.7}$$

In case f *has no sign, let* $q \geq q_0$*, with* $q_N \leq q < q_{N+1}$ *for some* $N \in \mathbb{N}_0$*, and* q_N *given in* (7.1.4)*. Assume* p *satisfies* (7.1.5)*. Then* (7.1.6) *holds for any* $k \geq (p+\alpha)/2$*, where* $\alpha > (3-p)/2^N + 1$ *(if* $N = 0$*,* $\alpha \geq 4-p$*).*

As a consequence of the regularity results on third derivatives, we obtain improved regularity properties of the stress field when p approaches the semilinear case $p = 2$.

Theorem 7.1.2. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain and* $u \in C^{1,\beta}_{loc}(\Omega)$ *be a weak solution of* (7.1.1)*. Assume that* $\tilde{\alpha} \geq 3$ *and we set* $q = 2(\tilde{\alpha}-1)$*. Suppose that*

$$\max\left\{2 - \frac{1}{2\tilde{\alpha}-1}, 2 - \frac{1}{C(n,q)}\right\} < p \leq 2. \tag{7.1.8}$$

where $C(n,q)$ *is given by* (7.1.3)*. Let* $f(x) \in W^{2,2(\tilde{\alpha}-1)/3}_{loc}(\Omega) \cap C^{1,\beta'}_{loc}(\Omega)$*. Then*

$$|\nabla u|^{p-2}\nabla u \in W^{1,\tilde{\alpha}}_{loc}(\Omega). \tag{7.1.9}$$

To accomplish this, we first extend a result in [120], which does not take into account the effects of the boundary of Ω. We have the following

Theorem 7.1.3. *Let* Ω *be a domain of* $\mathbb{R}^n$ *and* $u \in C^{1,\beta}_{loc}(\Omega)$ *be a weak solution to* (7.1.1)*, with* $f(x) \in W^{1,1}_{loc}(\Omega) \cap C^{0,\beta'}_{loc}(\Omega)$*. Let* $q \geq 2$ *and* p *be such that*

$$|p-2| < \frac{1}{C(n,q)},$$

with $C(n,q)$ *given by* (7.1.3)*. Then, if* $1 < p \leq 2$*, we have*

$$u \in W^{2,q}_{loc}(\Omega).$$

If $p > 2$ *then the same conclusion holds if* p *satisfies, in addition,* $q < \frac{p-1}{p-2}$*.*

Theorems 7.1.1 and Theorem 7.1.2 follow from more general estimates concerning the third derivatives. More precisely we have the following

Theorem 7.1.4. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain and let* $u \in C^{1,\beta}_{loc}(\Omega)$ *be a weak solution of* (7.1.1)*.*

For $\gamma > \max\{(p-2)/(p-1), 2-p\}$*, we set*

$$\begin{cases} q_N = 2(q_{N-1}-1) \\ q_0 = 3+\gamma. \end{cases} \tag{7.1.10}$$

Let $q > q_0$, with $q_N < q \leq q_{N+1}$ for some $N \in \mathbb{N}_0$ and p be such that:

$$2 - \frac{1}{C(n,q)} < p < \min\left\{2 + \frac{1}{q-1}, 2 + \frac{1}{C(n,q)}\right\}, \tag{7.1.11}$$

where $C(n,q)$ is given by (7.1.3). Assume

$$\alpha > \alpha(p,q,N) := \frac{(q-q_N)}{2^N q}(1-p) + \frac{3-p}{2^N} + 1. \tag{7.1.12}$$

Let $f(x) \in W^{2,q/(q-\gamma)}_{loc}(\Omega) \cap C^{1,\beta'}_{loc}(\Omega)$ and

$$f \geq \tau > 0 \quad in \quad \Omega\,, \quad (or\ f \leq \tau < 0 \quad in \quad \Omega\,).$$

Then, for any $\tilde{\Omega} \subset\subset \Omega$, for $\gamma \geq 1$, we have that

$$\int_{\tilde{\Omega}\setminus Z_u} |\nabla u|^{p-2+\alpha}|D^2u|^{\gamma-1}|D^3u|^2\,dx \leq C, \tag{7.1.13}$$

where $C = C(\gamma, \alpha, p, q, n, f, \tilde{\Omega}, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, N)$.

If $\max\{(p-2)/(p-1), 2-p\} < \gamma < 1$, for any $\tilde{\Omega} \subset\subset \Omega$, we deduce that

$$\int_{\tilde{\Omega}\setminus(Z_u \cup \{|D^2u|=0\})} |\nabla u|^{p-2+\alpha}|D^2u|^{\gamma-1}|D^3u|^2\,dx \leq C, \tag{7.1.14}$$

where $C = C(\gamma, \alpha, p, q, n, f, \tilde{\Omega}, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, N)$.

We also point out that, if we assume that f has no a sign, Theroem 7.1.4 holds, in the case $\alpha > (3-p)/2^N + 1$, for some $N \in \mathbb{N}$, if $N = 0$, $\alpha \geq 4 - p$ (see Theorem 7.3.3 in Section 7.3).

This chapter is structured as follows. In Section 7.2 we recall some preliminary results that will be useful to prove our result. In Section 7.3 we prove Theorem 7.1.1, Theorem 7.1.2, Theorem 7.1.3 and Theorem 7.1.4.

7.2 Preliminary results

In this section, we will state some results that will be useful for proving our main results.

In the sequel, we will use the notation $u_i := u_{x_i}$, for $i = 1, ..., n$, to indicate the partial derivative of u with respect to x_i. The second derivatives of u will be denoted with u_{ij}, $i, j = 1, ..., n$.

Let $x_0 \in \Omega$ and $B_{2R}(x_0) \subset\subset \Omega$. Now we consider the regularized problem

$$\begin{cases} -\operatorname{div}\left((\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}}\,\nabla u_\epsilon\right) = f(x) & \text{in } B_{2R}(x_0) \\ u_\epsilon = u & \text{on } \partial B_{2R}(x_0), \end{cases} \tag{7.2.1}$$

where $\epsilon \in [0,1)$. The existence of such a weak solution follows by classical minimization procedure. Notice that by standard regularity results [68, 101] the solution u_ε belongs to $C^2(B_{2R}(x_0))$, therefore u_ε is a classical solution of

$$-\Delta u_\varepsilon = (p-2)\frac{(D^2u_\varepsilon\nabla u_\varepsilon, \nabla u_\varepsilon)}{(\varepsilon + |\nabla u_\varepsilon|^2)} + \frac{f}{(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}}}. \tag{7.2.2}$$

Let us consider a compact set $K \subset\subset B_{2R}(x_0)$. We note that, by [68], u_ε is uniformly bounded in $C^{1,\beta}(K)$, for some $0 < \beta < 1$. Moreover, by Dirichlet datum in (7.2.1) and by uniqueness, it follows that

$$u_\varepsilon \to u \quad \text{in the norm } \|\cdot\|_{C^{1,\beta}(K)}. \tag{7.2.3}$$

We remind (see e.g. [101]) that it holds the Calderón-Zygmund inequality for elliptic operators

Lemma 7.2.1. *Let Ω be a bounded smooth domain of $\mathbb{R}^n$ and let $w \in W_0^{2,q}(\Omega)$, then there exists a positive constant $C(n,q)$ such that*

$$\|D^2 w\|_{L^q(\Omega)} \le C(n,q)\|\Delta w\|_{L^q(\Omega)}. \tag{7.2.4}$$

We recall a weighted Hessian regularity result and an integrability property of the inverse of the weight $|\nabla u|^{p-2}$.

Theorem 7.2.2 (see [57, 120])**.** *Let Ω be a domain of $\mathbb{R}^n$ and $p > 1$. Let u_ε be a weak solution of (7.2.1), with $f(x) \in W^{1,1}_{loc}(\Omega) \cap C^{0,\beta'}_{loc}(\Omega)$. Then, for any $\tilde\Omega \subset\subset \Omega$, it follows that*

$$\int_{\tilde\Omega} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2-\beta}{2}} \|D^2 u_\varepsilon\|^2 \, dx \le C \tag{7.2.5}$$

for any $0 \le \beta, \beta' < 1$, with $C = C(p, \beta, \tilde\Omega, f, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$.

Moreover, for $p > 2$ and for any fixed $1 \le q < \frac{p-1}{p-2}$, there exists a constant $C = C(p, q, \tilde\Omega, f, \|\nabla u\|_{L^\infty_{loc}(\Omega)}) > 0$ such that

$$\int_{\tilde\Omega} \frac{f^2}{(\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{q(p-2)}{2}}} \le C. \tag{7.2.6}$$

If $1 < p \le 2$, the inequality (7.2.6) holds for any $1 \le q < +\infty$.

We conclude this section recalling a result that will be very useful in the proofs of our results.

Theorem 7.2.3 (see [57])**.** *Let $\Omega \subset \mathbb{R}^n$ be domain, $p > 1$ and $u \in C^{1,\beta}_{loc}(\Omega)$ be a weak solution to (7.1.1) with $f \in W^{1,1}_{loc}(\Omega) \cap C^{0,\beta'}_{loc}(\Omega)$ and*

$$f \ge \tau > 0 \quad in \quad \Omega, \quad (or\ f \le \tau < 0 \quad in \quad \Omega).$$

Then, for any $\tilde\Omega \subset\subset \Omega$, we have that $|Z_u| = 0$ and

$$\int_{\tilde\Omega} \frac{1}{|\nabla u|^{(p-1)r}} \, dx \le C,$$

for any $r < 1$ with $C = C(p, r, \tilde\Omega, f, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ positive constant.

7.3 Proof of the main result

We start the section by extending a result proven in [120], for solutions $u \in W^{1,p}_{loc}(\Omega)$.

Proof of Theorem 7.1.3. We start with the case $p > 2$. Let $x_0 \in \Omega$ and $R > 0$ be fixed such that $B_{2R}(x_0) \subset\subset \Omega$. Let $\varphi \in C^\infty_c(B_{2R}(x_0))$ such that $\varphi = 1$ in $B_R(x_0)$, $\varphi = 0$ in $(B_{2R}(x_0))^c$ and $|\nabla\varphi| \leq \frac{2}{R}$ in the annular $B_{2R}(x_0) \setminus B_R(x_0)$.
Since $u_\varepsilon\varphi \in W^{2,q}_0(B_{2R}(x_0))$, using Lemma 7.2.1, and recalling that u_ε is solution of (7.2.2), we have

$$\begin{aligned}
&\|D^2(u_\epsilon\varphi)\|_{L^q(B_{2R}(x_0))} \leq C(n,q)\|\Delta(u_\epsilon\varphi)\|_{L^q(B_{2R}(x_0))} \\
&= C(n,q)\|\varphi\Delta u_\varepsilon + 2(\nabla u_\varepsilon, \nabla\varphi) + u_\varepsilon\Delta\varphi\|_{L^q(B_{2R}(x_0))} \\
&\leq C(n,q)\left\|(p-2)\frac{(\varphi D^2u_\varepsilon\nabla u_\varepsilon, \nabla u_\varepsilon)}{(\epsilon + |\nabla u_\epsilon|^2)} + \frac{\varphi f}{(\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}}}\right\|_{L^q(B_{2R}(x_0))} \\
&+ C(n,q,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)}),
\end{aligned} \tag{7.3.1}$$

where in the last inequality we have used the fact that $|\nabla u_\varepsilon| \leq C$, for a constant C not depending on ε (see [68]), and $C(n,q,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant.

Since $\varphi D^2u_\epsilon = D^2(u_\epsilon\,\varphi) - \nabla u_\epsilon \otimes \nabla\varphi - \nabla\varphi \otimes \nabla u_\epsilon - u_\epsilon D^2\varphi$, and using Theorem 7.2.2, in particular see (7.2.6), we deduce that

$$\begin{aligned}
\|D^2(u_\epsilon\varphi)\|_{L^q(B_{2R}(x_0))} &\leq C(n,q)(p-2)\|D^2(u_\epsilon\varphi)\|_{L^q(B_{2R}(x_0))} \\
&+ C(n,q,R,f,p,\|\nabla u\|_{L^\infty_{loc}(\Omega)}),
\end{aligned} \tag{7.3.2}$$

where $C(n,q,R,f,p,\|\nabla u\|_{L^\infty(\Omega)})$ is a positive constant.

Since $p-2 < \frac{1}{C(n,q)}$ and using the fact that

$$\|D^2u_\epsilon\|_{L^q(B_R(x_0))} \leq \|D^2(u_\epsilon\varphi)\|_{L^q(B_{2R}(x_0))}, \tag{7.3.3}$$

we infer that

$$\|D^2u_\epsilon\|_{L^q(B_R(x_0))} \leq C(n,q,R,f,p,\|\nabla u\|_{L^\infty_{loc}(\Omega)}). \tag{7.3.4}$$

where $C(n,q,R,f,p,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant not depending on ϵ.
Therefore we have

$$\sup_{\epsilon>0}\|u_\epsilon\|_{W^{2,q}(B_R(x_0))} < \infty.$$

By Rellich's theorem, up to a subsequence we get

$$u_\epsilon \rightharpoonup w \in W^{2,q}(B_R(x_0))$$

and almost everywhere in $B_R(x_0)$. Therefore we get

$$u \equiv w \in W^{2,q}(B_R(x_0)).$$

$\square$

Next we shall be intersted in the second linearized equation of (7.1.2) at any fixed solution u, which we can write as follows. For $\varphi \in C^\infty_c(\Omega \setminus Z_u)$, taking $\psi = \varphi_{ij} := (\partial^2\varphi)/(\partial x_i\partial x_j)$ in (7.1.2), we get

$$
\begin{aligned}
&\int_\Omega |\nabla u|^{p-2}(\nabla u_{ij},\nabla\varphi)+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u_i,\nabla\varphi)\\
+\quad &(p-2)(p-4)\int_\Omega |\nabla u|^{p-6}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u)(\nabla u,\nabla\varphi)\\
+\quad &(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij},\nabla u)(\nabla u,\nabla\varphi)\\
+\quad &(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u_j)(\nabla u,\nabla\varphi)\\
+\quad &(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u)(\nabla u_j,\nabla\varphi)=\int_\Omega f_{ij}\,\varphi,
\end{aligned}
\tag{7.3.5}
$$

where $f_{ij}:=(\partial^2 f)/(\partial x_i\partial x_j)$.

Now we are ready to prove

Proof of Theorem 7.1.4. To facilitate reading, we split the case $\gamma\geq 1$ from the case $\max\{(p-2)/(p-1),2-p\}<\gamma<1$.

Case $\gamma\geq 1$.

We start exploiting the second linearized equation (7.3.5). We use a regularization argument. For $\epsilon>0$ we define:

$$
G_\epsilon(t):=\begin{cases}0 & \text{if}\quad t\in[0,\epsilon]\\ (2t-2\epsilon) & \text{if}\quad t\in[\epsilon,2\epsilon]\\ t & \text{if}\quad t\in[2\epsilon,\infty)\end{cases}
\tag{7.3.6}
$$

if $t>0$, while $G_\epsilon(t):=-G_\epsilon(-t)$ if $t\leq 0$. Let $x_0\in\Omega$ and $B=B_R(x_0)\subset\subset\Omega$. Let us consider the cut-off function $\psi_R:=\psi\in C_c^\infty(B_{2R}(x_0))$ such that:

$$
\begin{cases}\psi=1 & \text{in}\quad B_R(x_0)\\ |\nabla\psi|\leq\frac{2}{R} & \text{in}\quad B_{2R}(x_0)\setminus B_R(x_0)\\ \psi=0 & \text{in}\quad B_{2R}(x_0)^c.\end{cases}
\tag{7.3.7}
$$

Now, let us define

$$
\varphi:=\frac{G_\epsilon(|\nabla u|)^{\alpha+1}}{|\nabla u|}\,|D^2u|^{\gamma-1}u_{ij}\,\psi^2,
\tag{7.3.8}
$$

so

$$
\begin{aligned}
\nabla\varphi=&\;|D^2u|^{\gamma-1}\frac{G_\varepsilon(|\nabla u|)^{\alpha+1}}{|\nabla u|}\psi^2\nabla u_{ij}+(\gamma-1)\frac{G_\varepsilon(|\nabla u|)^{\alpha+1}}{|\nabla u|}\psi^2u_{ij}|D^2u|^{\gamma-3}D^3u\cdot D^2u\\
&+G_\varepsilon(|\nabla u|)^{\alpha}((\alpha+1)G_\varepsilon'(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})|D^2u|^{\gamma-1}u_{ij}\psi^2|\nabla u|^{-2}D^2u\nabla u\\
&+2\frac{G_\varepsilon(|\nabla u|)^{\alpha+1}}{|\nabla u|}|D^2u|^{\gamma-1}u_{ij}\psi\nabla\psi,
\end{aligned}
\tag{7.3.9}
$$

where $D^3u\cdot D^2u$ denotes the vector $(\ldots,\sum_{k,l}u_{klj}u_{kl},\ldots)$, for $j=1,\ldots,n$.

Testing (7.3.8) in (7.3.5) we get

$$
\begin{aligned}
&\int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}|\nabla u_{ij}|^2\psi^2\,dx\\
&+(\gamma-1)\int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-3}(\nabla u_{ij}u_{ij}, D^3u\cdot D^2u)\psi^2\,dx\\
&+\int_\Omega |\nabla u|^{p-2} G_\varepsilon(|\nabla u|)^{\alpha}((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})\times\\
&\qquad\qquad\times(\nabla u_{ij}, D^2u\nabla u)|\nabla u|^{-2}|D^2u|^{\gamma-1}u_{ij}\psi^2\,dx\\
&+2\int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}(\nabla u_{ij},\nabla\psi)|D^2u|^{\gamma-1}u_{ij}\psi\,dx\\
&+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u_{ij})|D^2u|^{\gamma-1}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(\gamma-1)(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u_i, D^3u\cdot D^2u)u_{ij}\times\\
&\qquad\qquad\times|D^2u|^{\gamma-3}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u_i, D^2u\nabla u)|\nabla u|^{-2}\times\\
&\qquad\times G_\varepsilon(|\nabla u|)^{\alpha}((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})|D^2u|^{\gamma-1}u_{ij}\psi^2\,dx\\
&+2(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u_i,\nabla\psi)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}u_{ij}\psi\,dx\\
&+(p-2)(p-4)\int_\Omega |\nabla u|^{p-6}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u)(\nabla u,\nabla u_{ij})\times\\
&\qquad\qquad\times|D^2u|^{\gamma-1}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(\gamma-1)(p-2)(p-4)\int_\Omega |\nabla u|^{p-6}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u)\times\\
&\qquad\qquad\times(\nabla u, D^3u\cdot D^2u)u_{ij}|D^2u|^{\gamma-3}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(p-2)(p-4)\int_\Omega |\nabla u|^{p-6}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u)(\nabla u, D^2u\nabla u)|\nabla u|^{-2}\times\\
&\qquad\times G_\varepsilon(|\nabla u|)^{\alpha}((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})|D^2u|^{\gamma-1}u_{ij}\psi^2\,dx\\
&+2(p-2)(p-4)\int_\Omega |\nabla u|^{p-6}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u)(\nabla u,\nabla\psi)\times\\
&\qquad\qquad\times G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}u_{ij}\psi\,dx\\
&+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij},\nabla u)^2|D^2u|^{\gamma-1}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(\gamma-1)(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij}u_{ij},\nabla u)(D^3u\cdot D^2u,\nabla u)\times\\
&\qquad\qquad\times|D^2u|^{\gamma-3}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij},\nabla u)(\nabla u, D^2u\nabla u)|\nabla u|^{-2}\times\\
&\qquad\times G_\varepsilon(|\nabla u|)^{\alpha}((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})|D^2u|^{\gamma-1}u_{ij}\psi^2\,dx
\end{aligned}
\tag{7.3.10}
$$

$$
\begin{aligned}
&+2(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij},\nabla u)(\nabla u,\nabla\psi)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}u_{ij}\psi\,dx\\
&+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u_j)(\nabla u,\nabla u_{ij})|D^2u|^{\gamma-1}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(\gamma-1)(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u_j)(\nabla u,D^3u\cdot D^2u)\times\\
&\qquad\qquad\times u_{ij}|D^2u|^{\gamma-3}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u_j)(\nabla u,D^2u\nabla u)|\nabla u|^{-2}\times\\
&\qquad\times G_\varepsilon(|\nabla u|)^\alpha((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})|D^2u|^{\gamma-1}u_{ij}\psi^2\,dx\\
&+2(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u_j)(\nabla u,\nabla\psi)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}u_{ij}\psi\,dx\\
&+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u)(\nabla u_j,\nabla u_{ij})|D^2u|^{\gamma-1}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(\gamma-1)(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u)(\nabla u_j,D^3u\cdot D^2u)\times\\
&\qquad\qquad\times u_{ij}|D^2u|^{\gamma-3}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u)(\nabla u_j,D^2u\nabla u)|\nabla u|^{-2}\times\\
&\qquad\times G_\varepsilon(|\nabla u|)^\alpha((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})|D^2u|^{\gamma-1}u_{ij}\psi^2\,dx\\
&+2(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u)(\nabla u_j,\nabla\psi)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}u_{ij}\psi\,dx\\
&-\int_\Omega f_{ij}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}u_{ij}\psi^2\,dx=:\tilde I_1+\cdots+\tilde I_{25}=0
\end{aligned}
$$

Since $G_\varepsilon(t)\le t$ and $G'(t)\le 2$ for any $t>0$, we deduce that

$$
\begin{aligned}
&\tilde I_1+\tilde I_2+\tilde I_{13}+\tilde I_{14}=\int_\Omega |\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}|\nabla u_{ij}|^2\psi^2\\
+\quad&(\gamma-1)\int_\Omega |\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-3}(\nabla u_{ij}u_{ij},D^3u\cdot D^2u)\psi^2\,dx\\
+\quad&(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij},\nabla u)^2|D^2u|^{\gamma-1}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
+\quad&(\gamma-1)(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij}u_{ij},\nabla u)(D^3u\cdot D^2u,\nabla u)\times\\
&\qquad\qquad\times|D^2u|^{\gamma-3}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
\le\quad&C(\alpha,p,\gamma)\int_\Omega |\nabla u|^{p-3}G_\varepsilon(|\nabla u|)^\alpha|D^2u|^{\gamma+1}|D^3u|\psi^2\,dx\\
+\quad&C(p)\int_\Omega |\nabla u|^{p-2}G_\varepsilon(|\nabla u|)^\alpha|D^3u||\nabla\psi||D^2u|^\gamma\psi\,dx\\
+\quad&C(\alpha,p)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-4}|D^2u|^{\gamma+3}\psi^2\,dx\\
+\quad&C(p)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-3}|D^2u|^{\gamma+2}|\nabla\psi|\psi\,dx\\
&\qquad+\int_\Omega|\nabla u|^\alpha|f_{ij}||D^2u|^\gamma\psi^2\,dx,
\end{aligned}
\tag{7.3.11}
$$

where $C(p), C(\alpha, p)$ and $C(\alpha, p, \gamma)$ are positive constants.

Summing on $i, j = 1, ..., n$ we get

$$
\begin{aligned}
& \bar{I}_1 + \bar{I}_2 + \bar{I}_{13} + \bar{I}_{14} = \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-1} |D^3 u|^2 \psi^2 \\
+ \quad & (\gamma - 1) \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-3} |D^3 u \cdot D^2 u|^2 \psi^2 \, dx \\
+ \quad & (p - 2) \int_\Omega |\nabla u|^{p-4} \sum_{i,j=1}^n (\nabla u_{ij}, \nabla u)^2 |D^2 u|^{\gamma-1} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} \psi^2 \, dx \\
+ \quad & (\gamma - 1)(p - 2) \int_\Omega |\nabla u|^{p-4} (D^3 u \cdot D^2 u, \nabla u)^2 |D^2 u|^{\gamma-3} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} \psi^2 \, dx \\
\leq \quad & C(n, \alpha, p, \gamma) \int_\Omega |\nabla u|^{p-3} G_\varepsilon(|\nabla u|)^\alpha |D^2 u|^{\gamma+1} |D^3 u| \psi^2 \, dx \\
+ \quad & C(n, p) \int_\Omega |\nabla u|^{p-2} G_\varepsilon(|\nabla u|)^\alpha |D^3 u| |\nabla \psi| |D^2 u|^\gamma \psi \, dx \\
+ \quad & C(n, \alpha, p) \int_{\Omega \setminus Z_u} |\nabla u|^{p+\alpha-4} |D^2 u|^{\gamma+3} \psi^2 \, dx \\
+ \quad & C(n, p) \int_{\Omega \setminus Z_u} |\nabla u|^{p+\alpha-3} |D^2 u|^{\gamma+2} |\nabla \psi| \psi \, dx \\
& \qquad + C(n) \int_\Omega |\nabla u|^\alpha |D^2 f| |D^2 u|^\gamma \psi^2 \, dx,
\end{aligned} \tag{7.3.12}
$$

where $C(n), C(n, p), C(n, \alpha, p)$ and $C(n, \alpha, p, \gamma)$ are positive constants.

In the case $p \geq 2$ and $\gamma \geq 1$, by Cauchy-Schwarz inequality we have

$$
\begin{aligned}
& \bar{I}_1 + \bar{I}_2 + \bar{I}_{13} + \bar{I}_{14} \\
& \geq \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-1} |D^3 u|^2 \psi^2 \, dx.
\end{aligned} \tag{7.3.13}
$$

Using Cauchy-Schwarz inequality for $1 < p < 2$ and $\gamma \geq 1$, we note that

$$
\begin{aligned}
\bar{I}_{14} =& (\gamma - 1)(p - 2) \int_\Omega |\nabla u|^{p-4} (D^3 u \cdot D^2 u, \nabla u)^2 |D^2 u|^{\gamma-3} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} \psi^2 \, dx \\
\geq & (\gamma - 1)(p - 2) \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-3} |D^3 u \cdot D^2 u|^2 \psi^2 \, dx.
\end{aligned} \tag{7.3.14}
$$

Therefore, by (7.3.14) and using Cauchy-Schwarz inequality on $\bar{I}_{13}$, we get

$$
\begin{aligned}
& \bar{I}_1 + \bar{I}_2 + \bar{I}_{13} + \bar{I}_{14} \\
& \geq (p - 1) \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-1} |D^3 u|^2 \psi^2 \, dx.
\end{aligned} \tag{7.3.15}
$$

Using (7.3.13) and (7.3.15) in (7.3.11), and since $\gamma \geq 1$, there exists a positive constant $C(p)$ such that

$$
\begin{aligned}
&C(p)\int_\Omega |\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}|D^3u|^2\psi^2\,dx\\
&\quad\le C(n,\alpha,p,\gamma)\int_\Omega |\nabla u|^{p-3}G_\varepsilon(|\nabla u|)^{\alpha}|D^2u|^{\gamma+1}|D^3u|\psi^2\,dx\\
&\quad+C(n,p)\int_\Omega |\nabla u|^{p-2}G_\varepsilon(|\nabla u|)^{\alpha}|\nabla\psi||D^2u|^{\gamma}|D^3u|\psi\,dx\\
&\quad+C(n,\alpha,p)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-4}|D^2u|^{\gamma+3}\psi^2\,dx\\
&\quad+C(n,p)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-3}|D^2u|^{\gamma+2}|\nabla\psi|\psi\,dx\\
&\quad+C(n)\int_\Omega |\nabla u|^{\alpha}|D^2f||D^2u|^{\gamma}\psi^2\,dx=:I_1+I_2+I_3+I_4+I_5.
\end{aligned}
\tag{7.3.16}
$$

Now we estimate the right hand side of (7.3.16). We estimate the term I_3. Using a standard Young inequality we obtain

$$
\begin{aligned}
I_3&\le C(n,\alpha,p)\int_{B_{2R}\setminus Z_u}|\nabla u|^{p+\alpha-4}|D^2u|^{3+\gamma}\,dx\\
&=C(n,\alpha,p)\int_{B_{2R}\setminus Z_u}|\nabla u|^{p+\alpha-4}|\nabla u|^{\frac{p-2-\beta}{2}}|\nabla u|^{\frac{2-p+\beta}{2}}|D^2u||D^2u|^{2+\gamma}\,dx\\
&\le\frac{C(n,\alpha,p)}{2}\int_{B_{2R}\setminus Z_u}|\nabla u|^{p-2-\beta}|D^2u|^2\,dx\\
&+\frac{C(n,\alpha,p)}{2}\int_{B_{2R}\setminus Z_u}|\nabla u|^{2(p-4+\alpha)+2-p+\beta}|D^2u|^{4+2\gamma}\,dx\\
&\le C(n,\alpha,p,R,\beta,f,\|\nabla u\|_{L^\infty_{loc}(\Omega)})\\
&+C(n,\alpha,p)\int_{B_{2R}\setminus Z_u}|\nabla u|^{2(p-4+\alpha)+2-p+\beta}|D^2u|^{4+2\gamma}\,dx,
\end{aligned}
\tag{7.3.17}
$$

where in the last inequality we have used Theorem 7.2.2 and $C(n,\alpha,p,R,\beta,f,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant. Iterating this procedure N-times, we get

$$
\begin{aligned}
I_3&\le NC(n,\alpha,p,R,\beta,f,\|\nabla u\|_{L^\infty_{loc}(\Omega)})\\
&\quad+C(n,\alpha,p)\int_{B_{2R}\setminus Z_u}|\nabla u|^{2^N(p-4+\alpha)+\sum_{k=1}^N 2^{k-1}(2-p+\beta)}|D^2u|^{q_N}\,dx,
\end{aligned}
\tag{7.3.18}
$$

where q_N is given by

$$
\begin{cases}
q_N=2(q_{N-1}-1)\\
q_0=3+\gamma.
\end{cases}
\tag{7.3.19}
$$

Now by Young's inequality with exponents $(q/q_N, q/(q-q_N))$ we get

$$
\begin{aligned}
&\int_{B_{2R}\setminus Z_u}|\nabla u|^{2^N(p-4+\alpha)+\sum_{k=1}^N 2^{k-1}(2-p+\beta)}|D^2u|^{q_N}\,dx\\
&\quad\le C(q)\int_{B_{2R}\setminus Z_u}|\nabla u|^{\frac{q}{q-q_N}\left(2^N(p-4+\alpha)+\sum_{k=1}^N 2^{k-1}(2-p+\beta)\right)}\,dx+C(q)\int_{B_{2R}}|D^2u|^q\,dx
\end{aligned}
\tag{7.3.20}
$$

where $C(q)$ is a positive constant.

If

$$\frac{q}{q-q_N}\left(2^N(p-4+\alpha)+\sum_{k=1}^{N}2^{k-1}(2-p+\beta)\right)>-p+1, \tag{7.3.21}$$

and $q>q_N$, from Theorem 7.2.3, we have

$$\int_{B_{2R}}|\nabla u|^{\frac{q}{q-q_N}\left(2^N(p-4+\alpha)+\sum_{k=1}^{N}2^{k-1}(2-p+\beta)\right)}\,dx\le C(N,\alpha,p,q,f,\beta,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)}), \tag{7.3.22}$$

where $C(N,\alpha,p,q,f,\beta,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant. Moreover, since $q>q_N$, from Theorem 7.1.3 we deduce that

$$\int_{B_{2R}}|D^2u|^q\,dx\le C(n,p,q,f,R), \tag{7.3.23}$$

where $C(n,p,q,f,R)$ is a positive constant.

By (7.3.22) and (7.3.23) we estimate

$$I_3\le C(\alpha,p,q,n,f,R,\beta,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N), \tag{7.3.24}$$

where $C(\alpha,p,q,n,f,R,\beta,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N)$ is a positive constant.

We note that (7.3.21) yields, for β close to 1,

$$\alpha>\alpha(p,q,N):=\alpha_N:=\frac{(q-q_N)}{2^Nq}(1-p)+\frac{3-p}{2^N}+1.$$

We note that, if $p<2+1/(q-1)$, that is $q<(p-1)/(p-2)$ we have,

$$\alpha_k<\alpha_{k-1}\quad\forall k\in\mathbb{N},$$

and

$$\alpha>\alpha_N>1\quad\forall N\in\mathbb{N}_0.$$

For the term I_1, using weighted Young's inequality and by (7.3.24) we get

$$\begin{aligned}I_1&\le\theta\int_\Omega|\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}|D^3u|^2\psi^2\,dx\\&+C(n,\gamma,p,\alpha,\theta)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-4}|D^2u|^{3+\gamma}\psi^2\,dx\\&\le\theta\int_\Omega|\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}|D^3u|^2\psi^2\,dx\\&+C(\theta,\gamma,\alpha,p,q,n,f,R,\beta,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N),\end{aligned} \tag{7.3.25}$$

where $C(\theta,\gamma,\alpha,p,q,n,f,R,\beta,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N)$ is a positive constant.

For the term I_2, since $\alpha>1$, by (7.3.7) and from Theorem 7.1.3 we deduce that

$$
\begin{aligned}
I_2 &\le \theta \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-1} |D^3 u|^2 \psi^2 \, dx \\
&+ C(n,p,\theta) \int_\Omega |\nabla u|^{p+\alpha-2} |D^2 u|^{1+\gamma} |\nabla \psi|^2 \, dx \\
&\le \theta \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-1} |D^3 u|^2 \psi^2 \, dx \\
&+ C(n,p,\theta,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)}) \int_{B_{2R}} |D^2 u|^{1+\gamma} \, dx \\
&\le \theta \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-1} |D^3 u|^2 \psi^2 \, dx \\
&+ C(\theta,\alpha,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N),
\end{aligned} \tag{7.3.26}
$$

where $C(\theta,\alpha,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N)$ is a positive constant. From Theorem 7.2.2, by (7.3.24), (7.3.26) and using standard Young's inequality we estimate

$$
\begin{aligned}
I_4 &\le C(n,p,R) \int_{B_{2R}} |\nabla u|^{p+\alpha-3} |D^2 u|^{2+\gamma} \, dx \\
&\le \frac{C(n,p,R)}{2} \int_{B_{2R}} |\nabla u|^{p+\alpha-2} |D^2 u|^{1+\gamma} \, dx + \frac{C(n,p,R)}{2} \int_{B_{2R}} |\nabla u|^{p+\alpha-4} |D^2 u|^{3+\gamma} \, dx \\
&\le C(\alpha,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N),
\end{aligned} \tag{7.3.27}
$$

where $C(\alpha,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N)$ is a positive constant.
For the last term I_5, by assumptions on f and from Theorem 7.1.3 we get

$$
\begin{aligned}
I_5 &\le C(n,\alpha,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)}) \int_{B_{2R}} |D^2 f| |D^2 u|^\gamma \, dx \\
&\le C(\alpha,R,f,n,p,q,\|\nabla u\|_{L^\infty_{loc}(\Omega)}),
\end{aligned} \tag{7.3.28}
$$

where $C(\alpha,R,f,n,p,q,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant.
Using (7.3.24), (7.3.25), (7.3.26), (7.3.27) and (7.3.28) in (7.3.16), we get:

$$
\begin{aligned}
(C(p)-2\theta) &\int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-1} |D^3 u|^2 \psi^2 \, dx \\
&\le C(\theta,\alpha,\gamma,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N),
\end{aligned} \tag{7.3.29}
$$

where $C(\theta,\alpha,\gamma,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N)$ is a positive constant.
Now, by Fatou's Lemma and choosing θ sufficiently small such that $C(p)-2\theta>0$, we get:

$$
\int_{B_R(x_0)\setminus Z_u} |\nabla u|^{p-2+\alpha} |D^2 u|^{\gamma-1} |D^3 u|^2 \, dx \le C(\theta,\gamma,\alpha,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N). \tag{7.3.30}
$$

Using a covering argument, we are done.
Case $\max\{(p-2)/(p-1), 2-p\} < \gamma < 1$.
Let us define

$$
\varphi := \frac{G_\epsilon(|\nabla u|)^{\alpha+1}}{|\nabla u|} G_\tau(|D^2 u|) |D^2 u|^{\gamma-2} u_{ij} \, \psi^2, \tag{7.3.31}
$$

where $G_\tau, G_\varepsilon, \psi$ are defined in (7.3.6) and (7.3.7). So we obtain

$$
\begin{aligned}
\nabla\varphi = {} & G_\tau(|D^2u|)|D^2u|^{\gamma-2}\frac{G_\varepsilon(|\nabla u|)^{\alpha+1}}{|\nabla u|}\psi^2\nabla u_{ij} \\
& + \frac{G_\varepsilon(|\nabla u|)^{\alpha+1}}{|\nabla u|}\psi^2 u_{ij}|D^2u|^{\gamma-4}((\gamma-2)G_\tau(|D^2u|)+G'_\tau(|D^2u|)|D^2u|)D^3u\cdot D^2u \\
& + G_\varepsilon(|\nabla u|)^{\alpha}((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})\times \\
& \qquad\qquad \times G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi^2|\nabla u|^{-2}D^2u\nabla u \\
& + 2\frac{G_\varepsilon(|\nabla u|)^{\alpha+1}}{|\nabla u|}G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi\nabla\psi,
\end{aligned}
\tag{7.3.32}
$$

where $D^3u\cdot D^2u$ denotes the vector $(..., \sum_{k,l}u_{klj}u_{kl}, ...)$, for $j=1,...,n$. Now we set $h_\tau(t) := (\gamma-2)G_\tau(t)+G'_\tau(t)t$, for $t\geq 0$.

Testing (7.3.31) in (7.3.5) we get

$$
\begin{aligned}
& \int_\Omega |\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}G_\tau(|D^2u|)|D^2u|^{\gamma-2}|\nabla u_{ij}|^2\psi^2\,dx \\
& + \int_\Omega |\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-4}h_\tau(|D^2u|)(\nabla u_{ij}u_{ij}, D^3u\cdot D^2u)\psi^2\,dx \\
& + \int_\Omega |\nabla u|^{p-2}G_\varepsilon(|\nabla u|)^{\alpha}((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})\times \\
& \qquad\qquad \times(\nabla u_{ij}, D^2u\nabla u)|\nabla u|^{-2}G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi^2\,dx \\
& + 2\int_\Omega |\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}(\nabla u_{ij},\nabla\psi)G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi\,dx \\
& + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u_{ij})G_\tau(|D^2u|)|D^2u|^{\gamma-2}\times \\
& \qquad\qquad \times G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx \\
& + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u_i, D^3u\cdot D^2u)u_{ij}\times \\
& \qquad\qquad \times|D^2u|^{\gamma-4}h_\tau(|D^2u|)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx \\
& + (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u_i, D^2u\nabla u)|\nabla u|^{-2}G_\varepsilon(|\nabla u|)^{\alpha}\times \\
& \qquad \times((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi^2\,dx \\
& + 2(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u,\nabla u_j)(\nabla u_i,\nabla\psi)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\times \\
& \qquad\qquad \times G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi\,dx \\
& + (p-2)(p-4)\int_\Omega |\nabla u|^{p-6}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u)(\nabla u,\nabla u_{ij})\times \\
& \qquad\qquad \times G_\tau(|D^2u|)|D^2u|^{\gamma-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx \\
& + (p-2)(p-4)\int_\Omega |\nabla u|^{p-6}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u)(\nabla u, D^3u\cdot D^2u)\times \\
& \qquad\qquad \times u_{ij}|D^2u|^{\gamma-4}h_\tau(|D^2u|)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx
\end{aligned}
\tag{7.3.33}
$$

$$
\begin{aligned}
&+ (p-2)(p-4)\int_\Omega |\nabla u|^{p-6}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u)(\nabla u, D^2u\nabla u)|\nabla u|^{-2}G_\varepsilon(|\nabla u|)^\alpha\times\\
&\qquad\times((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi^2\,dx\\
&+ 2(p-2)(p-4)\int_\Omega |\nabla u|^{p-6}(\nabla u,\nabla u_j)(\nabla u_i,\nabla u)(\nabla u,\nabla\psi)\times\\
&\qquad\times G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi\,dx\\
&+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij},\nabla u)^2G_\tau(|D^2u|)|D^2u|^{\gamma-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij}u_{ij},\nabla u)(D^3u\cdot D^2u,\nabla u)\times\\
&\qquad\times |D^2u|^{\gamma-4}h_\tau(|D^2u|)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij},\nabla u)(\nabla u, D^2u\nabla u)|\nabla u|^{-2}G_\varepsilon(|\nabla u|)^\alpha\times\\
&\qquad\times((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi^2\,dx\\
&+ 2(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_{ij},\nabla u)(\nabla u,\nabla\psi)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\times\\
&\qquad\times G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi\,dx\\
&+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u_j)(\nabla u,\nabla u_{ij})G_\tau(|D^2u|)|D^2u|^{\gamma-2}\times\\
&\qquad G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u_j)(\nabla u, D^3u\cdot D^2u)\times\\
&\qquad\times u_{ij}|D^2u|^{\gamma-4}h_\tau(|D^2u|)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u_j)(\nabla u, D^2u\nabla u)|\nabla u|^{-2}G_\varepsilon(|\nabla u|)^\alpha\times\\
&\qquad\times((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi^2\,dx\\
&+ 2(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u_j)(\nabla u,\nabla\psi)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\times\\
&\qquad\times G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi\,dx\\
&+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u)(\nabla u_j,\nabla u_{ij})G_\tau(|D^2u|)|D^2u|^{\gamma-2}\times\\
&\qquad\times G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u)(\nabla u_j, D^3u\cdot D^2u)\times\\
&\qquad\times u_{ij}|D^2u|^{\gamma-4}h_\tau(|D^2u|)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx\\
&+ (p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u)(\nabla u_j, D^2u\nabla u)|\nabla u|^{-2}G_\varepsilon(|\nabla u|)^\alpha\times\\
&\qquad\times((\alpha+1)G'_\varepsilon(|\nabla u|)-G_\varepsilon(|\nabla u|)|\nabla u|^{-1})G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi^2\,dx\\
&+ 2(p-2)\int_\Omega |\nabla u|^{p-4}(\nabla u_i,\nabla u)(\nabla u_j,\nabla\psi)G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\times\\
&\qquad\times G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi\,dx\\
&- \int_\Omega f_{ij}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}G_\tau(|D^2u|)|D^2u|^{\gamma-2}u_{ij}\psi^2\,dx =: I_1+\cdots+I_{25}=0
\end{aligned}
$$

Since $G(t) \leq t$ and $G'(t) \leq 2$ for any $t \geq 0$, we deduce that

$$\begin{aligned}
& I_1 + I_2 + I_{13} + I_{14} = \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} G_\tau(|D^2 u|)|D^2 u|^{\gamma-2} |\nabla u_{ij}|^2 \psi^2 \\
+ \quad & \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-4} h_\tau(|D^2 u|)(\nabla u_{ij} u_{ij}, D^3 u \cdot D^2 u)\psi^2 \, dx \\
+ \quad & (p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u_{ij}, \nabla u)^2 G_\tau(|D^2 u|)|D^2 u|^{\gamma-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} \psi^2 \, dx \\
+ \quad & (p-2) \int_\Omega |\nabla u|^{p-4} (\nabla u_{ij} u_{ij}, \nabla u)(D^3 u \cdot D^2 u, \nabla u) \times \\
& \qquad\qquad \times |D^2 u|^{\gamma-4} h_\tau(|D^2 u|) G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} \psi^2 \, dx \\
\leq \quad & C(\alpha, p, \gamma) \int_\Omega |\nabla u|^{p-3} G_\varepsilon(|\nabla u|)^\alpha G_\tau(|D^2 u|)|D^2 u|^\gamma |D^3 u| \psi^2 \, dx \\
+ \quad & C(p) \int_\Omega |\nabla u|^{p-3} G_\varepsilon(|\nabla u|)^\alpha |D^2 u|^{\gamma-1} |h_\tau(|D^2 u|)||D^3 u \cdot D^2 u| \psi^2 \, dx \\
+ \quad & C(p) \int_\Omega |\nabla u|^{p-4} G_\varepsilon(|\nabla u|)^\alpha |D^2 u|^{\gamma-1} |h_\tau(|D^2 u|)||(D^3 u \cdot D^2 u, \nabla u)| \psi^2 \, dx \\
+ \quad & C(p) \int_\Omega |\nabla u|^{p-2} G_\varepsilon(|\nabla u|)^\alpha |D^3 u||\nabla \psi| G_\tau(|D^2 u|)|D^2 u|^{\gamma-1} \psi \, dx \\
+ \quad & C(\alpha, p) \int_{\Omega \setminus Z_u} |\nabla u|^{p+\alpha-4} |D^2 u|^{\gamma+3} \psi^2 \, dx \\
+ \quad & C(p) \int_{\Omega \setminus Z_u} |\nabla u|^{p+\alpha-3} |D^2 u|^{\gamma+2} |\nabla \psi| \psi \, dx \\
& \qquad + \int_\Omega |\nabla u|^\alpha |f_{ij}||D^2 u|^\gamma \psi^2 \, dx,
\end{aligned} \tag{7.3.34}$$

where $C(p), C(\alpha, p)$ and $C(\alpha, p, \gamma)$ are positive constants.

Summing on $i, j = 1, ..., n$ we get

$$\begin{aligned}
& \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} G_\tau(|D^2 u|)|D^2 u|^{\gamma-2} |D^3 u|^2 \psi^2 \\
& + \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} |D^2 u|^{\gamma-4} h_\tau(|D^2 u|))|D^3 u \cdot D^2 u|^2 \psi^2 \, dx \\
& + (p-2) \int_\Omega |\nabla u|^{p-4} \sum_{i,j=1}^n (\nabla u_{ij}, \nabla u)^2 G_\tau(|D^2 u|)|D^2 u|^{\gamma-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} \psi^2 \, dx \\
& + (p-2) \int_\Omega |\nabla u|^{p-4} (D^3 u \cdot D^2 u, \nabla u)^2 |D^2 u|^{\gamma-4} h_\tau(|D^2 u|) G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} \psi^2 \, dx \\
& \leq C(n, \alpha, p, \gamma) \int_\Omega |\nabla u|^{p-3} G_\varepsilon(|\nabla u|)^\alpha G_\tau(|D^2 u|)|D^2 u|^\gamma |D^3 u| \psi^2 \, dx \\
& + C(n, p) \int_\Omega |\nabla u|^{p-3} G_\varepsilon(|\nabla u|)^\alpha |D^2 u|^{\gamma-1} |h_\tau(|D^2 u|)||D^3 u \cdot D^2 u| \psi^2 \, dx \\
& + C(n, p) \int_\Omega |\nabla u|^{p-4} G_\varepsilon(|\nabla u|)^\alpha |D^2 u|^{\gamma-1} |h_\tau(|D^2 u|)||(D^3 u \cdot D^2 u, \nabla u)| \psi^2 \, dx
\end{aligned} \tag{7.3.35}$$

$$
\begin{aligned}
&+C(n,p)\int_\Omega |\nabla u|^{p-2}G_\varepsilon(|\nabla u|)^\alpha |D^3u||\nabla\psi|G_\tau(|D^2u|)|D^2u|^{\gamma-1}\psi\,dx\\
&+C(n,\alpha,p)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-4}|D^2u|^{\gamma+3}\psi^2\,dx\\
&+C(n,p)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-3}|D^2u|^{\gamma+2}|\nabla\psi|\psi\,dx\\
&\qquad\qquad + C(n)\int_\Omega |\nabla u|^\alpha |D^2f||D^2u|^\gamma\psi^2\,dx =: J_1+\cdots+J_7,
\end{aligned}
$$

where $C(n), C(n,p), C(n,\alpha,p)$ and $C(n,\alpha,p,\gamma)$ are positive constants.

Now we estimate the right-hand side of (7.3.35). We estimate the term J_1. By weighted Young's inequality we get

$$
\begin{aligned}
J_1 &\le \theta\int_\Omega |\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx\\
&+C(n,\gamma,p,\alpha,\theta)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-4}|D^2u|^{3+\gamma}\psi^2\,dx,
\end{aligned}
\tag{7.3.36}
$$

where $C(n,\gamma,p,\alpha,\theta)$ is a positive constant.

For the term J_2, since $|h_\tau(t)|\le(2-\gamma)G_\tau(t)+G_\tau'(t)t$, for any $t\ge 0$, by (7.3.36) and using a weighted Young's inequality we obtain

$$
\begin{aligned}
J_2 &\le C(n,p,\gamma)\int_\Omega |\nabla u|^{p-3}G_\varepsilon(|\nabla u|)^\alpha G_\tau(|D^2u|)|D^2u|^\gamma|D^3u|\psi^2\,dx\\
&+C(n,p)\int_\Omega |\nabla u|^{p-3}G_\varepsilon(|\nabla u|)^\alpha|D^2u|^{\gamma-1}G_\tau'(|D^2u|)|D^2u|||D^3u\cdot D^2u|\psi^2\,dx\\
&\le\theta\int_\Omega|\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx\\
&+C(n,\gamma,p,\theta)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-4}|D^2u|^{3+\gamma}\psi^2\,dx\\
&+\theta\int_\Omega|\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-4}G_\tau'(|D^2u|)|D^2u||D^3u\cdot D^2u|^2\psi^2\,dx\\
&+C(n,p,\theta)\int_{\Omega\setminus Z_u}|\nabla u|^{p+\alpha-4}|D^2u|^{3+\gamma}\psi^2\,dx
\end{aligned}
\tag{7.3.37}
$$

where $C(n,\gamma,p,\theta)$ and $C(n,p,\theta)$ are positive constants. We estimate the term J_3. By (7.3.36) and by weighted Young inequality we get

$$
\begin{aligned}
J_3 &\leq C(n,p,\gamma) \int_\Omega |\nabla u|^{p-3} G_\varepsilon(|\nabla u|)^\alpha G_\tau(|D^2u|)|D^2u|^\gamma |D^3u|\psi^2\, dx \\
&+ C(n,p) \int_\Omega |\nabla u|^{p-4} G_\varepsilon(|\nabla u|)^\alpha |D^2u|^{\gamma-1} G'_\tau(|D^2u|)|D^2u|||(D^3u\cdot D^2u,\nabla u)|\psi^2\, dx \\
&\leq \theta \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1} |\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\, dx \\
&+ C(n,\gamma,p,\theta) \int_{\Omega\setminus Z_u} |\nabla u|^{p+\alpha-4}|D^2u|^{3+\gamma}\psi^2\, dx \\
&+ \theta \int_\Omega |\nabla u|^{p-4} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-4} G'_\tau(|D^2u|)|D^2u|(D^3u\cdot D^2u,\nabla u)^2\psi^2\, dx \\
&+ C(n,p,\theta) \int_{\Omega\setminus Z_u} |\nabla u|^{p+\alpha-4}|D^2u|^{3+\gamma}\psi^2\, dx,
\end{aligned}
\tag{7.3.38}
$$

where $C(n,\gamma,p,\theta)$ and $C(n,p,\theta)$ are positive constant. For the term J_4, by weighted Young's inequality we have

$$
\begin{aligned}
J_4 &\leq \theta \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\, dx \\
&+ C(n,p,\theta) \int_\Omega |\nabla u|^{p-2+\alpha}|D^2u|^{\gamma+1}|\nabla\psi|^2\, dx,
\end{aligned}
\tag{7.3.39}
$$

where $C(n,p,\theta)$ is a positive constant. Now we set $h_{\tau,\theta}(t) := (\gamma-2)G_\tau(t) + (1-\theta)G'_\tau(t)t$, for any $t \geq 0$. By (7.3.36), (7.3.37), (7.3.38) and (7.3.39) we get

$$
\begin{aligned}
& (1-4\theta) \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\, dx \\
+ & \int_\Omega |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-4} h_{\tau,\theta}(|D^2u|)|D^3u\cdot D^2u|^2\psi^2\, dx \\
+ & (p-2)\int_\Omega |\nabla u|^{p-4}\sum_{i=1}^n (\nabla u_{ij},\nabla u)^2 G_\tau(|D^2u|)|D^2u|^{\gamma-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\, dx \\
+ & (p-2)\int_\Omega |\nabla u|^{p-4}(D^3u\cdot D^2u,\nabla u)^2|D^2u|^{\gamma-4} h_{\tau,\theta}(|D^2u|) G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\, dx \\
\leq & C(n,\alpha,p,\gamma,\theta)\int_{\Omega\setminus Z_u} |\nabla u|^{p+\alpha-4}|D^2u|^{\gamma+3}\psi^2\, dx \\
+ & C(n,p)\int_{\Omega\setminus Z_u} |\nabla u|^{p+\alpha-3}|D^2u|^{\gamma+2}|\nabla\psi|\psi\, dx \\
+ & C(n,p,\theta)\int_\Omega |\nabla u|^{p-2+\alpha}|D^2u|^{\gamma+1}|\nabla\psi|^2\, dx \\
& \qquad + C(n)\int_\Omega |\nabla u|^\alpha |D^2f||D^2u|^\gamma\psi^2\, dx,
\end{aligned}
\tag{7.3.40}
$$

where $C(n,\alpha,p,\gamma,\theta)$ is a positive constant.

Now we split the domain Ω into three subdomains as follows

$$
\Omega = \{|D^2u| \leq \tau\} \cup \{\tau < |D^2u| < 2\tau\} \cup \{|D^2u| \geq 2\tau\} =: \Omega_0 \cup \Omega_1 \cup \Omega_2.
$$

Now we estimate the left-hand side of (7.3.40). Recalling the definition G_τ (see (7.3.6)), we get

$$
\begin{aligned}
&\tilde{J}_1 + \cdots + \tilde{J}_8 \\
:=&(1-4\theta)\int_{\Omega_1} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx \\
&+\int_{\Omega_1} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-4} h_{\tau,\theta}(|D^2u|)|D^3u\cdot D^2u|^2\psi^2\,dx \\
&+(p-2)\int_{\Omega_1} |\nabla u|^{p-4}\sum_{i,j=1}^n (\nabla u_{ij},\nabla u)^2 G_\tau(|D^2u|)|D^2u|^{\gamma-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx \\
&+(p-2)\int_{\Omega_1} |\nabla u|^{p-4}(D^3u\cdot D^2u,\nabla u)^2|D^2u|^{\gamma-4} h_{\tau,\theta}(|D^2u|) G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx \\
&+(1-4\theta)\int_{\Omega_2} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx \\
&+\int_{\Omega_2} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-4} h_{\tau,\theta}(|D^2u|)|D^3u\cdot D^2u|^2\psi^2\,dx \\
&+(p-2)\int_{\Omega_2} |\nabla u|^{p-4}\sum_{i,j=1}^n (\nabla u_{ij},\nabla u)^2 G_\tau(|D^2u|)|D^2u|^{\gamma-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx \\
&+(p-2)\int_{\Omega_2} |\nabla u|^{p-4}(D^3u\cdot D^2u,\nabla u)^2|D^2u|^{\gamma-4} h_{\tau,\theta}(|D^2u|) G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx
\end{aligned}
\tag{7.3.41}
$$

We consider the case $p<2$. For $\max\{(p-2)/(p-1), 2-p\} < \gamma < 1$ fixed and $\theta > 0$ sufficiently small, we note that $h_{\tau,\theta} > 0$ in Ω_1 and $h_{\tau,\theta}(t) = (\gamma-1-\theta)t = (\gamma-1-\theta)G_\tau(t) < 0$ in Ω_2. Using Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&(p-2)\int_{\Omega_1} |\nabla u|^{p-4}(D^3u\cdot D^2u,\nabla u)^2|D^2u|^{\gamma-4} h_{\tau,\theta}(|D^2u|) G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\psi^2\,dx \\
&\geq (p-2)\int_{\Omega_1} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-4} h_{\tau,\theta}(|D^2u|)|D^3u\cdot D^2u|^2\psi^2\,dx.
\end{aligned}
\tag{7.3.42}
$$

By (7.3.42) and Cauchy-Schwarz inequality we have

$$
\begin{aligned}
&\tilde{J}_1 + \cdots + \tilde{J}_8 \\
\geq&(p-1-4\theta)\int_{\Omega_1} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx \\
&+(p-1)\int_{\Omega_1} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-4} h_{\tau,\theta}(|D^2u|)|D^3u\cdot D^2u|^2\psi^2\,dx \\
&+(\gamma+p-2-5\theta)\int_{\Omega_2} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx \\
&\geq (\gamma+p-2-5\theta)\int_{\Omega_2} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx,
\end{aligned}
\tag{7.3.43}
$$

where in the last inequality we have used the fact that $h_{\tau,\theta} > 0$ in Ω_1, for θ sufficiently small.

In the case $p \geq 2$, by Cauchy-Schwarz inequality we have

$$\begin{aligned}
&\tilde{J}_1 + \cdots + \tilde{J}_8 \\
\geq&(1-4\theta)\int_{\Omega_1} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx \\
&+(\gamma+(p-2)(\gamma-1-\theta)-5\theta)\int_{\Omega_2} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\times \\
&\qquad\qquad\times G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx \\
\geq&(\gamma+(p-2)(\gamma-1-\theta)-5\theta)\int_{\Omega_2} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}\times \\
&\qquad\qquad\times G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx.
\end{aligned} \tag{7.3.44}$$

Using (7.3.43) and (7.3.44) in (7.3.40), and since $\gamma > \max\{(p-2)/(p-1), 2-p\}$, for $\theta > 0$ sufficiently small, there exists a positive constant $C(\gamma, p, \theta)$ such that

$$\begin{aligned}
&C(\gamma,p,\theta)\int_{\Omega_2} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1} G_\tau(|D^2u|)|D^2u|^{\gamma-2}|D^3u|^2\psi^2\,dx \\
\leq\;& C(n,\alpha,p,\gamma,\theta)\int_{\Omega\setminus Z_u} |\nabla u|^{p+\alpha-4}|D^2u|^{\gamma+3}\psi^2\,dx \\
+\;& C(n,p)\int_{\Omega\setminus Z_u} |\nabla u|^{p+\alpha-3}|D^2u|^{\gamma+2}|\nabla\psi|\psi\,dx \\
+\;& C(n,p,\theta)\int_{\Omega} |\nabla u|^{p-2+\alpha}|D^2u|^{\gamma+1}|\nabla\psi|^2\,dx \\
&\qquad+C(n)\int_{\Omega} |\nabla u|^{\alpha}|D^2 f||D^2u|^{\gamma}\psi^2\,dx,
\end{aligned} \tag{7.3.45}$$

By Fatou Lemma, for $\tau \to 0$ we obtain

$$\begin{aligned}
&C(\gamma,p,\theta)\int_{\Omega\setminus\{|D^2u|=0\}} |\nabla u|^{p-2} G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}|D^3u|^2\psi^2\,dx \\
\leq\;& C(n,\alpha,p,\gamma,\theta)\int_{\Omega\setminus Z_u} |\nabla u|^{p+\alpha-4}|D^2u|^{\gamma+3}\psi^2\,dx \\
+\;& C(n,p)\int_{\Omega\setminus Z_u} |\nabla u|^{p+\alpha-3}|D^2u|^{\gamma+2}|\nabla\psi|\psi\,dx \\
+\;& C(n,p,\theta)\int_{\Omega} |\nabla u|^{p-2+\alpha}|D^2u|^{\gamma+1}|\nabla\psi|^2\,dx \\
&\qquad+C(n)\int_{\Omega} |\nabla u|^{\alpha}|D^2 f||D^2u|^{\gamma}\psi^2\,dx.
\end{aligned} \tag{7.3.46}$$

Proceeding as in the case $\gamma \geq 1$, we get the thesis.

□

Theorem 7.1.4 allows us to state the following

Theorem 7.3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain and $u \in C^{1,\beta}_{loc}(\Omega)$ be a solution of (7.1.1). Let $f(x) \in W^{2,2}_{loc}(\Omega) \cap C^{1,\beta'}_{loc}(\Omega)$ and*

$$f \geq \tau > 0 \quad in \quad \Omega, \quad (or\ f \leq \tau < 0 \quad in \quad \Omega).$$

Then for $\varepsilon > 0$ sufficiently small there exists $\delta_0 = \delta_0(\varepsilon) > 0$, such that for any $p \in (2-\delta_0, 2+\delta_0)$, we have

$$|D^3u|^{\tilde{q}} \in L^1_{loc}(\Omega), \tag{7.3.47}$$

with $\tilde{q} < 1 - \varepsilon$.

Proof. We note that, from Theorem 7.2.3, the Lebesgue measure of Z_u is zero. Moreover, since $D^2u \in W^{1,1}_{loc}(\Omega \setminus Z_u)$, using Stampacchia's Theorem, we deduce that $D^3u = 0$ a.e. in $\{x \in \Omega \setminus Z_u : |D^2u| = 0\}$. We fix $x_0 \in \Omega$ and $R > 0$ such that $B := B_{2R}(x_0) \subset\subset \Omega$. Then, taking $\beta < 1$ we obtain

$$\begin{aligned}\int_B |D^3u|^{\tilde{q}}\,dx = \int_{B\setminus(Z_u\cup\{|D^2u|=0\})} &|D^3u|^{\tilde{q}}|\nabla u|^{\frac{p-2+\alpha}{2}\tilde{q}}|\nabla u|^{\frac{2-p-\alpha}{2}\tilde{q}}|D^2u|^{\frac{(\gamma-1)}{2}\tilde{q}}|D^2u|^{\frac{(1-\gamma)}{2}\tilde{q}}\times\\ &\times|\nabla u|^{\frac{(p-2-\beta)(1-\gamma)}{4}\tilde{q}}|\nabla u|^{\frac{(2-p+\beta)(1-\gamma)}{4}\tilde{q}}\,dx,\end{aligned} \tag{7.3.48}$$

where α and γ are given in Theorem 7.1.4.

Choosing $0 < \tilde{q} < 4/(3-\gamma)$, with $\max\{(p-2)/(p-1), 2-p\} < \gamma < 1$ (see Theorem 7.1.4) we can use a Holder inequality with exponents $\left(\frac{2}{\tilde{q}}, \frac{4}{(1-\gamma)\tilde{q}}, \frac{4}{4-2\tilde{q}-(1-\gamma)\tilde{q}}\right)$, so we get

$$\begin{aligned}\int_{B\setminus(Z_u\cup\{|D^2u|=0\})}|D^3u|^{\tilde{q}}\,dx \le &\left(\int_{B\setminus(Z_u\cup\{|D^2u|=0\})}|\nabla u|^{p-2+\alpha}|D^2u|^{\gamma-1}|D^3u|^2\,dx\right)^{\frac{\tilde{q}}{2}}\times\\ &\times\left(\int_B|\nabla u|^{p-2-\beta}|D^2u|^2\,dx\right)^{\frac{(1-\gamma)\tilde{q}}{4}}\times\\ &\times\left(\int_B|\nabla u|^{-\left[\frac{(p-2-\beta)(1-\gamma)}{4}\tilde{q}+\frac{p-2+\alpha}{2}\tilde{q}\right]\frac{4}{4-2\tilde{q}-(1-\gamma)\tilde{q}}}\,dx\right)^{\frac{4-2\tilde{q}-(1-\gamma)\tilde{q}}{4}}.\end{aligned} \tag{7.3.49}$$

Using Theorem 7.1.4, we have

$$\int_{B\setminus(Z_u\cup\{|D^2u|=0\})}|\nabla u|^{p-2+\alpha}|D^2u|^{\gamma-1}|D^3u|^2\,dx \le C(\gamma,\alpha,p,q,n,f,B,\|\nabla u\|_{L^\infty_{loc}(\Omega)},N), \tag{7.3.50}$$

From Theorem 7.2.2, we obtain

$$\int_B|\nabla u|^{p-2-\beta}|D^2u|^2\,dx \le C(p,\beta,B,f,\|\nabla u\|_{L^\infty_{loc}(\Omega)}). \tag{7.3.51}$$

By applying Theorem 7.2.3 to the third integral we get

$$\left[\frac{(p-2-\beta)(1-\gamma)}{4}+\frac{p-2+\alpha}{2}\right]\frac{4\tilde{q}}{4-\tilde{q}(3-\gamma)} < p-1, \tag{7.3.52}$$

that is, taking β close to 1

$$\tilde{q} < \frac{4(p-1)}{6p+2\gamma(2-p)+2\alpha-10}. \tag{7.3.53}$$

We consider the case $p < 2$. For $\gamma \approx 2-p$, we have

$$\tilde{q} < \frac{2(p-1)}{p^2 - p + \alpha - 1}. \tag{7.3.54}$$

In the case $p \geq 2$, for $\gamma \approx 1$, we get

$$\tilde{q} < \frac{2(p-1)}{2p + \alpha - 3}. \tag{7.3.55}$$

Now recalling (7.1.12) (see Theorem 7.1.4), for α close to 1, and therefore for ε sufficiently small, there exists $\delta_0 = \delta_0(\varepsilon)$ such that (7.3.54) and (7.3.55) are fulfilled for any $\tilde{q} < 1 - \varepsilon$ and for any $p \in (2 - \delta_0, 2 + \delta_0)$. This concludes the proof.

□

Now we state a result which will be useful to us later

Proposition 7.3.2. *Let $u_\varepsilon \in C^{1,\beta}_{loc}(\Omega)$ be a weak solution of the regularized problem (7.2.1). Let f, q, p, α obey the assumptions of Theorem 7.1.4. Then for $\gamma \geq 1$, and for any $\tilde{\Omega} \subset\subset \Omega$, we have that*

$$\int_{\tilde{\Omega}} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2+\alpha}{2}} |D^2 u_\varepsilon|^{\gamma-1} |D^3 u_\varepsilon|^2 \, dx \leq C, \tag{7.3.56}$$

for all $\varepsilon \in (0,1)$, where C is a positive constant not depending on ε.

Proof. The proof is similar to that of Theorem 7.1.4. However, we will give a sketch of the proof. First we consider the second linearized equation of (7.2.1) at any fixed solution u_ε, which satisfies

$$\int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} (\nabla u_\varepsilon, \nabla \psi) \, dx = \int_\Omega f \psi \, dx \quad \forall \psi \in C^\infty_c(\Omega). \tag{7.3.57}$$

For $\varphi \in C^\infty_c(\Omega)$, taking $\psi := \varphi_{ij}$ in (7.3.57), we get

$$\begin{aligned}
& \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} (\nabla u_{\varepsilon,ij}, \nabla \varphi) \\
+ \quad & (p-2) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-4}{2}} (\nabla u_\varepsilon, \nabla u_{\varepsilon,j})(\nabla u_{\varepsilon,i}, \nabla \varphi) \\
+ \quad & (p-2)(p-4) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-6}{2}} (\nabla u_\varepsilon, \nabla u_{\varepsilon,j})(\nabla u_{\varepsilon,i}, \nabla u_\varepsilon)(\nabla u_\varepsilon, \nabla \varphi) \\
+ \quad & (p-2) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-4}{2}} (\nabla u_{\varepsilon,ij}, \nabla u_\varepsilon)(\nabla u_\varepsilon, \nabla \varphi) \\
+ \quad & (p-2) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-4}{2}} (\nabla u_{\varepsilon,i}, \nabla u_{\varepsilon,j})(\nabla u_\varepsilon, \nabla \varphi) \\
+ \quad & (p-2) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-4}{2}} (\nabla u_{\varepsilon,i}, \nabla u_\varepsilon)(\nabla u_{\varepsilon,j}, \nabla \varphi) = \int_\Omega f_{ij} \, \varphi.
\end{aligned} \tag{7.3.58}$$

Let us define

$$\varphi := (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{\alpha}{2}} |D^2 u_\varepsilon|^{\gamma-1} u_{\varepsilon,ij} \, \psi^2, \tag{7.3.59}$$

where ψ is defined in (7.3.7). Substituting (7.3.59) in (7.3.58), following the proof of Theorem 7.1.4, we can deduce

$$\begin{aligned} C(p) &\int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2+\alpha}{2}} |D^2 u_\varepsilon|^{\gamma-1} |D^3 u_\varepsilon|^2 \psi^2 \, dx \\ &\leq C(n,\alpha,p,\gamma) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-3+\alpha}{2}} |D^2 u_\varepsilon|^{\gamma+1} |D^3 u_\varepsilon| \psi^2 \, dx \\ &+ C(n,p) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2+\alpha}{2}} |\nabla \psi| |D^2 u_\varepsilon|^{\gamma} |D^3 u_\varepsilon| \psi \, dx \\ &+ C(n,\alpha,p) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p+\alpha-4}{2}} |D^2 u_\varepsilon|^{\gamma+3} \psi^2 \, dx \\ &+ C(n,p) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-3+\alpha}{2}} |D^2 u_\varepsilon|^{\gamma+2} |\nabla \psi| \psi \, dx \\ &+ C(n) \int_\Omega (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{\alpha}{2}} |D^2 f| |D^2 u_\varepsilon|^{\gamma} \psi^2 \, dx, \end{aligned} \tag{7.3.60}$$

where $C(n,\alpha,p,\gamma)$, $C(n,\alpha,p)$, $C(n,p)$, $C(p)$ and $C(n)$ are positive constants. Proceeding as in the proof of Theorem 7.1.4, we get the thesis.

□

In the case in which the function f does not have a sign we can state the following

Theorem 7.3.3. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $u \in C^{1,\beta}_{loc}(\Omega)$ be a weak solution of (7.1.1). For $\gamma > \max\{(p-2)/(p-1), 2-p\}$, we set*

$$\begin{cases} q_N = 2(q_{N-1} - 1) \\ q_0 = 3 + \gamma. \end{cases} \tag{7.3.61}$$

Let $q \geq q_0$, with $q_N \leq q < q_{N+1}$ for some $N \in \mathbb{N}_0$ and p be such that:

$$2 - \frac{1}{C(n,q)} < p < \min\left\{2 + \frac{1}{q-1}, 2 + \frac{1}{C(n,q)}\right\}, \tag{7.3.62}$$

where $C(n,q)$ is given by (7.1.3). Assume

$$\alpha > \alpha(p,N) := \frac{3-p}{2^N} + 1, \tag{7.3.63}$$

if $N = 0$, $\alpha \geq 4 - p$. Let $f(x) \in W^{2,q/(q-\gamma)}_{loc}(\Omega) \cap C^{1,\beta'}_{loc}(\Omega)$. Then, for any $\tilde\Omega \subset\subset \Omega$, for $\gamma \geq 1$, we have that

$$\int_{\tilde\Omega \setminus Z_u} |\nabla u|^{p-2+\alpha} |D^2 u|^{\gamma-1} |D^3 u|^2 \, dx \leq C, \tag{7.3.64}$$

where $C = C(\gamma,\alpha,p,N,q,n,f,\tilde\Omega,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$.

If $\max\{(p-2)/(p-1), 2-p\} < \gamma < 1$, for any $\tilde\Omega \subset\subset \Omega$, we deduce that

$$\int_{\tilde\Omega \setminus (Z_u \cup \{|D^2 u| = 0\})} |\nabla u|^{p-2+\alpha} |D^2 u|^{\gamma-1} |D^3 u|^2 \, dx \leq C, \tag{7.3.65}$$

where $C = C(\gamma,\alpha,p,q,n,f,\tilde\Omega,\|\nabla u\|_{L^\infty_{loc}(\Omega)}, N)$.

Proof. We consider the case $\gamma > 1$, the case $\max\{(p-2)/(p-1), 2-p\} < \gamma < 1$ is similar. The proof is similar to that of the Theorem 7.1.4. The key distinction lies in the non-application of Theorem 7.2.3. However, we will give a sketch of the proof for the reader's convenience.

Let us define

$$\varphi := \frac{G_\epsilon(|\nabla u|)^{\alpha+1}}{|\nabla u|}\,|D^2u|^{\gamma-1}u_{ij}\,\psi^2, \tag{7.3.66}$$

where G_ϵ and ψ are defined in (7.3.6) and (7.3.7). Substituting (7.3.66) in (7.3.5), following the proof of the Theorem 7.1.4, we obtain (7.3.16), that is

$$\begin{aligned} C(p)&\int_\Omega |\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha+1}|\nabla u|^{-1}|D^2u|^{\gamma-1}|D^3u|^2\psi^2\,dx\\ &\leq C(n,\alpha,p,\gamma)\int_\Omega |\nabla u|^{p-3}G_\epsilon(|\nabla u|)^{\alpha}|D^2u|^{\gamma+1}|D^3u|\psi^2\,dx\\ &+C(n,p)\int_\Omega |\nabla u|^{p-2}G_\epsilon(|\nabla u|)^{\alpha}|\nabla\psi||D^2u|^{\gamma}|D^3u|\psi\,dx\\ &+C(n,\alpha,p)\int_\Omega |\nabla u|^{p+\alpha-4}|D^2u|^{\gamma+3}\psi^2\,dx\\ &+C(n,p)\int_\Omega |\nabla u|^{p+\alpha-3}|D^2u|^{\gamma+2}|\nabla\psi|\psi\,dx\\ &+C(n)\int_\Omega |\nabla u|^{\alpha}|D^2f||D^2u|^{\gamma}\psi^2\,dx =: I_1+I_2+I_3+I_4+I_5, \end{aligned} \tag{7.3.67}$$

where $C(n,\alpha,p,\gamma)$, $C(n,\alpha,p)$, $C(n,p)$, $C(p)$ and $C(n)$ are positive constants.
Now we estimate the term I_3. Using a standard Young inequality we obtain

$$\begin{aligned} I_3 &\leq C(n,\alpha,p)\int_{B_{2R}} |\nabla u|^{p+\alpha-4}|D^2u|^{3+\gamma}\,dx\\ &= C(n,\alpha,p)\int_{B_{2R}} |\nabla u|^{p+\alpha-4}|\nabla u|^{\frac{p-2-\beta}{2}}|\nabla u|^{\frac{2-p+\beta}{2}}|D^2u||D^2u|^{2+\gamma}\,dx\\ &\leq \frac{C(n,\alpha,p)}{2}\int_{B_{2R}} |\nabla u|^{p-2-\beta}|D^2u|^2\,dx\\ &+\frac{C(n,\alpha,p)}{2}\int_{B_{2R}} |\nabla u|^{2(p-4+\alpha)+2-p+\beta}|D^2u|^{4+2\gamma}\,dx\\ &\leq C(n,\alpha,p,R,\beta,f,\|\nabla u\|_{L^\infty_{loc}(\Omega)})\\ &+C(n,\alpha,p)\int_{B_{2R}} |\nabla u|^{2(p-4+\alpha)+2-p+\beta}|D^2u|^{4+2\gamma}\,dx, \end{aligned} \tag{7.3.68}$$

where in the last inequality we have used Theorem 7.2.2 and
$C(n,\alpha,p,R,\beta,f,\|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant. Iterating this procedure N-times, we get

$$\begin{aligned} I_3 &\leq NC(n,\alpha,p,R,\beta,f,\|\nabla u\|_{L^\infty_{loc}(\Omega)})\\ &\quad+C(n,\alpha,p)\int_{B_{2R}} |\nabla u|^{2^N(p-4+\alpha)+\sum_{k=1}^{N}2^{k-1}(2-p+\beta)}|D^2u|^{q_N}\,dx, \end{aligned} \tag{7.3.69}$$

where q_N is given by

$$\begin{cases} q_N = 2(q_{N-1}-1)\\ q_0 = 3+\gamma. \end{cases} \tag{7.3.70}$$

For $\beta \approx 1$, we note that $\alpha > (3-p)/2^N + 1$ yields

$$2^N(p-4+\alpha) + \sum_{k=1}^{N} 2^{k-1}(2-p+\beta) \geq 0. \tag{7.3.71}$$

We note that if $N = 0$, then we have $\alpha \geq 4 - p$.

By (7.3.71), using Theorem 7.1.3 we get

$$\begin{aligned} I_3 \leq & NC(n, \alpha, p, R, \beta, f, \|\nabla u\|_{L^\infty_{loc}(\Omega)}) \\ & + C(n, \alpha, p, \|\nabla u\|_{L^\infty_{loc}(\Omega)}) \int_B |D^2 u|^{q_N}\, dx \\ & \leq C(n, q, \alpha, p, f, q, B, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, N), \end{aligned} \tag{7.3.72}$$

where $B := B_{2R}(x_0) \subset\subset \Omega$, with $x_0 \in \Omega$, and $C(n, q, \alpha, p, f, q, B, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, N)$ is a positive constant. Using (7.3.67) and proceeding as in the proof of Theorem 7.1.4, we obtain the thesis.

$\square$

Remark 7.3.4. *Under the assumptions of Theorem 7.3.3, for a weak solution u_ε of the regularized problem (7.2.1), the following inequality, for $\gamma \geq 1$, holds*

$$\int_{\tilde{\Omega}} (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2+\alpha}{2}} |D^2 u_\varepsilon|^{\gamma-1} |D^3 u_\varepsilon|^2\, dx \leq C, \tag{7.3.73}$$

for all $\varepsilon \in [0, 1)$, where C is a positive constant not depending on ε. The case $\varepsilon > 0$ follows by the same arguments used in the proof of Theorem 7.3.3 and Proposition 7.3.2.

Now we are ready to prove Theorem 7.1.1

Proof of Theorem 7.1.1. First we suppose that $f \geq \tau > 0$ in Ω. For $\varepsilon > 0$ sufficiently small we set

$$V_{\epsilon,i} := h_\epsilon(|\nabla u|)|\nabla u|^{k-2} u_i, \tag{7.3.74}$$

where $k > 0$, $h_\epsilon(t) := \epsilon\, h(\frac{t}{\epsilon})$ and $h \in C^2(\mathbb{R}^+)$ is such that

$$h(t) := \begin{cases} 0 & \text{if} \quad t \in [0, 1] \\ t & \text{if} \quad t \in [2, \infty), \end{cases} \tag{7.3.75}$$

with $h(t) \leq t$ for $t \in [1, 2]$. We note that there exists a positive constant $\tilde{C}$ such that $h'_\epsilon(t) \leq \tilde{C}$ and $h''_\epsilon(t) \leq \frac{\tilde{C}}{\epsilon}$ for any $t \geq 0$.

Then, we have

$$\begin{aligned} \frac{\partial V_{\epsilon,i}}{\partial x_j} = & h_\epsilon(|\nabla u|)|\nabla u|^{k-2} u_{ij} + (k-2) h_\epsilon(|\nabla u|)|\nabla u|^{k-4} \langle \nabla u_j, \nabla u \rangle u_i \\ & + h'_\epsilon(|\nabla u|)|\nabla u|^{k-3} \langle \nabla u_j, \nabla u \rangle u_i. \end{aligned} \tag{7.3.76}$$

We set $B := B_R(x_0) \subset\subset \Omega$, with $x_0 \in \Omega$. Moreover, for $\tilde{\delta} > 0$, yet to be determined, sufficiently small, we set $r := 1 + \tilde{\delta}$. Since $h_\epsilon(t) \leq t$ and $h'_\epsilon(t) \leq \tilde{C}$, using a Holder inequality with exponents $(\frac{2}{r}, \frac{2}{2-r})$ we deduce that

$$\begin{aligned}\int_B \left|\frac{\partial V_{\epsilon,i}}{\partial x_j}\right|^r dx &\leq C(k) \int_{B\setminus Z_u} |\nabla u|^{r(k-1)} |D^2 u|^r \, dx \\ &\leq C(k)|B|^{\frac{2-r}{2}} \left(\int_{B\setminus Z_u} |\nabla u|^{2(k-1)} |D^2 u|^2 \, dx\right)^{\frac{r}{2}},\end{aligned} \tag{7.3.77}$$

where $C(k)$ is a positive constant. Now we note that it is possible to find β close to 1 such that

$$2(k-1) \geq p - 2 - \beta \iff k > \frac{p-1}{2}. \tag{7.3.78}$$

So using Theorem 7.2.2 we have that

$$\int_B \left|\frac{\partial V_{\epsilon,i}}{\partial x_j}\right|^r dx \leq C, \tag{7.3.79}$$

with C not depending on ϵ. Moreover

$$\begin{aligned}\frac{\partial V_{\epsilon,i}}{\partial x_j \partial x_l} &= h'_\epsilon(|\nabla u|)\langle \nabla u_l, \nabla u\rangle |\nabla u|^{k-3} u_{ij} + (k-2) h_\epsilon(|\nabla u|)|\nabla u|^{k-4} \langle \nabla u_l, \nabla u\rangle u_{ij} \\ &+ h_\epsilon(|\nabla u|)|\nabla u|^{k-2} u_{ijl} + (k-2) h'_\epsilon(|\nabla u|)\langle \nabla u_l, \nabla u\rangle |\nabla u|^{k-5} \langle \nabla u_j, \nabla u\rangle u_i \\ &+ (k-2)(k-4) h_\epsilon(|\nabla u|)|\nabla u|^{k-6} \langle \nabla u_l, \nabla u\rangle \langle \nabla u_j, \nabla u\rangle u_i \\ &+ (k-2) h_\epsilon(|\nabla u|)|\nabla u|^{k-4} \left(\langle \nabla u_{jl}, \nabla u\rangle + \langle \nabla u_j, \nabla u_l\rangle\right) u_i \\ &+ (k-2) h_\epsilon(|\nabla u|)|\nabla u|^{k-4} \langle \nabla u_j, \nabla u\rangle u_{il} \\ &+ h''_\epsilon(|\nabla u|)\langle \nabla u_j, \nabla u\rangle \langle \nabla u_l, \nabla u\rangle |\nabla u|^{k-4} u_i \\ &+ h'_\epsilon(|\nabla u|) \left(\langle \nabla u_{jl}, \nabla u\rangle + \langle \nabla u_j, \nabla u_l\rangle\right) |\nabla u|^{k-3} u_i \\ &+ (k-3) h'_\epsilon(|\nabla u|)\langle \nabla u_j, \nabla u\rangle \langle \nabla u_l, \nabla u\rangle |\nabla u|^{k-5} u_i \\ &+ h'_\epsilon(|\nabla u|)\langle \nabla u_j, \nabla u\rangle |\nabla u|^{k-3} u_{il}.\end{aligned} \tag{7.3.80}$$

Since $h_\epsilon(t) \leq t$, $h'_\epsilon(t) \leq \tilde{C}$ and $h''_\epsilon(t) \leq \tilde{C}/t$, we deduce that

$$\begin{aligned}&\int_B \left|\frac{\partial V_{\epsilon,i}}{\partial x_j \partial x_l}\right|^r dx \\ &\leq C(k) \int_{B\setminus Z_u} |\nabla u|^{r(k-2)} |D^2 u|^{2r} \, dx + C(k) \int_{B\setminus Z_u} |\nabla u|^{r(k-1)} |D^3 u|^r \, dx =: I_1 + I_2.\end{aligned} \tag{7.3.81}$$

We estimate the term I_1. By applying a standard Young inequality we get

$$\begin{aligned}I_1 &= C(k) \int_B |\nabla u|^{r(k-2)} |D^2 u|^{2r} |D^2 u||D^2 u|^{-1} |\nabla u|^{\frac{p-2-\beta}{2}} |\nabla u|^{\frac{2-p+\beta}{2}} \, dx \\ &\leq \frac{C(k)}{2} \int_B |\nabla u|^{p-2-\beta} |D^2 u|^2 \, dx + \frac{C(k)}{2} \int_B |\nabla u|^{2r(k-2)+2-p+\beta} |D^2 u|^{4r-2} \, dx\end{aligned} \tag{7.3.82}$$

Iterating this procedure $\tilde{N}$-times we obtain

$$\begin{aligned}I_1 &\leq C(k, \tilde{N}) \int_B |\nabla u|^{p-2-\beta} |D^2 u|^2 \, dx \\ &+ C(k, \tilde{N}) \int_B |\nabla u|^{2^{\tilde{N}} r(k-2) + \sum_{s=1}^{\tilde{N}} 2^{s-1}(2-p+\beta)} |D^2 u|^{\tilde{q}_{\tilde{N}}} \, dx,\end{aligned} \tag{7.3.83}$$

where $\tilde{q}_{\tilde{N}}$ is given by

$$\begin{cases} \tilde{q}_{\tilde{N}} = 2(\tilde{q}_{\tilde{N}-1} - 1) \\ q_0 = 2r, \end{cases} \tag{7.3.84}$$

and $C(k, \tilde{N})$ is a positive constant.

For $\beta \approx 1$, we get

$$2^{\tilde{N}} r(k-2) + \sum_{s=1}^{\tilde{N}} 2^{s-1}(2-p+\beta) \geq 0 \iff k > \frac{3-p}{2^{\tilde{N}} r} + \frac{p-3+2r}{r}. \tag{7.3.85}$$

Now we fixed $k > (\alpha+1)/2$. We note that, under assumptions on p (see (7.1.5)), we have $(\alpha+1)/2 \geq p-1$. Recalling $r = 1+\tilde{\delta}$, for any $\tilde{N}$ sufficiently large, there exists $\tilde{\delta} = \tilde{\delta}(\tilde{N}) > 0$, sufficiently small such that $\tilde{q}_{\tilde{N}} \leq 4$. Moreover, by (7.3.85), for any $\tilde{N}$ sufficiently large, there exists $\tilde{\varepsilon} > 0$ sufficiently small

$$k > p - 1 + \tilde{\varepsilon}. \tag{7.3.86}$$

Using Theorem 7.2.2 and Theorem 7.1.3 we get

$$I_1 \leq C + C \int_B |D^2 u|^{\tilde{q}_{\tilde{N}}} \, dx \leq C, \tag{7.3.87}$$

where C is a positive constant not depending on ϵ.
We estimate the term I_2. Using a Holder inequality with exponents $\left(\frac{2}{r}, \frac{2}{2-r}\right)$ we get

$$\begin{aligned} I_2 &= \int_B |\nabla u|^{r(k-\frac{1}{2}(p+\alpha))} |\nabla u|^{\frac{r}{2}(p-2+\alpha)} |D^3 u|^r \, dx \\ &\leq \left(\int_B |\nabla u|^{\frac{2r}{2-r}(k-\frac{1}{2}(p+\alpha))} \, dx \right)^{\frac{2-r}{2}} \left(\int_B |\nabla u|^{p-2+\alpha} |D^3 u|^2 \, dx \right)^{\frac{r}{2}}. \end{aligned} \tag{7.3.88}$$

We note that for any $\tilde{N}$ sufficiently large there exist $\hat{\varepsilon} = \hat{\varepsilon}(\tilde{N})$ sufficiently small, such that

$$\frac{2r}{2-r}(k - \frac{1}{2}(p+\alpha)) > 1 - p \iff k > \frac{\alpha+1}{2} + \hat{\varepsilon}. \tag{7.3.89}$$

By (7.3.89), using Theorem 7.2.3 and Theorem 7.1.4 we get

$$I_2 \leq C, \tag{7.3.90}$$

where C is a positive constant not depending on ϵ.

So by (7.3.87) and (7.3.90), for any $k > (\alpha+1)/2$, we have that

$$\int_B \left| \frac{\partial V_{\epsilon,i}}{\partial x_j \partial x_l} \right|^r \, dx \leq C, \tag{7.3.91}$$

where C is a positive constant not depending on ϵ.
Since $W^{2,r}_{loc}(\Omega)$ is reflexive space, there exists $\tilde{V} \in W^{2,r}_{loc}(\Omega)$ such that

$$V_{\epsilon,i} \rightharpoonup \tilde{V} \quad \text{for } \epsilon \to 0.$$

By the compact embedding, $V_{\epsilon,i} \to \tilde{V}$ in $L^q(\Omega)$ with $q < 2^*$ and up to subsequence $V_{\epsilon,i} \to \tilde{V}$ a.e. in Ω. Since $V_{\epsilon,i} \to |\nabla u|^{k-1} u_i$ a.e. in Ω, then

$$\tilde{V} = |\nabla u|^{k-1} u_i \in W^{2,r}_{loc}(\Omega) \subset W^{2,1}_{loc}(\Omega).$$

In the case f has no sign we cannot apply Theorem 7.2.3. So by similar computations we get

$$\begin{aligned}&\int_B \left|\frac{\partial V_{\epsilon,i}}{\partial x_j \partial x_l}\right|^r \, dx \\ &\quad \leq C(k)\int_B |\nabla u|^{r(k-2)}|D^2 u|^{2r}\, dx + C(k)\int_B |\nabla u|^{r(k-1)}|D^3 u|^r\, dx =: I_1 + I_2.\end{aligned} \tag{7.3.92}$$

Now the term I_1 can be estimated as in the previous case. For the term I_2, from Theorem 7.3.3 and for $k \geq (p+\alpha)/2$, applying a Holder inequality with exponents $(\frac{2}{r}, \frac{2}{2-r})$ we get

$$\begin{aligned}I_2 &= \int_B |\nabla u|^{r(k-\frac{1}{2}(p+\alpha))}|\nabla u|^{\frac{r}{2}(p-2+\alpha)}|D^3 u|^r\, dx \\ &\leq \left(\int_B |\nabla u|^{\frac{2r}{2-r}(k-\frac{1}{2}(p+\alpha))}\, dx\right)^{\frac{2-r}{2}} \left(\int_B |\nabla u|^{p-2+\alpha}|D^3 u|^2\, dx\right)^{\frac{r}{2}} \leq C,\end{aligned} \tag{7.3.93}$$

where C is a positive constant not depending on ε. Proceeding as in the previous case we get the thesis. $\square$

Using Theorem 7.3.3 we can prove

Proof of Theorem 7.1.2. Let $x_0 \in \Omega$ and $R > 0$ such that $B := B_{2R}(x_0) \subset\subset \Omega$. Let us consider $u_\epsilon \in u + W_0^{1,p}(\Omega)$ be a solution of

$$\int_B (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} (\nabla u_\epsilon, \nabla\varphi)\, dx = \int_B f\varphi\, dx \quad \forall \varphi \in C_c^\infty(\Omega). \tag{7.3.94}$$

We remark that if $f \in C^{1,\beta'}_{loc}(\Omega)$, by standard regularity result [68],[101] $u_\epsilon \in C^3(B)$. We fix $i = 1, ..., n$ and we use $\varphi_i \in C_c^\infty(\Omega)$ in (7.3.94). Integrating by parts we get

$$\begin{aligned}&\int_B (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-2}{2}} (\nabla u_{\epsilon,i}, \nabla\varphi)\, dx \\ &\quad + (p-2)\int_B (\epsilon + |\nabla u_\epsilon|^2)^{\frac{p-4}{2}} (\nabla u_\epsilon, \nabla u_{\epsilon,i})(\nabla u_\epsilon, \nabla\varphi)\, dx = \int_B f_i\varphi\, dx.\end{aligned} \tag{7.3.95}$$

We prove (7.1.9) in $B_R(x_0)$. The result will follow by a covering argument.
For $i, k, l \in \{1, ..., n\}$ let us define

$$\varphi := (\epsilon + |\nabla u_\epsilon|^2)^{\frac{(\tilde\alpha-1)(p-2)}{2}} u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2, \tag{7.3.96}$$

where $\tilde\alpha \geq 3$ and ψ is the standard cut-off function defined in (7.3.7).
So

$$\begin{aligned}\nabla\varphi &= (\tilde\alpha-1)(p-2)(\epsilon + |\nabla u_\epsilon|^2)^{\frac{(\tilde\alpha-1)(p-2)-2}{2}} D^2 u_\epsilon \nabla u_\epsilon u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2 \\ &\quad + (\epsilon + |\nabla u_\epsilon|^2)^{\frac{(\tilde\alpha-1)(p-2)}{2}} \nabla u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2 \\ &\quad + (\tilde\alpha-2)(\epsilon + |\nabla u_\epsilon|^2)^{\frac{(\tilde\alpha-1)(p-2)}{2}} u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde\alpha-3}\,\mathrm{sign}(u_{\epsilon,kl})\nabla u_{\epsilon,kl}\psi^2 \\ &\quad + (\epsilon + |\nabla u_\epsilon|^2)^{\frac{(\tilde\alpha-1)(p-2)}{2}} u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde\alpha-2} 2\psi\nabla\psi.\end{aligned} \tag{7.3.97}$$

Substituting (7.3.97) in (7.3.95) we get

$$
\begin{aligned}
&(\tilde{\alpha}-1)(p-2)\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}(\nabla u_{\epsilon,i}, D^2u_\epsilon\nabla u_\epsilon)\times\\
&\qquad\qquad\qquad\times(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde{\alpha}-1)(p-2)-2}{2}}u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi^2\,dx\\
&+\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}|\nabla u_{\epsilon,i}|^2(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde{\alpha}-1)(p-2)}{2}}|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi^2\,dx\\
&+(\tilde{\alpha}-2)\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}(\nabla u_{\epsilon,i},\nabla u_{\epsilon,kl})(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde{\alpha}-1)(p-2)}{2}}\times\\
&\qquad\qquad\qquad\times u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde{\alpha}-3}\operatorname{sign}(u_{\epsilon,kl})\psi^2\,dx\\
&+2\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2}{2}}(\nabla u_{\epsilon,i},\nabla\psi)(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde{\alpha}-1)(p-2)}{2}}u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi\,dx\\
&+(\tilde{\alpha}-1)(p-2)^2\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-4}{2}}(\nabla u_\epsilon,\nabla u_{\epsilon,i})(\nabla u_\epsilon, D^2u_\epsilon\nabla u_\epsilon)\times\\
&\qquad\qquad\qquad\times(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde{\alpha}-1)(p-2)-2}{2}}u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi^2\,dx\\
&+(p-2)\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-4}{2}}(\nabla u_\epsilon,\nabla u_{\epsilon,i})^2(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde{\alpha}-1)(p-2)}{2}}|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi^2\,dx\\
&+(\tilde{\alpha}-2)(p-2)\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-4}{2}}(\nabla u_\epsilon,\nabla u_{\epsilon,i})(\nabla u_\epsilon,\nabla u_{\epsilon,kl})\times\\
&\qquad\qquad\qquad\times(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde{\alpha}-1)(p-2)}{2}}u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde{\alpha}-3}\operatorname{sign}(u_{\epsilon,kl})\psi^2\,dx\\
&+2(p-2)\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-4}{2}}(\nabla u_\epsilon,\nabla u_{\epsilon,i})(\nabla u_\epsilon,\nabla\psi)\times\\
&\qquad\qquad\qquad\times(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde{\alpha}-1)(p-2)}{2}}u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi\,dx\\
&=\int_B f_i(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde{\alpha}-1)(p-2)}{2}}u_{\epsilon,i}|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi^2\,dx.
\end{aligned}
\tag{7.3.98}
$$

Summing over $i=1,...,n$, we get

$$
\begin{aligned}
&(\tilde{\alpha}-1)(p-2)\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde{\alpha}(p-2)-2}{2}}|D^2u_\epsilon\nabla u_\epsilon|^2|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi^2\,dx\\
&+\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde{\alpha}(p-2)}{2}}|D^2u_\epsilon|^2|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi^2\,dx\\
&+(\tilde{\alpha}-2)\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde{\alpha}(p-2)}{2}}(D^2u_\epsilon\nabla u_\epsilon,\nabla u_{\epsilon,kl})|u_{\epsilon,kl}|^{\tilde{\alpha}-3}\operatorname{sign}(u_{\epsilon,kl})\psi^2\,dx\\
&+2\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde{\alpha}(p-2)}{2}}(D^2u_\epsilon\nabla u_\epsilon,\nabla\psi)|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi\,dx\\
&+(\tilde{\alpha}-1)(p-2)^2\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde{\alpha}(p-2)-4}{2}}(\nabla u_\epsilon, D^2u_\epsilon\nabla u_\epsilon)^2|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi^2\,dx\\
&+(p-2)\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{\alpha(p-2)-2}{2}}\sum_{i=1}^{n}(\nabla u_\epsilon,\nabla u_{\epsilon,i})^2|u_{\epsilon,kl}|^{\tilde{\alpha}-2}\psi^2\,dx\\
&+(\tilde{\alpha}-2)(p-2)\int_B (\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde{\alpha}(p-2)-2}{2}}(\nabla u_\epsilon, D^2u_\epsilon\nabla u_\epsilon)(\nabla u_\epsilon,\nabla u_{\epsilon,kl})\times\\
&\qquad\qquad\times|u_{\epsilon,kl}|^{\tilde{\alpha}-3}\operatorname{sign}(u_{\epsilon,kl})\psi^2\,dx
\end{aligned}
\tag{7.3.99}
$$

$$+ 2(p-2)\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)-2}{2}}(\nabla u_\epsilon, D^2u_\epsilon\nabla u_\epsilon)(\nabla u_\epsilon,\nabla\psi)|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi$$
$$-\int_B(\nabla f,\nabla u_\epsilon)(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde\alpha-1)(p-2)}{2}}|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2\,dx =: I_1+\cdots+I_9=0$$

Since $p<2$, using the Cauchy-Schwarz inequality we get

$$\begin{aligned} I_1 =&(\tilde\alpha-1)(p-2)\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)-2}{2}}|D^2u_\epsilon\nabla u_\epsilon|^2|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2\,dx\\ &\geq(\tilde\alpha-1)(p-2)\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)}{2}}|D^2u_\epsilon|^2|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2\,dx \end{aligned} \tag{7.3.100}$$

In the same way, we deduce

$$\begin{aligned} I_6 =&(p-2)\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)-2}{2}}\sum_{i=1}^n(\nabla u_{\epsilon,i},\nabla u_\epsilon)^2|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2\,dx\\ &\geq(p-2)\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)}{2}}|D^2u_\epsilon|^2|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2\,dx \end{aligned} \tag{7.3.101}$$

Using (7.3.100) and (7.3.101) in (7.3.99) we get

$$\begin{aligned} &[(p-1)+(\tilde\alpha-1)(p-2)]\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)}{2}}|D^2u_\epsilon|^2|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2\,dx\\ &\leq I_1+I_2+I_5+I_6\\ &\leq(\tilde\alpha-2)(p-1)\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)+1}{2}}|D^2u_\epsilon||u_{\epsilon,kl}|^{\tilde\alpha-3}|\nabla u_{\epsilon,kl}|\psi^2\,dx\\ &+(2+2|p-2|)\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)+1}{2}}|D^2u_\epsilon||u_{\epsilon,kl}|^{\tilde\alpha-2}|\nabla\psi|\psi\,dx\\ &+\int_B|\nabla f|(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde\alpha-1)(p-2)+1}{2}}|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2\,dx \end{aligned} \tag{7.3.102}$$

Taking $p>2-\frac{1}{\tilde\alpha}$ we get

$$\begin{aligned} &\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)}{2}}|D^2u_\epsilon|^2|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2\,dx\\ &\leq C(\tilde\alpha,p)\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)+1}{2}}|D^2u_\epsilon||u_{\epsilon,kl}|^{\tilde\alpha-3}|\nabla u_{\epsilon,kl}|\psi^2\,dx\\ &+C(p)\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{\tilde\alpha(p-2)+1}{2}}|D^2u_\epsilon||u_{\epsilon,kl}|^{\tilde\alpha-2}|\nabla\psi|\psi\,dx\\ &+\int_B|\nabla f|(\epsilon+|\nabla u_\epsilon|^2)^{\frac{(\tilde\alpha-1)(p-2)+1}{2}}|u_{\epsilon,kl}|^{\tilde\alpha-2}\psi^2\,dx\\ &=:\tilde I_1+\tilde I_2+\tilde I_3, \end{aligned} \tag{7.3.103}$$

where $C(\tilde\alpha,p)$ and $C(p)$ are positive constants.

Now we estimate the term $\tilde I_1$ of (7.3.103). From Proposition 7.3.2 (see also Remark 7.3.4), and using a standard Young inequality we get:

$$\begin{aligned} \tilde I_1\leq&\frac{C(\tilde\alpha,p)}{2}\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{p-2+\alpha}{2}}|D^2u_\varepsilon|^{\gamma-1}|D^3u_\varepsilon|^2\psi^2\,dx\\ &+\frac{C(\tilde\alpha,p)}{2}\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{2\tilde\alpha(p-2)+4-p-\alpha}{2}}|D^2u_\varepsilon|^{2\tilde\alpha-3-\gamma}\psi^2\,dx\\ &\leq C(\gamma,\alpha,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},\tilde\alpha)\\ &+\frac{C(\tilde\alpha,p)}{2}\int_B(\epsilon+|\nabla u_\epsilon|^2)^{\frac{2\tilde\alpha(p-2)+4-p-\alpha}{2}}|D^2u_\varepsilon|^{2\tilde\alpha-3-\gamma}\,dx, \end{aligned} \tag{7.3.104}$$

where α, γ are given in Theorem 7.3.3, and $C(\gamma, \alpha, p, q, n, f, R, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, \tilde{\alpha})$ is a positive constant.

From Theorem 7.3.3 (see also Remark 7.3.4) we can choose $\alpha = \alpha(p, 0) = 4 - p$, and for $\beta \approx 1$, we obtain

$$2\tilde{\alpha}(p-2) + 4 - p - \alpha \geq p - 2 - \beta \iff p > 2 - \frac{1}{2\tilde{\alpha} - 1}. \tag{7.3.105}$$

For $\gamma = 2\tilde{\alpha} - 5$, by Theorem 7.2.2, we have that

$$\tilde{I}_1 \leq C(\gamma, \alpha, \beta, p, q, n, f, R, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, \tilde{\alpha}) \tag{7.3.106}$$

We note that, since $q \geq 3 + \gamma = 2(\tilde{\alpha} - 1) \geq 4$ and p satisfies (7.1.8), we can apply Theorem 7.3.3. Furthermore, from Theorem 7.3.3, we note that $f \in W^{2,q/(q-\gamma)}_{loc}(\Omega) = W^{2,2(\tilde{\alpha}-1)/3}_{loc}(\Omega)$.

We estimate the term $\tilde{I}_2$. Since p fulfils (7.1.8), by definition of ψ and using Theorem 7.1.3 we get

$$\begin{aligned} \tilde{I}_2 &\leq C(R, \tilde{\alpha}, p, \|\nabla u\|_{L^\infty_{loc}(\Omega)}) \int_B |D^2 u_\varepsilon|^{\tilde{\alpha}-1}\, dx \\ &\leq C(p, q, n, f, R, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, \tilde{\alpha}) \end{aligned} \tag{7.3.107}$$

where $C(p, q, n, f, R, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, \tilde{\alpha})$ is a positive constant.

For the term $\tilde{I}_3$, by assumptions on f and p, we deduce

$$\begin{aligned} \tilde{I}_3 &\leq C(\tilde{\alpha}, p, \|\nabla u\|_{L^\infty_{loc}(\Omega)}) \int_B |\nabla f| |D^2 u_\epsilon|^{\tilde{\alpha}-2}\, dx \\ &\leq C(\tilde{\alpha}, p, q, n, f, R, \|\nabla u\|_{L^\infty_{loc}(\Omega)}), \end{aligned} \tag{7.3.108}$$

where $C(\tilde{\alpha}, p, q, n, f, R, \|\nabla u\|_{L^\infty_{loc}(\Omega)})$ is a positive constant. By (7.3.106), (7.3.107) and (7.3.108) we get

$$\int_B (\epsilon + |\nabla u_\epsilon|^2)^{\frac{\tilde{\alpha}(p-2)}{2}} |D^2 u_\epsilon|^2 |u_{\epsilon,kl}|^{\tilde{\alpha}-2} \psi^2\, dx \leq C(\gamma, \alpha, p, q, n, f, R, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, \tilde{\alpha}), \tag{7.3.109}$$

where $C(\gamma, \alpha, p, q, n, f, R, \|\nabla u\|_{L^\infty_{loc}(\Omega)}, \tilde{\alpha})$ is a positive constant.

Let us consider a compact set $K \subset\subset B \setminus Z_u$. By Schauder estimates we remark that

$$\|u_\varepsilon\|_{C^{2,\tilde{\beta}}(K)} \leq C,$$

where $\tilde{\beta} \in (0, 1)$ and C is a positive constant not depending on ε.

Setting now

$$V_\varepsilon := (\varepsilon + |\nabla u_\varepsilon|^2)^{\frac{p-2}{2}} \nabla u_\varepsilon,$$

by (7.3.109), we deduce $V_\varepsilon \in W^{1,\tilde{\alpha}}_{loc}(\Omega)$, and it is uniformly bounded in the space. There exists $\tilde{V} \in W^{1,\tilde{\alpha}}_{loc}(\Omega)$ such that

$$V_\varepsilon \rightharpoonup \tilde{V} \quad \text{for } \varepsilon \to 0.$$

By the compact embedding $V_\varepsilon \to \tilde{V}$ in $L^q_{loc}(\Omega)$ with $q < 2^*$ and up to subsequence $V_\varepsilon \to \tilde{V}$ a.e.. $V_\varepsilon \to |\nabla u|^{p-2} \nabla u$ a.e. therefore

$$\tilde{V} = |\nabla u|^{p-2} \nabla u \in W^{1,\tilde{\alpha}}_{loc}(\Omega)$$

providing the desired regularity result. Let us also emphasize that, by Fatou's Lemma, we also deduce that

$$\int_{B_R(x_0)\setminus Z_u} |\nabla u|^{\tilde{\alpha}(p-2)}|D^2u|^{\tilde{\alpha}}\,dx \le C(\gamma,\alpha,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},\tilde{\alpha}),$$

where $C(\gamma,\alpha,p,q,n,f,R,\|\nabla u\|_{L^\infty_{loc}(\Omega)},\tilde{\alpha})$ is a positive constant. $\square$

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